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Optimal Distribution of Players into Two Alliances in Misère Subtraction Game

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Abstract: In combinatorial game theory, multi-player games expands on the research of two-person games and are widely applicable to computer science and business. The given paper studies the misère version of the subtraction game (Nim with one pile of counters) for two alliances extending the results of Kelly in [2, 3]; Liu and Zhao in [5], Liu and Wang in [6] and Suetsugu in [7]. The main result of the current paper determines the optimal distribution of the players among two alliances that minimizes N, the smallest number of counters such that the larger alliance has a winning strategy for all the games with more than N counters.

MSC: 91A06.

Keywords: Multi-player impartial game, Misère Subtraction game, Alliances.

1. Introduction

1.1. Overview

Interactions among several players occur in several industry applications, including computer game systems and conflict situations in business. Although classical combinatorial game theory for two players has been comprehensively analyzed in [1] by Berlekamp, Conway and Guy, the research is expanding to multi-player games. The analysis of multi-player games, although a lot more practical and fascinating, is extremely complex and the winning moves depend on the relationships of the players. Researchers in the field have established two main ways to consider players' preferences:

(1). Organizing the players into two groups (two alliances)

(2). https://www.overleaf.com/project/5c2cce456b20297dba4725b22) using preference rankings (based on alliance matrix).

The preference or alliance matrix was first introduced in 2012 by Krawec [4] and after that researched by Liu and Zhao [5], Liu and Wang in [6] and Suetsugu in [7]. The given paper investigates the first preference selection and studies games with two alliances. In particular, we will study the best way to organize the $2n+1$ players into two alliances of sizes $n+1$ and n in order for both alliances to win the largest number of games. The game is unevenly balanced and the larger alliance has an advantage. Hence, the goal of the paper is to minimize the starting number of counters N such that the larger alliance wins all the games with more than N counters.

In the introductory chapter we will introduce the essential notation that has been previously used in the literature (see [2] and [3]). Additionally, we will state the preliminary results that are used in the next chapter. These theorems were proved

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by Kelly in [2, 3] and Liu and Zhao in [5] in the case of two alliances. The main chapter, Chapter 2 proves the formulas that formally show how to choose the optimal distribution of players for each alliance.

1.2. Notations and Terminology

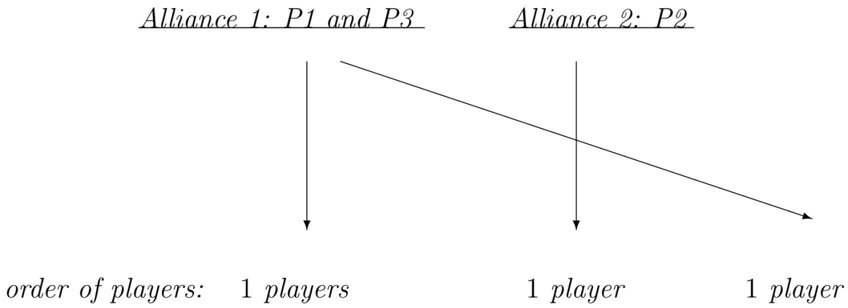
In the case of two evenly distributed alliances, we have proved in paper [2] that the game resembles the two player impartial subtraction game. In this article we consider games with odd number $2n+1$ of players. The alliances in the game are formed by any selection of $n+1$ and n players; these alliances are labelled alliance 1 and alliance 2. Let N denote the starting number of counters of a game. Players take turns by removing $1, .., m\,(m > 1)$ counters at their turn. From [2], the set of turns, where we start with player one, and all $2n+1$ players have their turn, is called a **cycle**. We will consider misère version of the subtraction game as in previous literature ([2, 3, 5, 6]. The misère play theory for subtraction games is similar to the normal play convention of the subtraction game. However, in multi-player subtraction game we avoid the situation where alliance members might compete against each other and hence play against each others interests. Hence, we will work with the misère subtraction game and the game ends when all the coins have vanished providing a clear winning alliance.
The following two examples will illustrate how the formation of alliances among different players influences the outcome of the game. Examples 1.1 and 1.2 analyze three player misère subtraction game, where each player can remove 1 or 2 counters, hence $n = 1$ and $m = 2$.

Example 1.1. *Suppose the first player P1 and the second player P2 play as a team (alliance) against the third player P3.*

	Alliance 1: P1 and P2	Alliance 2: P3
	↓	↓
order of players:	2 *players*	1 *player*

Suppose the starting number of counters in the game is $N = 1$. The first player P1 loses immediately.
If $N = 2$, then the second player P2 loses.
Let $N = 3, 4$, or 5. The first player P1 and second player P2 can leave one counter after their turns and player P3 loses.
Let $N = 6$. The first player P1 and second player P2 can leave only two to four counters after their turns, hence player P3 wins by leaving one or two counters.
Next, we show that if $N \geq 7$, then the players P1 and P2 can choose how many counters to leave from any three consecutive numbers. For $N = 7$, the players P1 and P2 should take one counter each to leave five counters after their turns. No matter how many counters player P3 takes, there will be three or more left, hence P1 or P2 will win.
For $N = 8$ or 9, if the players P1 and P2 form an alliance, they can leave five counters after their turns and player P3 will lose.
In the games with $N \geq 10$, one can justify recursively that the players P1 and P2 form an alliance, they will be able to leave $N-2$, $N-3$ or $N-4$ counters on their turn. However, there are no three consecutive numbers $N \geq 7$ where player P3 wins, therefore player P3 will lose. We see that Alliance 1 (players 1 and 2) loses games with $N = 1, 2$ and 6 counters, while Alliance 2 (player 3) loses games $N = 3, 4, 5$ and 7 or more counters.

Example 1.2. *Suppose the first player P1 and the third player P3 form alliance against player P2.*



Let $N = 1$. As in Example 1.1, the first player P1 loses immediately.

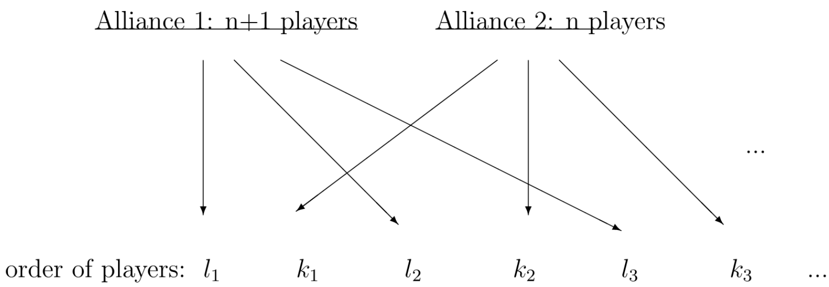
Let $N = 2$ or 3. The first player can leave one counter after her turn, and the second player P2 loses.

Let $N = 4$ or 5. No matter how many counters the first player P1 removes, the second player P2 can leave $1 - 2$ counters after her turn. Hence either player P1 or player P3 lose.

Let $N \geq 6$. If $N = 6$, the winning strategy for the alliance (P1 and P3) is to start the game by removing one counter. No matter what the second player' move is, to support her alliance, the player P3 can leave two counters. The first player will remove one counter and leave one counter, hence the second player (alliance) will lose the game. In the games with $N > 6$, the alliance can choose the winning number of counters and alliance 2 will lose. We see that Alliance 1 (players 1 and 3) loses games with $N = 1, 4$ and 5 counters, while Alliance 2 (player 2) loses games $N = 2, 3$ and 6 or more counters.

In Example 1.1, the losing alliance or player will lose all games starting with $N \geq 7$ counters, while in Example B the losing alliance or player will lose all games starting with $N \geq 6$ counters. In both of these examples the winning alliance consists of two members and the losing alliance is the single player. In summary, it is preferable for P1 to team up with P3 instead of P2 as they will win all the games with six or more counters.

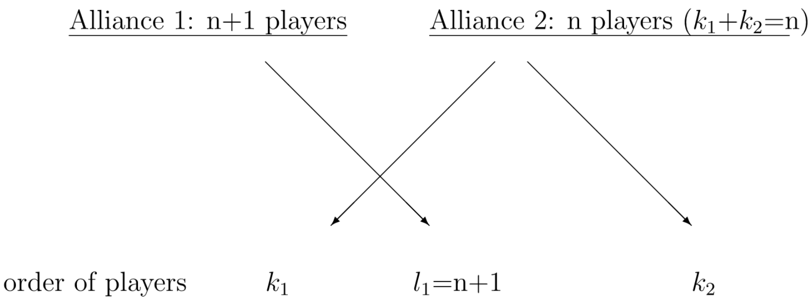
These two examples lead to the goal of this paper. Suppose we want to minimize the starting number of counters N, the cutoff point such that the larger alliance always wins with more than N counters. Specifically, we will be investigating how should the players be distributed to achieve this goal:



The question was introduced by Kelly in [3] and was partially answered in the same paper for the games where two alliances are divided into just two groups of consecutive players proved. Here, we will generalize the results of that paper. Let's restate the Theorem from [3] for just the two groups of players:

Theorem 1.3. *Consider the misère subtraction game for $2n+1$ players, where each player removes 1 or 2 counters on their turn. If the first $n + 1$ players form one alliance and the last n players form another alliance, then the game is reducible to a two player misère game, where the first player takes $n + 1, ..., 2(n + 1)$ counters and the second player takes $n, ..., 2n$ counters. Moreover, the largest game that the first alliance loses is with $3n^2 + 2n + 1$ counters.*

The theorem concludes that the larger alliance has always an advantage. In addition to this result, if the game is played with large enough number of counters, the bigger alliance will win the game no matter who starts. After this introductory result by Kelly, Liu and Zhao [5] studied subtraction games for multiple players and modified the formulas in [2] and [3] in the case the smaller alliance starts the game as illustrated below.



For the rest of the paper we assume WLOG and as in [2] that Alliance 2 has the last turn in each cycle. Next, we will restate the results of Liu and Zhao in [5], which states that the larger alliance has an advantage if the game is played with alliances that are distributed in more than two uneven subgroups for the games where the number of counters $m > 2$.

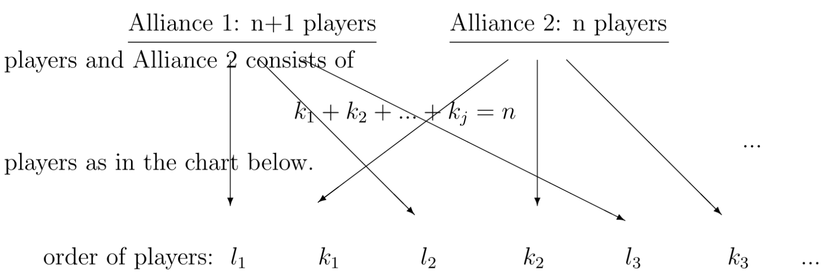
The theorem below determines the winning numbers of counters for two alliances clustered in several groups. We will use this version for our work in next chapter. The result identifies the winning patterns in the first cycle for each alliance with a predetermined groupings of alliances. In particular, we expand Theorem 8 above (from [3]) to games where alliance members are in $j > 1$ clusters and can remove at least one and at most $m \geq 2$ counters at their turn. To describe the grouping of the alliances, we consider games where the alliances are divided into $j > 1$ groups. Alliance 1 consists of

$$l_1 + l_2 + ... + l_j = n + 1$$

players and Alliance 2 consists of

$$k_1 + k_2 + ... + k_j = n$$

players as in the chart below.



Next, we will introduce the notation for the number of counters each alliance can remove at his or her turn.

Suppose Alliance 1 can take

$l_1, ..., m\, l_m$ counters on his first turn;

$l_2, ..., m\, l_2$ counters on his second turn;

$\vdots$

$l_j, ..., m\, l_j$ counters on his j.-th turn in each cycle, and Alliance 2 can take

$$
\begin{aligned}
&k_1, ..., m\,k_1 \text{ counters on his first turn;}\\
&k_2, ..., m\,k_2 \text{ counters on his second turn;}\\
&\quad\vdots\\
&k_j, ..., m\,k_j \text{ counters on his j.-th turn}
\end{aligned}
$$

in each cycle. The following result holds.

Theorem 1.4. *Consider the misère subtraction game for* $2n+1$ *players, where each player removes* 1 *or* 2 *counters on their turn. Suppose the following conditions are satisfied for every* $t \geq 1$

$$l_t < m\,l_t + k_t < l_t + m\,k_t + l_{(t+1)} < ... < m\,l_t + k_t + ... + m\,l_j + k_j, \tag{1}$$

and

$$k_t < m\,k_t + l_{(t+1)} < k_t + m\,l_{(t+)1} + k_{(t+1)} < ... < m\,k_t + l_t + ... + m\,k_j + l_1, \tag{2}$$

then the maximum number of counters taken in each unsafe interval for alliances in the first cycle can be calculated using the following formulas:

$$
\begin{aligned}
N_{11} &= l_1\\
N_{12} &= m\,l_1 + k_1\\
N_{13} &= l_1 + m\,k_1 + l_2\\
N_{14} &= m\,l_1 + k_1 + m\,l_2 + k_2\\
\vdots\quad & \qquad\qquad \vdots\\
N_{1(2j-1)} &= l_1 + m\,k_1 + l_2 + ... + m\,k_{j-1} + l_j\\
N_{1(2j)} &= m\,l_1 + k_1 + m\,l_2 + ... + m\,l_j + k_j.
\end{aligned} \tag{3}
$$

Proof. This is essentially a two-player game where we combine all possible moves for players in given alliance. One can easily see that $N_{11} = l_1$. If the game starts with more than l_1 counters, but no more than $N_{12} = m\,l_1 + k_1$ counters, Alliance 1 can force Alliance 2 to lose the game by leaving k_1 or less counters. Since the conditions (1) and (2) hold, then $N_{12} > N_{11}$. Similarly, at t-th turn for Alliance 1, in order to win the game, they can take x counters, where $l_t < x \leq m\,l_t + k_t$ or $l_t + mk_t + l_{(t+1)} < x \leq m\,l_t + k_t + m\,l_{(t+1)} + k_{(t+1)}...$ or $l_{t1} + mk_{t1} + l_{(t+1)1} + ... + l_{j1} + mk_{j1} < x \leq m\,l_{t1} + k_{t1} + m\,l_{(t+1)1} + k_{(t+1)1} + ... + m\,l_{j1} + k_{j1}$ and $N_{1t+1} > N_{1t}$ i.e. they cannot skip unsafe intervals. Hence if the condition (1) holds then Alliance 1 will not skip any unsafe intervals in the first cycle and will win the games with x counters.

On the other hand, at t-th turn for Alliance 2, they will win the games with x counters, where $k_{t1} < x \leq m\,k_{t1} + l(t+1)1$ or $k_{t1} + ml_{(t+1)1} + k_{(t+1)1} < x \leq m\,k_{t1} + l_{(t+1)1} + m\,k_{(t+1)1} + l_{(t+2)1}$ ... or $k_{t1} + ml_{(t+1)} + k_{(t+1)} + ... + k_j + mk_1 x \leq m\,k_t + l_{(t+1)} + m\,k_{(t+1)} + l_{(t+2)} + ... + m\,k_j + l_1$, and they cannot skip unsafe intervals. Therefore the conditions (1) and (2) will lead to the formulas (3), that provides the distribution of winning number of counters for alliances in the first cycle if no unsafe intervals are skipped. $\square$

This theorem generalizes the results of Liu and Zhao in [5] and corrects the proof of Proposition 8 in [2] by requiring the conditions (1) and (2) to be satisfied for every t. In most groupings of alliances, the conditions of the Theorem 1 are satisfied. For example, if the alliances distribute their members in non-increasing clusters.

Definition 1.5. *We denote the length of the unsafe intervals t, $1 \leq t \leq j$ that Alliance 1 loses in the first cycle by*

$$R_t = N_{1(2t-1)} - N_{1(2(t-1))},$$

where $1 \leq t \leq j$ stands for the t-th turn and $N_{10} = 0$.

If the conditions (1) and (2) are satisfied, the Propositions 12 and 13 in [2] hold, which implies that R_t reduces in each cycle by $m-1$. In particular, if the largest unsafe interval of Alliance 1 losses in the first cycle has length k, then Alliance 1 wins all the games starting from cycle r, where r is an integer such that

$$r > \frac{k}{m-1} + 1.$$

2. Optimal Distribution of Alliances Clustered in Several Groups

In this paper, we will find a distribution of alliances that minimizes N, the number of counters, such that the larger alliance has a winning strategy for all the games with more than N counters.

Suppose the subtraction game is played with $2n+1$ players. Our goal in this chapter is to determine how to distribute the players into two alliances (of sizes $n+1$ and n) in order to minimize N, where N denotes the smallest number of counters, such that the larger alliance has a winning strategy for all the games that start with more than N counters.

We start with several observations about the length of the unsafe intervals R_t, $1 \leq t \leq j$ that Alliance 1 loses in the first cycle.

First proposition shows that a specific distribution of alliances leads to a game with only two unsafe intervals in the first cycle.

Proposition 2.1. *Let j denote the number of turns each alliance has in one cycle. Suppose both alliances group their members so that the largest clusters is the first. In particular, $l_1 = n+2-j$ and $k_1 = n+1-j$ and the remaining turns for each alliance are taken by single players. The given distribution of alliances creates a two player game.*

Proof. Using the formulas (3), one can calculate

$$\begin{aligned}
&N_{11} = n+2-j,\\
&N_{12} = m(n+2-j) + n+1-j = (m+1)(n+1-j) + m, \quad \textit{and}\\
&N_{13} = n+2-j + m(n+1-j) + 1 = (m+1)(n+1-j) + 2.
\end{aligned}$$

Hence $N_{12} - N_{13} = m-1 \geq 0$ which implies that Alliance 2 loses all the games from $N_{11}+1$ up to N_{14}.

Furthermore, rewriting the formulas (3) recursively, we obtain that for all the unsafe intervals $j > 1$:

$$\begin{aligned}
N_{1(2j-1)} &= N_{1(2(j-1)-1)} + k_{j-1} + l_j = N_{1(2(j-1)-1)} + m + 1 \quad \textit{and}\\
N_{1(2j)} &= N_{1(2(j-1))} + l_j + k_j = N_{1(2(j-1))} + m + 1.
\end{aligned}$$

Therefore for any $1 < t \leq j$, $N_{1(2t+1)} - N_{1(2t)} = N_{13} - N_{12} \leq 0$. In other words, Alliance 1 loses the games up to $n+2-j$, but after that Alliance 2 loses all the games with more than $n+2-j$ counters. Hence, the game has two unsafe intervals in the first cycle. In particular, the first unsafe interval Alliance 1 loses in the first cycle is from 1 up to $N_{11} = n+2-j$ counters and the first unsafe interval Alliance 2 loses is from $n+2-j+1$ up to $mn+m+n$ counters. The subsequent unsafe intervals follow similar pattern as was observed in [3]. $\square$

Proposition 2.2. *The length of each unsafe interval that Alliance 1 loses in the first cycle can be calculated by* $R_1 = l_1$

$$R_t = N_{1(2t-1)} - N_{1(2(t-1))} = (m-1)[k_1 + k_2 + ... + k_{(t-1)} - l_1 - l_2 - ... - l_{(t-1)}] + l_t, \tag{4}$$

for $1 < t \leq j$ *and the length of each unsafe interval that Alliance 2 loses in the first cycle can be calculated by*

$$N_{1(2t)} - N_{1(2t-1)} = (m-1)[l_1 + l_2 + ...l_t - k_1 - k_2 - ... - k_{(t-1)}] + k_t, \tag{5}$$

for $1 \leq t \leq j$.

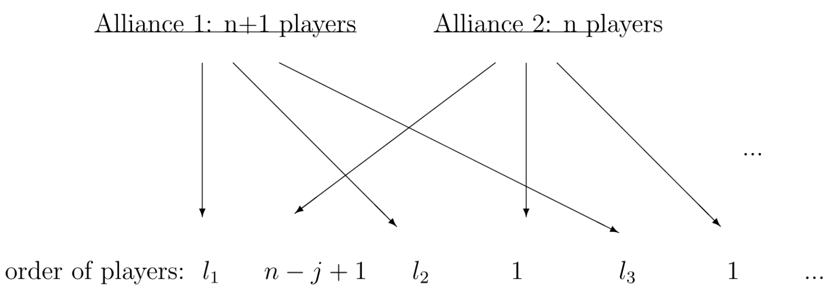
Proof. The direct calculation from formulas (3) gives us (4) and (5). $\square$

The above Proposition indicates that the length of the corresponding unsafe intervals for Alliance 1 reduces by $m-1$ in each cycle under the conditions stated in the previous section and in Proposition 12 in [2]. Therefore, to minimize N, Alliance 1 needs to minimize $\max\{R_t, 1 \leq t \leq j\}$ and subsequently they need to minimize the corresponding t where this is occurring. The primary goal for Alliance 2 is to maximize $\max\{R_t, 1 \leq t \leq j\}$ and then maximize the corresponding t.

Proposition 2.3. *For Alliance 2, the optimal choice is to take* $k_1 = n-j+1$, $k_2 = 1, ..., k_j = 1$.

Proof. It follows from (4) in Proposition 3 that since the length of the t-th unsafe interval of Alliance 1 losses $R_t, 1 \leq t \leq j-1$, contains the term k_t $(1 \leq t \leq j-1)$ with a positive sign in (4) (on the other hand, it occurs with a negative sign in (5)). $\square$

The chart below illustrates Alliance 2's best choice.



The best choices for Alliance 1 are not as straightforward, which is observed in the next few propositions.

Proposition 2.4. *The optimal choice for the Alliance 1 is to take* $l_1 \geq \lceil \frac{(m-1)(n-j+1)+2}{m} \rceil$.

Proof. Since the length of t-th unsafe interval of Alliance 1 losses is $R_1 = l_1$ and $R_2 = (m-1)(n-j+1-l_1) + l_2$, then in order to $R_1 \geq R_2$, one obtains that $(m-1)(n-j+1) + l_2 \leq ml_1$ and the proposition follows. $\square$

Proposition 2.5. *If* $m \geq n-j+1$*, then the optimal choice for the Alliance 1 is to take* $l_1 = n-j+1$, $l_2 = 2, ..., l_j = 1$.

Proof. The length of t-th unsafe interval of Alliance 1 losses is $R_1 = l_1$ and $R_2 = (m-1)(n-j+1-l_1)+l_2$. If we assume that $l_1 < n-j+1$ or equivalently, $m-1 \geq n-j$ we obtain the inequality $R_2 > m-1+l_2 = n-j+1 > l_1$ or $R_2 > R_1$. Hence we will assume that $l_1 \geq n-j+1$ which gives two options: $l_1 = n-j+2$ (and the other alliance members are $l_2 = ... = l_j = 1$.) or $l_1 = n-j+1$ (and the other alliance members are $l_2 = 2, l_3 = ... = l_j = 1$. Since $R_1 = l_1$, the second case is optimal. □

Moreover, Proposition 2.4 implies that if k is large then $\lim\limits_{n\to\infty} l_1 = n-j+1 = k_1$.
Note. For the rest of the paper we will assume that $m < n-j+1$. This case, although more challenging, yields exciting results.
To minimize N, Alliance 1 wants to choose R_1 as small as possible and the lengths of the unsafe intervals $R_t \geq R_1, t > 1$. The Propositions 2.6, 2.7 and 2.8 determine the necessary conditions for this.

Proposition 2.6. *If $l_{(t+1)} \leq l_t$, then the sequence $\{R_t\}$ is non-increasing for $t \geq 1$.*

Proof. Direct computation from the formulas (3) gives us:

$$\begin{aligned} R_{t+1} - R_t &= N_{1(2t+1)} - N_{1(2t)} - N_{1(2(t-1)+1)} + N_{12(t-1))} \\ &= (m-1)(k_t - l_t) + l_{\{t+1\}} - l_t \leq l_{\{t+1\}} - l_t \leq 0\,(t > 1), \end{aligned}$$

since $k_t - l_t < 0$. □

The next two propositions observe that if Alliance 1 has a single member in the partition, then the subsequent lengths of the unsafe intervals do not decrease.

Proposition 2.7. *If there exists t such that $l_t = 1$, then $R_{t+1} = R_t + l_{t+1} - 1$.*

Proof. The computation $R_{t+1} - R_t = (m-1)(1-1) + l_{t+1} - 1$ leads to the desired result. □

Proposition 2.8. *If $R_{t+1} \leq R_t$, $\forall t > 1$ and there $\exists t$ such that $l_t = 1$, then $l_{t+1} = l_{t+2} = ... = l_j = 1$.*

Proof. Compute $R_{t+1} - R_t = (m-1)(1-1) + l_{\{t+1\}} - l_t = l_{\{t+1\}} - 1 \leq 0$. Additionally, we obtain $l_{t+1} \leq (m-1)(l_t - 1) + l_t)$ i.e. $1 = l_{t+1} \leq l_t = 1$. This can be repeated for any index greater than t. □

Proposition 2.9. *The sequence $\{R_t\}$ is non-increasing for $t \geq 2$ if and only if*

$$l_{t+1} \leq l_t + (m-1)(l_t - 1), \;\; for\, t \geq 2. \tag{6}$$

Proof. The non-increasing sequence R_t must satisfy for

$$R_{t+1} - R_t = (m-1)(1-l_t) + l_{\{t+1\}} - l_t \leq 0\,(t \geq 2),$$

which simplifies to $l_{t+1} \leq l_t + (m-1)(l_t - 1)$. □

Therefore l_{t+1} is less than or equal to l_t plus some small non-negative term. We observe from (4), that $l_t, 1 \leq t \leq j-1$ has a negative sign in the formula for $R_t, 1 \leq t \leq j$ and since we want to minimize the length of the unsafe intervals $R_t, 1 \leq t \leq j$, then from now on WLOG we will choose $l_t, 1 \leq t \leq j-1$ in decreasing order with the exception of the last term l_j. Next we will determine a restriction for the size of the second alliance l_2.

Proposition 2.10. *The inequality $R_2 \le R_1$ holds if and only if*

$$l_2 \le ml_1 - (m-1)(n-j+1)). \tag{7}$$

Proof. Direct computation from the formulas (3) verifies the proposition: $R_2 - R_1 = [(m-1)(n-j+1-l_1)+l_2] - l_1 \le 0$ and hence $l_2 \le ml_1 - (m-1)(n-j+1)$. □

The main theorem below summarizes all the conditions required for the best distribution of players for Alliance 1.

Theorem 2.11. *The optimal distribution of players for Alliance 1 is:*

$$\begin{aligned} & l_1 \ge \lceil \frac{(m-1)(n-j+1)+2}{m} \rceil, \\ & l_2 \ge 2, \ \ and \\ & l_{t+1} \le l_t + (m-1)(l_t - 1), \ \ for\, t \ge 2 \end{aligned} \tag{8}$$

Proof. To minimize l_1, the following conditions must be satisfied:

$$\min l_1 = R_1; \quad R_1 = l_1 \ge R_2 = (m-1)(n-j+1-l_1) + l_2;$$

Proposition 2.10 shows that there is a positive linear relationship between l_1 and l_2. Hence we need to select l_2 as small as possible to minimize l_1. Additionally, Proposition 2.10 showcases that $l_1 - l_2 \ge (m-1)(n-j+1-l_1) \ge 0$ and hence $l_2 \le l_1$. In summary, to minimize R_2 and to maintain the inequality $R_2 \le R_1$, we need to choose smallest possible value for l_2. By Proposition 2.8, we cannot use $l_2 = 1$ as this leads to the distribution $l_1 = n+j$ and $l_j = 1$, $t > 2$, which maximizes the first unsafe interval. Hence to minimize l_1, we choose the smallest possible l_2 that is greater than 1. Proposition 2.9 confirms the last inequality for the rest of the terms l_t, $t \ge 3$. □

In practice, to satisfy the conditions of Theorem 12 and to shorten the length of each unsafe interval, the most efficient solution for Alliance 1 is to select l_1 satisfying the condition (8); $l_2 = 2$ if possible and the rest of the terms l_t, $t \ge 3$ approximately equal to each other.

For completeness, we will add the case of $m = 1$ for the theorem.

Remark 2.12. *If m=1, Alliance 1 minimizes N by to choosing*

$$l_1 = \lceil \frac{(n+1)}{j} \rceil,$$

and

$$l_1 \ge l_2 \ge l_3 \ge ... \ge l_j.$$

To verify the remark one observes that our goal is to minimize $R_1 = l_1$, $R_2 = l_2$, $R_3 = l_3, \dots$. The optimal solution to it is to take the terms equal to each other and arrange these in non-increasing order. The final Examples 2.13 and 2.14 illustrate how to best distribute the alliance members according to the formulas (8).

Example 2.13. *For $n = 12$, $j = 3$, $m = 3$, we can use Theorem 12 to obtain:*

$$k_1 = n - j + 1 = 10, \, k_2 = 1, \, k_3 = 1$$

and

$$l_1 = \lceil \frac{(m-1)(n-j+1)+2}{m} \rceil = \lceil \frac{2 \times 10 + 2}{3} \rceil = 8, \, l_2 = 2, \, l_3 = 3.$$

Example 2.14. *If we choose different m: $n = 12$, $j = 3$, $m = 2$, we have that*

$$k_1 = n - j + 1 = 10,\ k_2 = 1,\ k_3 = 1.$$

We cannot select

$$l_1 = \lceil \frac{(m-1)(n-j+1)+2}{m} \rceil = \lceil \frac{1 \times 10 + 2}{2} \rceil = 6,\ l_2 = 2, \quad l_3 = 5 \geq l_2 + (m-1)(l_2 - 1) = 2 + 1 = 3.$$

Therefore, we will choose the distribution

$$l_1 = \lceil \frac{(m-1)(n-j+1)+2}{m} \rceil + 1 = \lceil \frac{1 \times 10 + 2}{2} \rceil + 1 = 7,\ l_2 = 3,\ l_3 = 3.$$

These examples showcase that even small differences in the initial setup influence the optimal choices for alliances. In summary, the best distribution of players for alliances can always be determined from the Proposition 2.3 and from the formulas (8) in Theorem 2.11.

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