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Gregus Type Fixed Point Theorems Satisfying Strict Contractive Condition of Integral Type

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Abstract: In the present paper, we prove some common fixed point theorems of Gregus type for two pairs of self mappings satisfying strict contractive condition of integral type by using the weak subsequential continuity property with compatibility of type (E) in metric spaces. Our results improve some previous known results and relevant literature.

MSC: 47H10, 54H25.

Keywords: Gregus type fixed point, weakly subsequentially continuous, compatible of type (E).

1. Introduction

In the present paper will prove two common fixed point theorems of Gregus type for four mappings which satisfying strict contractive condition of integral type in metric spaces by using subsequential continuity and compatibility of type (E) due to Singh et al. [29].

2. Preliminaries

Definition 2.1. *Two self mappings A and S of a metric space (X, d) are said to be compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$ and $\lim_{n\to\infty} A^2x_n = \lim_{n\to+\infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.*

Remark 2.2. *If $At = St$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not true. Generally compatibility of type (E) implies compatibility of type (B).*

Definition 2.3. *Two self mappings A and S of a metric space (X, d) are A-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$. Also, the pair $\{A, S\}$ is said to be S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$.*

Notice that if A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true. Pant[20] introduced the notion of reciprocal continuity as follows:

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Definition 2.4. *Self mappings A and S of a metric space (X,d) are said to be reciprocally continuous, if* $\lim_{n\to\infty} ASx_n = At$ *and* $\lim_{n\to\infty} SAx_n = St$, *whenever $\{x_n\}$ is a sequence in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, *for some $t \in X$.*

In 2009, H. Bouhadjera and C. Godet Thobie [6] introduced the concept of subsequential continuity as follows:

Definition 2.5. *Two self mappings A and S of a metric space (X,d) is called to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, *for some $t \in X$ and satisfy* $\lim_{n\to\infty} ASx_n = At$ *and* $\lim_{n\to\infty} SAx_n) = St$.

Clearly that continuous or reciprocally continuous maps are subsequentially continuous, but the converse may be not.

Example 2.6. *Let $X = [0,\infty)$ and d is the euclidian metric, we define A, S as follows:*

$$Ax = \begin{cases} 2+x, & 0 \le x \le 2 \\ \frac{x+2}{2}, & x > 2 \end{cases}, \quad Sx = \begin{cases} 2-x, & 0 \le x < 2 \\ 2x-2, & x \ge 2 \end{cases}$$

Clearly that A and S are discontinuous at 2. We consider a sequence $\{x_n\}$ such that for each $n \ge 1$: $x_n = \frac{1}{n}$, clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2$, *also we have:*

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(2-\frac{1}{n}) = 4 = A(2),$$
$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(2+\frac{1}{n}) = 2 = S(2),$$

then $\{A,S\}$ is subsequentially continuous.
On other hand, let $\{y_n\}$ be a sequence which defined or each $n \ge 1$ by: $y_n = 2 + \frac{1}{n}$, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = 2,$$

but

$$\lim_{n\to\infty} ASy_n = \lim_{n\to\infty} A(2+\frac{2}{n}) = 2 \ne A(2),$$
$$\lim_{n\to\infty} SAy_n = \lim_{n\to\infty} S(4+\frac{1}{n}) = 6 \ne S(2),$$

then A and S are never reciprocally continuous.

Definition 2.7. *Let f and S to be two self mappings of a metric space (X,d), the pair $\{f,S\}$ is said to be weakly subsequentially continuous if there exists a sequence $\{x_n\}$ such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, *for some $z \in X$ and* $\lim_{n\to\infty} ASx_n = Az$, $\lim_{n\to\infty} SAx_n) = Sz$.

Notice that subsequentially continuous or reciprocally continuous mappings are weakly subsequentially continuous, but the converse may be not.

Example 2.8. *Let $X = [0,8]$ and d is the euclidian metric, we define A, S as follows:*

$$Ax = \begin{cases} \frac{x+4}{2}, & 0 \le x \le 4 \\ x+1, & 4 \le x \le 8 \end{cases}, \quad Sx = \begin{cases} 8-x, & 0 \le x \le 4 \\ x-2, & 4 \le x \le 8 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \ge 1$: $x_n = 4 - e^{-n}$, clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 4$, *also we have:*

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(4+e^{-n}) = 5,$$
$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(4-\frac{1}{2}e^{-n}) = 4 = S(4),$$

then $\{A,S\}$ is S-subsequentially continuous.

3. Main results

Theorem 3.1. *Let $A, B, S, T : X \to X$, be self mappings of a metric space (X, d) such for all x, y in X we have:*

$$\left(1 + a\left(\int_0^{d(Ax,By)} \varphi(t)\right)^p\right)\left(\int_0^{d(Sx,Ty)} \varphi(t)dt\right)^p < a\Big(\Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p + \Big(\int_0^{d(Sx,By)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p\Big) + \alpha\Big(\int_0^{d(Ax,By)} \varphi(t)dt\Big)^p\Big) + \beta\tau\Big(\Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^p\Big) \quad (1)$$

where a, α, β are non negative numbers such $\alpha+\beta < 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, if the pair $\{A, S\}$ is wsc and compatible of type (E) as well as $\{B, T\}$, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that $\{A, S\}$ is A-subsequentially continuous, there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ and $\lim_{n\to\infty} ASx_n = Az$, also the pair is compatible of type (E) implies that $\lim_{n\to\infty} ASx_n = Sz$, also the pair $\{f, S\}$ is compatible implies that $\lim_{n\to\infty} ASx_n = Sz$ and $\lim_{n\to\infty} SAx_n = Sz$, which implies $Sz = fz =$ and z is a coincidence point for f and S.

Similarly for B and T, suppose that $\{B, T\}$ is B-subsequentially continuous so there exists a sequence $\{y_n\} \in X$ such

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = t,$$

and

$$\lim_{n\to\infty} BTy_n = Bt,$$

also the pair $\{g, T\}$ is compatible of type (E) implies

$$\lim_{n\to\infty} BTy_n = \lim_{n\to\infty} T^2 y_n = Tt$$

and

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} B^2 y_n = Bt$$

which implies that $Bt = Tt$.

Firstly, we prove $Az = Bt$, if not by using (1) we get

$$\Big(1 + a\Big(\int_0^{d(Az,Bt)} \varphi(t)\Big)^p\Big)\Big(\int_0^{d(Sz,Tt)} \varphi(t)dt\Big)^p < a\Big(\int_0^{d(Az,Tt)} \varphi(t)dt\Big)^p\Big(\int_0^{d(Bt,Sz)} \varphi(t)dt\Big)^p + \alpha\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p + \beta\tau\Big(0, 0, \Big(\int_0^{d(Az,Tt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Sz,Bt)} \varphi(t)dt\Big)^p\Big)$$

since $Az = Sz$ and $Bt = Tt$, we get:

$$\begin{aligned}\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p &= \Big(\int_0^{d(\{Az\},\{Bt\})} \varphi(t)dt\Big)^p \\ &< \alpha(\int_0^{d(Az,Bt)} \varphi(t))^p + \beta\tau\Big(0, 0, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \\ &< (\alpha+\beta)\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p\end{aligned}$$

$$< \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p,$$

which is a contradiction, then $Az = Bt$. Now, we prove $z = Az$, if not by using (1) we get:

$$\begin{aligned}\Big(1 + a(\int_0^{d(Ax_nBt)} \varphi(t)\Big)^p \Big(\int_0^{d(Sx_n,Tt)} \varphi(t)dt\Big)^p < a\Bigg[\Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(Bt,Tt)} \varphi(t)dt\Big)^p \\ + \Big(\int_0^{d(Bt,Sx_n)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(Ax_n,Tt)} \varphi(t)dt\Big)^p\Bigg] + \alpha\Big(\int_0^{d(Ax_n,Bt)} \varphi(t)dt\Big)^p \\ + \beta\tau\Big(\Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt)^p, 0, \Big(\int_0^{d(Ax_n,Tt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Sx_n,Bt)} \varphi(t)dt\Big)^p\Big)\end{aligned}$$

letting $n \to \infty$, we get:

$$\begin{aligned}\Big(1 + a(\int_0^{d(z,Az)} \varphi(t)\Big)^p \Big(\int_0^{d(z,Az)} \varphi(t)dt\Big)^p \leq a\Big(\int_0^{d(Bt,z)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(z,Tt)} \varphi(t)dt)\Big)^p + \alpha\Big(\int_0^{d(z,Az)} \varphi(t)dt\Big)^p \\ + \beta\tau\Big(0, 0, \Big(\int_0^{d(z,Tt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,t)} \varphi(t)dt\Big)^p\Big),\end{aligned}$$

since $Az = Sz = Bt$, we get

$$\Big(\int_0^{d(z,Az)} \varphi(t)dt\Big)^p \leq (\alpha + \beta)\Big(\int_0^{d(z,Az)} \varphi(t)dt\Big)^p < \Big(\int_0^{d(z,Az)} \varphi(t)dt\Big)^p,$$

which is a contradiction, then $z = Az = Sz$. Nextly we prove $z = t$, if not by using we get

$$\begin{aligned}\Big(1 + a(\int_0^{d(Ax_n,By_n)} \varphi(t))^p\Big)\Big(\int_0^{d(Sx_n,Ty_n)} \varphi(t)dt\Big)^p < a\Bigg[\Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(By_n,Ty_n)} \varphi(t)dt\Big)^p \\ + \Big(\int_0^{d(Ax_n,Ty_n)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(By_n,Sx_n)} \varphi(t)dt\Big)^p\Bigg] + \alpha\Big(\int_0^{d(Ax_n,By_n)} \varphi(t)dt\Big)^p \\ + \beta\tau\Big(\Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By_n,Ty_n)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax_n,Ty_n)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Sx_n,By_n)} \varphi(t)dt\Big)^p\Big)\end{aligned}$$

letting $n \to \infty$, we get:

$$\begin{aligned}\Big(1 + a\Big(\int_0^{d(z,t)} \varphi(t)\Big)^p\Big)\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p \leq a\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p . \Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p \\ + \alpha\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p + \beta\tau\Big(\Big(0, 0, \Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p,\end{aligned}$$

and so we have:

$$\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p \leq \alpha\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p + \beta\tau\Big(\Big(0, 0, \Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p,$$

which implies that

$$\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p \leq (\alpha + \beta)\Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p < \Big(\int_0^{d(z,t)} \varphi(t)dt\Big)^p,$$

which is a contradiction, then z is a common fixed point for A, B, S and T. For the uniqueness, suppose there is another fixed point w, by using (1) we get:

$$\Big(1 + a(\int_0^{d(Az,Bw)} \varphi(t))^p\Big)(\int_0^{d(Sz,Tw)} \varphi(t)dt)^p < a\Big(\int_0^{d(Az,Tw)} \varphi(t)dt)^p\Big) . \Big(\int_0^{d(Bw,Sz)} \varphi(t)dt\Big)^p\Big) + \alpha\Big(\int_0^{d(Az,Bw)} \varphi(t)dt\Big)^p\Big)$$

$$+\beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tw)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Bw,Sz)}\varphi(t)dt\Big)^p\Big),$$

since z and w are fixed points, get:

$$\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p=\Big(\int_0^{H(\{z\},\{w\})}\varphi(t)dt\Big)^p\leq(\alpha+\beta)\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p<\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p$$

which is a contradiction, then $z=w$. □

Theorem 3.1 improves Theorem 2 of Chauhan et al. [7] and some main results of Djoudi and Aliouche [11] and Theorem 2.5 in [25]. If $\alpha=0$, we obtain the following corollary:

Corollary 3.2. *Let $A,B,S,T:X\to X$, be self mappings such:*

$$\Big(\int_0^{d(Sx,Ty)}\varphi(t)\Big)^p<\alpha\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)^p\Big)$$
$$+\beta\tau\Big(\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^p\Big)$$

where α,β are non negative numbers such $\alpha+\beta<1$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which defined in Theorem 2.1. Suppose that the pairs $\{A,S\}$ and $\{B,T\}$ are A-subsequentially continuous and A-compatible of type(E), then A,B,S and T have a unique common fixed point in X.

Corollary 3.1 improves Corollary 2 of Chauhan et al. [7] and Corollary 2 in [10].

If we take

$$\tau(x_1,x_2,x_3,x_4)=\max\{x_1,x_2,\sqrt{x_1x_3},\sqrt{x_3x_4}\},$$

we get to the following corollary:

Corollary 3.3. *Let A,B,S and T self mappings of metric space (X,d) such:*

$$\Big(1+a\big(\int_0^{d(Ax,By)}\varphi(t)\big)\Big)\int_0^{d(Sx,Ty)}\varphi(t)dt<a\Big(\big(\int_0^{d(Ax,Sx)}\varphi(t)dt\big).\big(\int_0^{d(By,Ty)}\varphi(t)dt\big)+\big(\int_0^{d(Sx,By)}\varphi(t)dt\big).\big(\int_0^{d(Ax,Ty)}\varphi(t)dt\big)\Big)$$
$$+\alpha\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)+(1-\alpha)\max\Bigg\{\int_0^{d(Ax,Sx)}\varphi(t)dt,\int_0^{d(By,Ty)}\varphi(t)dt,\Big(\int_0^{d(Ax,Ty)}\varphi dt\Big)^{\frac{1}{2}}.$$
$$\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^{\frac{1}{2}},\Big(\int_0^{d(fx,Ty)}\varphi(t)dt\Big)^{\frac{1}{2}}.\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^{\frac{1}{2}}\Bigg\},$$

where $0\leq\alpha<1$, and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which defined in theorem 2.1. Suppose that

(1). the pair $\{A,S\}$ is A-subsequentially continuous and A-compatible of type (E),

(2). the pair and $\{B,T\}$ is B-subsequentially continuous and B-compatible of type (E),

then A,B,S and T have a unique common fixed point in X.

Corollary 3.2 improves and generalizes Theorem 2.5 of Pathak and Shahazad in [25].

Corollary 3.4. *Let A,B,S and T be self mappings such:*

$$(1+ad^p(Ax,By))d^p(Sx,Ty)<a\Big[d^p(Ax,Sx).d^p(Bt,Ty)+d^p(Ax,Sx).d^p(Ax,Sx)\Big]$$
$$+\alpha d^p(Ax,By)+\beta\tau\Big(d^p(Ax,Sx),d^p(By,Ty),d(Ax,Ty),d^p(By,Sx)\Big),$$

if the two following conditions hold:

(1). the pair $\{A,S\}$ and $\{B,T\}$ are subsequentially continuous,

(2). the pair $\{A,S\}$ is A-compatible or S-compatible of type (E),

(3). the pair $\{B,T\}$ is B-compatible or T compatible of type (E),

then A,B,S and T have a unique common fixed point in X.

LetΛ be a set of all continuous function $\Lambda:\mathbb{R}_+^5\to\mathbb{R}_+$, such $\lambda(0,0,x,x,x)=kx$, where $0<k<1$.

Theorem 3.5. *Let A,B,S and T be self mappings on metric space (X,d) such for all x,y in X we have:*

$$\Big(1+a\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p<\lambda\Big(\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p,$$
$$\Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p\Big),\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^p\Big) \tag{2}$$

where $\lambda\in\Lambda$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon>0$, $\int_0^\varepsilon\varphi(t)dt>0$, assume that the two pairs $\{A,S\}$ and $\{B,T\}$ are wsc and compatible of type (E), then $A,B,,S$ and T have a unique common fixed point in X.

Proof. As in proof of Theorem 2.1 z is a coincidence point for A and S and t is a coincidence point for B and T, where

$$\lim_{n\to\infty}By_n=t\ \text{ and }\ \lim_{n\to\infty}Ax_n=z,$$

we claim $Az=Bt$, if not by using (2) we get

$$\Big(\int_0^{d(Az,Bt)}\varphi(t)\Big)^p=\Big(\int_0^{d(Sz,Tt)}\varphi(t)dt\Big)^p<\lambda(0,0,\Big(\int_0^{d(Az,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Sz,Bt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p\Big)$$

since $d(Sz,Bt)\leq d(fz,gt)$ and $d(Az,Tt)\leq d(Az,Bt)$, we get:

$$\begin{aligned}\Big(\int_0^{d(Az,Bt)}\varphi(t)\Big)^p&<\lambda(0,0,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p\Big)\\&<k\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p\\&<\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p\end{aligned}$$

which is a contradiction, then $Az=Bt$. Now, we prove $z=Az$, if not by using (1) we get:

$$\begin{aligned}\Big(\int_0^{d(Sx_n,Tt)}\varphi(t)dt\Big)^p&\leq\Big(1+a\Big(\int_0^{d(Ax_n,Bt)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx_n,Tt)}\varphi(t)dt\Big)^p\\&<\lambda\Big(\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Bt,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p,\\&\Big(\int_0^{d(Bt,Sx_n)}\varphi(t)dt\Big)^p,\Big(\Big(\int_0^{d(Ax_n,Bt)}\varphi(t)dt\Big)^p\Big),\end{aligned}$$

letting $n\to\infty$, we get

$$\Big(1+a\Big(\int_0^{d(z,Bt)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(z,Tt)}\varphi(t)dt\Big)^p\leq\lambda\Big(0,0,\Big(\int_0^{d(z,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Bt,z)}\varphi(t)dt\Big)^p\Big),\Big(\int_0^{d(z,Bt)}\varphi(t)dt\Big)^p\Big),$$

consequently we get

$$\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p \leq \lambda\Big(0,0,\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p,\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p,\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p\Big)$$
$$= k\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p < \Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p,$$

which is a contradiction, then $z = Az = Sz$, nextly we claim $z = t$, if not by using (2) we get

$$\Big(1+a\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx_n,Ty_n)}\varphi(t)dt\Big)^p < \lambda\Big(\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p,$$
$$\Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By_n,Sx_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt)^p\Big),$$

letting $n \to \infty$, we get

$$\Big(1+a\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p \leq \lambda\Big(0,0,\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p,\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p\Big),$$
$$= k\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p\Big) < \Big(\int_0^{d(z,t)}\varphi(t)\Big)^p,$$

which is a contradiction, then z is a common fixed point for A, B, S and T. For the uniqueness, it is similar as in proof of Theorem 3.1. □

Corollary 3.6. *For four self-mappings A, B, S and T on metric space (X, d) ,which satisfying for all $x, y \in X$:*

$$(1 + ad^p(Ax, By))d^p(Sx, Ty) < \lambda(d^p(Ax, Sx), d^p(By, Ty), d^p(Ax, Ty), d^p(By, Sx), d^p(Ax, By)),$$

if the two pairs $\{A, S\}, \{B, T\}$ are A-subsequentially continuous and A-compatible of type (E), then A, B, S and T have a unique common fixed point in X.

Corollary 3.6 improves Corollary 4 in [7] and generalizes Theorem 3.1 of [9].

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