Int. J. Math. And Appl., 9(2)(2021), 115–118
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On Almost Class (Q) and Class (M, n) Operators

Wanjala Victor[1,*] and Beatrice Adhiambo Obiero[1]

1 Department of Mathematics and Computing, Rongo University, Kitere Hills, Kenya.

Abstract: In this paper we investigate Some basic properties of n-Perinormal operators and its relation to other classes of operators. We equally introduce a new class of operators, Almost Class (Q) operators. This is achieved by relaxing the conditions for (Q) we generalize this class to the class of n and (n,m)-Almost Class (Q) and a result is given on the class of (n,m)-Almost Class (Q) operator.

Keywords: n-perinormal, n-power-hyponormal, quasi n-power-hyponormal operators, Almost Class (Q), Class (Q) operators.

1. Introduction

Throughout this paper H represents Hilbert space and $B(H)$ the banach algebra of bounded linear operators acting on the Hilbert space H. An operator $T \in B(H)$ is said to be in class (M, n) if $T^{*n}T^n \geq (T^*T)^n$, n-power-hyponormal if $T^nT^* \leq T^*T^n$, normal if $T^*T = TT^*$, quasi n-power-hyponormal if $T^*((T^*T^n) - (T^nT^*))T \geq 0$, Almost Class (Q) if $T^{*2}T^2 \geq (T^*T)^2$. We note that Almost Class (Q) class coincides with class (M, n) when $n = 2$ and also with quasi-hyponormal operator. T is said to be n-Almost Class (Q) if $T^{*2}T^{2n} \geq (T^*T^n)^2$ and (n, m)-Almost Class (Q) if $T^{*2m}T^{2n} \geq (T^{*m}T^n)^2$ for all positive integers n and m.

2. Main Results

Proposition 2.1. *If $T \in (M, n)$ and $S \in B(H)$ is unitarily equivalent to T, then $S \in (M, n)$.*

Proof. If $T \in (M, n)$ with S being unitarily equivalent to T, then it implies existence of a unitary operator $U \in B(H)$ such that $S = U^*TU$ and $S^* = U^*T^*U$, hence;

$$(U^*T^*U)^n(U^*TU)^n \geq (U^*T^*UU^*TU)^n$$
$$(U^*T^*U)^n(U^*TU)^n - (U^*T^*UU^*TU)^n \geq 0$$
$$U^*T^{*n}UU^*T^nU - (U^*T^*UU^*TU)^n \geq 0$$
$$U^*T^{*n}T^nU - (U^*T^*TU)^n \geq 0$$
$$T^{*n}T^n \geq (T^*T)^n$$

Hence $S \in (M, n)$. ∎

* *E-mail: wanjalavictor421@gmail.com*

Proposition 2.2. *If T is (n,m)-Almost Class (Q) with S being unitarily equivalent to T, then S is also (n,m)-Almost Class (Q).*

Proof. If T is in (n,m)-Almost Class (Q) with S being unitarily equivalent to T, then it implies existence of a unitary operator $U \in B(H)$ such that $S = U^*TU$ and $S^* = U^*T^*U$, hence;

$$(U^*T^*U)^m(U^*T^*U)^m(U^*TU)^n(U^*TU)^n \geq (U^*T^{*m}UU^*T^nU)^2$$
$$(U^*T^*U)^m(U^*T^*U)^m(U^*TU)^n(U^*TU)^n - (U^*T^{*m}UU^*T^nU)^2 \geq 0$$
$$U^*T^{*2m}UU^*T^{2n}U - (U^*T^{*m}UU^*T^nU)^2 \geq 0$$
$$U^*T^{*2m}T^{2n}U - (U^*T^{*m}T^nU)^2 \geq 0$$
$$T^{*2m}T^{2n} \geq (T^{*m}T^n)^2$$

Hence S is an (n,m)-Almost Class (Q) operator. □

Proposition 2.3. *Let $T \in (M,n)$, then if follows that $T^* \in (M,n)$*

Proof. Since $T \in (M,n)$;

$$(T^*T)^n - T^{*n}T^n \leq 0$$
$$(T^*T)^{n*} - (T^{*n}T^n)^* \leq 0$$
$$(TT^*)^n - T^nT^{*n} \leq 0$$
$$T^nT^{*n} \geq (TT^*)^n$$

Hence $T^* \in (M,n)$. □

Corollary 2.4. *Let T and T^* be two Class (M,n) operators, then T is n-power Class (Q).*

Theorem 2.5. *Let S and T be commuting (M,n) operators with $T^*S = ST^*$, then $ST \in (M,n)$.*

Proof.

$$\begin{aligned}(ST)^{*n}(ST)^n &= S^{*n}T^{*n}S^nT^n \\ &\leq S^{*n}T^nT^{*n}S^n \\ &= T^nS^{*n}S^nT^{*n} \\ &\leq T^nS^{*n}S^nT^{*n}\end{aligned}$$

Hence $(ST)^{*n}(ST)^n \geq ((ST)^*(ST))^n$. Hence $ST \in (M,n)$. □

Theorem 2.6. *If T is n-power hyponormal operator, then $T \in (M,n)$.*

Proposition 2.7. *If $T \in (M,3)$ and T is an isometry, then $T \in (M,2)$.*

Proof. Since $T \in (M,3)$;

$$T^{*3}T^3 \geq (T^*T)^3$$

$$= (T^{*2}T^2)T^*T \geq (T^{*2}T)^2T^*T$$
$$= T^{*2}T^2 \geq (T^*T)^2 \quad \text{Since T is an isometry}$$

Since T is n-power hyponormal operator;

$$T^nT^* \leq T^*T^n \tag{1}$$

pre-multiplying and post-multiplying the inequality 1 on the left hand side by T^n and T^* on the right

$$T^nT^nT^* \leq T^*T^nT^* \tag{2}$$

post-multiplying the inequality 3 on the left hand side by T^* and T^n on the right

$$T^nT^nT^*T^* \leq T^*T^nT^*T^n \tag{3}$$

□

Proposition 2.8. *If $T \in B(H)$ is both in Class $(M, 2)$ and Class (Q) then T is normal.*

Proof.

$$\begin{aligned}(T^*T - TT^*)^*(T^*T - TT^*) &= (TT^*T - T^*T)^*(T^*T - TT^*)\\
0 &\leq TT^*T^*T - (TT^*)^2 - (T^*T)^2 + T^*TTT^*\\
0 &\leq T^2T^{*2} - (TT^*)^2 - (T^*T)^2 + T^{*2}T^2 \quad (\text{since } T \in (M, 2))\\
0 &\leq T^2T^{*2} - (T^*T)^2 \quad (\text{Since } T \in \text{Class (Q)})\\
&= 0\end{aligned}$$

Hence by [6], $T^*T - TT^* = 0$ and hence T is normal. □

Theorem 2.9. *Let $T_j \dots T_q$ be in class (M, n) operators. Then $T_j \oplus \cdots \oplus T_q \in (M, n)$ operators.*

Proof. Since $T_j \dots T_q$ is in (M, n), we have:

$$\begin{aligned}(T_j \oplus \cdots \oplus T_q)^{*n}(T_j \oplus \cdots \oplus T_q)^n &\geq ((T_j \oplus \cdots \oplus T_q)^*(T_j \oplus \cdots \oplus T_q))^n\\
&= (T_j^{*n} \oplus \cdots \oplus T_q^{*n})(T_j^n \oplus \cdots \oplus T_q^n)\\
&\geq (T_j^{*n} \oplus \cdots \oplus T_q^{*n})(T_1^n \oplus \cdots \oplus T_j^n)\\
&= T_j^{*n}T_j^n \oplus \cdots \oplus T_q^{*n}T_q^n\\
&\geq T_j^{*n}T_j^n \oplus \cdots \oplus T_q^{*n}T_q^n\\
&= (T_j^{*n} \oplus \cdots \oplus T_q^{*n})(T_j^n \oplus \cdots \oplus T_q^n)\\
&\geq (T_j^{*n} \oplus \cdots \oplus T_q^{*n})(T_j^n \oplus \cdots \oplus T_q^n)\\
&= (T_j \oplus \cdots \oplus T_q)^{*n}(T_j \oplus \cdots \oplus T_q)^n\\
&\geq ((T_1 \oplus \cdots \oplus T_j)^*(T_1 \oplus \cdots \oplus T_j))^n\end{aligned}$$

Hence $T_j \oplus \cdots \oplus T_q \in (M, n)$ operators. □

Theorem 2.10. *Let $T_j \dots T_q \in (M, n)$ operators. Then $T_j \otimes \cdots \otimes T_q \in (M, n)$.*

Proof. Let $x_j \dots x_q \in H$, it follows that;

$$\begin{aligned}
(T_j \otimes \cdots \otimes T_q)^{*n}(T_j \otimes \cdots \otimes T_q)^n(x_j \otimes \cdots \otimes x_q) &\geq ((T_j \otimes \cdots \otimes T_q)^*(T_j \otimes \cdots \otimes T_q))^n(x_j \otimes \cdots \otimes x_q) \\
&= (T_j^{*n} \otimes \cdots \otimes T_q^{*n})(T_j^n \otimes \cdots \otimes T_q^n)(x_j \otimes \cdots \otimes x_q) \\
&\geq (T_j^{*n} \otimes \cdots \otimes T_q^{*n})(T_j^n \otimes \cdots \otimes T_q^n)(x_j \otimes \cdots \otimes x_q) \\
&= T_j^{*n}T_j^n x_j \otimes \cdots \otimes T_q^{*n}T_q^n x_q \\
&\geq T_j^{*n}T_j^n x_j \otimes \cdots \otimes T_q^{*n}T_q^n x_q \\
&= (T_j^{*n} \otimes \cdots \otimes T_q^{*n})(T_j^n \otimes \cdots \otimes T_q^n)(x_j \otimes \cdots \otimes x_q) \\
&\geq (T_j^{*n} \otimes \cdots \otimes T_q^{*n})(T_j^n \otimes \cdots \otimes T_q^n)(x_j \otimes \cdots \otimes x_q) \\
&= (T_j \otimes \cdots \otimes T_q)^{*n}(T_j \otimes \cdots \otimes T_q)^n(x_j \otimes \cdots \otimes x_q) \\
&\geq ((T_j \otimes \cdots \otimes T_q)^*(T_j \otimes \cdots \otimes T_q))^n(x_j \otimes \cdots \otimes x_q)
\end{aligned}$$

Hence $T_j \otimes \cdots \otimes T_q \in (M, n)$. □

References

[1] T. Furuta, *Invitation to Linear Operators from Matrices Bounded Linear Operators on a Hilbert Space*, Taylor and Francis, London, (2001).

[2] A. A. Jibril, *On Almost Similar Operators*, Arabian J. Sci. Engg., 21(1996), 442-449.

[3] A. A. Jibril, *On operators for which $T^{*2}T^2 = (T^*T)^2$*, International Mathematics Forum, 46(2010), 2255-2262.

[4] A. A. Jibril, *On operators whose squares are 2-normal*, Journal of Mathematical Sciences : Advances and Applications, 8(2011), 61-72.

[5] Mesaoud Guesba and Mostefa Nadir, *On n-power-hyponormal operators*, Global journal of Pure and Applied Mathematics, 12(2016), 473-479.

[6] S. K. Berberian, *Introduction to Hilbert space*, Chelsea Publishing Company, New-York, (1976).