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On (α,β)-Class (Q) Operators

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Abstract: In this paper, we introduce a new class of operator, the class of (α,β)-Class (Q) operator acting on a complex Hilbert space H. An operator $T \in B(H)$ is said to be (α,β)-Class (Q) if $\alpha^2 T^{*2}T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2}T^2$ for $0 \leq \alpha \leq 1 \leq \beta$. We look at some properties that this class are priviledged to enjoy.

Keywords: Class (Q), Normal, (α,β)-normal, Hypernormal and (α,β)-Class (Q) operators.

1. Introduction

Throughout this paper, H denotes the usual Hilbert space over the complex field and $B(H)$ the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H. In recent years the class of normal operators has been expounded and generalized widely. This has been done by relaxing some conditions of normality and introducing classes such as (α,β)-normal as covered in [4]. This was later extended to the class of p-(α,β)-normal which was covered in [1]. In this paper, we extend the concept of (α,β) to class (Q) operators.

Definition 1.1. *An operators $T \in B(H)$ is said to be :*

(1). Class (Q) if $T^{*2}T^2 = (T^*T)^2$.

(2). (α,β)-normal if $\beta^2 T^*T \geq TT^* \geq \alpha^2 T^*T$.

(3). Normal if $T^*T = TT^*$.

(4). n-perinormal if $T^{*n}T^n \geq (T^*T)^n$.

(5). (α,β)-Class (Q) operator if $\alpha^2 T^{*2}T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2}T^2$.

If $\beta = 1$, we observe from the right inequality that this class coincides with the class of 2-perinormal operators.

2. Main Results

Theorem 2.1. *If $T \in (\alpha,\beta)$-Class (Q), then so is;*

(1). λT for any real λ.

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(2). Any $S \in B(H)$ that is unitarily equivalent to T.

Proof.

(1). Suppose $T \in (\alpha, \beta)$-Class (Q), then

$$\alpha^2 T^{*2} T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2} T^2$$
$$\alpha^2 (\lambda T)^{*2} (\lambda T)^2 \leq ((\lambda T)^* \lambda T)^2 \leq \beta^2 (\lambda T)^{*2} (\lambda T)^2$$
$$\alpha^2 (\lambda^*)^2 (\lambda)^2 T^{*2} T^2 \leq (\lambda^*)^2 (\lambda)^2 (T^*T)^2 \leq (\lambda^*)^2 (\lambda)^2 \beta^2 T^{*2} T^2$$
$$\alpha^2 T^{*2} T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2} T^2$$

and hence $\lambda T \in (\alpha, \beta)$-Class (Q).

(2). Let $S \in B(H)$ be unitarily equivalent to T, then there exists a unitary operator $U \in B(H)$ such that $S = U^*TU$ and $S^* = U^*T^*U$. Then;

$$\begin{aligned} \alpha^2 S^{*2} S^2 &\leq (S^*S)^2 \leq \beta^2 S^{*2} S^2 \\ &= \alpha^2 U^*T^*UU^*T^*US^2 \leq (U^*T^*UU^*TU)^2 \leq \beta^2 U^*T^*UU^*T^*US^2 \\ &= \alpha^2 U^*T^{*2}US^2 \leq (U^*T^*TU)^2 \leq \beta^2 U^*T^{*2}US^2 \\ &= \alpha^2 U^*T^{*2}UU^*TUU^*TU \leq (U^*T^*TU)^2 \leq \beta^2 U^*T^{*2}UU^*TUU^*TU \\ &= \alpha^2 U^*T^{*2}T^2U \leq (U^*T^*TU)^2 \leq \beta^2 U^*T^{*2}T^2U. \end{aligned}$$

Hence the proof. □

Theorem 2.2. *If $T \in B(H)$ is an (α, β)-normal, then $T \in (\alpha, \beta)$-Class (Q).*

Proof. Let $T \in (\alpha, \beta)$-normal, then

$$\beta^2 T^*T \geq TT^* \geq \alpha^2 T^*T \tag{1}$$

pre-multiplying and post-multiplying both sides of the inequality 1,

$$= \beta^2 T^*T^*TT \geq T^*TT^*T \geq \alpha^2 T^*T^*TT$$
$$= \beta^2 T^{*2}T^2 \geq (T^*T)^2 \geq \alpha^2 T^{*2}T^2$$

□

Theorem 2.3. *$T \in (\alpha, \beta)$-Class (Q), then T^* is also (α, β)-Class (Q) for $\alpha\beta = 1$.*

Proof. Since $T \in (\alpha, \beta)$-Class (Q),

$$\alpha^2 T^{*2} T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2} T^2 \tag{2}$$
$$\alpha^4 T^{*2} T^2 \leq \alpha^2 (T^*T)^2 \leq \alpha^2 \beta^2 T^{*2} T^2 \tag{3}$$
$$\alpha^2 \beta^2 T^{*2} T^2 \leq \beta^2 (T^*T)^2 \leq \beta^4 T^{*2} T^2 \tag{4}$$

From (3) and (4) we have;

$$\alpha^2 T^2 T^{*2} \leq \alpha^2 \beta^2 (TT^*)^2 \leq \beta^2 T^2 T^{*2} \tag{5}$$

Hence T^* is also (α, β)-Class (Q). □

Corollary 2.4. *If T and T^* are two (α,β)-Class (Q) for $\alpha,\beta=1$, then T is Class (Q).*

Theorem 2.5. *If T is (α,β)-Class (Q) and P is a unitary operator such that $TP=PT$, then $K=TP$ is also (α,β)-Class (Q).*

Proof.

$$\alpha^2K^{*2}K^2 \le (K^*K)^2 \le \beta^2K^{*2}K^2$$
$$\alpha^2(TP)^{*2}(TP)^2 \le (T^*P^*TP)^2 \le \beta^2(TP)^{*2}(TP)^2$$
$$\alpha^2T^{*2}P^{*2}T^2P^2 \le T^*P^*T^*P^*TPTP \le \beta^2T^{*2}P^{*2}T^2P^2$$
$$\alpha^2T^{*2}P^{*2}P^2T^2 \le T^*TP^*PT^*TP^*P \le \beta^2T^{*2}P^{*2}P^2T^2$$
$$\alpha^2T^{*2}T^2 \le T^*TT^*T \le \beta^2T^{*2}T^2$$
$$\alpha^2T^{*2}T^2 \le (T^*T)^2 \le \beta^2T^{*2}T^2.$$

□

Theorem 2.6. *If S and T are commuting (α,β)-Class (Q) operators with $T^*S=S^*T$, then ST is an (α,β)-Class (Q).*

Proof.

$$\begin{aligned}\alpha^2(ST)^2(ST)^{*2} &= S^2T^2S^{*2}T^{*2}\\ &\le S^2T^2S^{*2}T^{*2}\\ &\le (ST)^2(ST)^{*2}\\ &\le (ST)^{*2}(ST)^2\\ &\le ((ST)^*(ST))^2\end{aligned}$$

and

$$\begin{aligned}((ST)^*(ST))^{*2} &= S^{*2}T^{*2}S^2T^2\\ &\le \beta^2S^{*2}T^{*2}S^2T^2\\ &\le \beta^2(ST)^{*2}(ST)^2.\end{aligned}$$

Hence $\alpha^2(ST)^2(ST)^{*2} \le ((ST)^*(ST))^2 \le \beta^2(ST)^{*2}(ST)^2$. □

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