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New Theorem on Triangles-more Generalized Than Pythagoras Theorem

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Abstract: This paper establishes a basic equation $a^n + b^n = c^n$ applicable for any triangle, having a, b and c as the sides with 'c' being the longest side and 'n' is a number varying from 1 to infinity. Here, a, b, c and n need not always be integers. It also arrives at a relation between largest angle θ (opposite to the longest side 'c') and sides of the triangle with the equation based on cosine rule.The paper graphically and mathematically illustrates the relation between the angle θ and 'n', for different values of 'n' and 'r' (where 'r' is the ratio of sides b/a) for the range of both 'n' and 'r' varying from 1 to infinity. The paper also shows that Pythagoras theorem is a particular case of the above fundamental equation, when $n = 2$. The paper clearly illustrates with an example that the above fundamental equation is valid even when any one (or two or all) of the sides a, b or c will become non-integer values for all powers of $n > 2$. This gives a clear way of understanding the Fermat's Last Theorem.

MSC: 51P99, 60A99.

Keywords: Geometry, Triangle, New theorem.

1. Introduction

There are various new theorems on triangles and quadrilaterals [1, 2]. Some papers discuss use of such properties in building Egyptian pyramids [3, 4]. The triangle properties in combination of circles are also have been established in some papers [5]. The basis for such properties are derived from properties of straight lines [6]. Adding to all those, this paper gives much more fundamental equation which is applicable to all types of triangles. It is important to note that perfect cuboid problem [7] is different, as it deals with integers only. Figure 1 shows a right angled triangle in which $a = 3$ units, $b = 4$ units and $c = 5$ units. According to Pythagoras theorem, we have $a^2 + b^2 = c^2$. By substituting the values, we have $3^2 + 4^2 = 5^2$. Comparing it with the basic equation $a^n + b^n = c^n$, we can see that Pythagoras theorem is a particular case of this equation, when $n = 2$. When "n" is not equal to 2, then obviously the triangle will not be a right angled triangle. What will be the value of "n" for any other angle? The basic equation will establish a relation between "n" and the largest angle of the triangle, with proof.

1.1. New Theorem

Theorem 1.1. *For every triangle ABC with sides a,b,c with c being the longest side, we can always find 'n' for the basic equation $a^n + b^n = c^n$, where 'n' is a number varying from 1 to infinity.*

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Proof. The basic equation to be proved is

$$a^n + b^n = c^n. \tag{1}$$

Let us consider the cosine rule.

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2 * a * b} \tag{2}$$

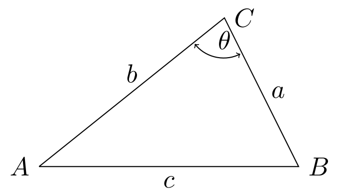

Figure 1. **Cosine rule**

Let us assume that a is the smallest side and $b/a = r$. Hence, equation (1) becomes $a^n(1 + r^n) = c^n$. Therefore $c = (a^n(1 + r^n))^{\frac{1}{n}}$. Hence $c^2 = (a^n(1 + r^n))^{\frac{2}{n}}$. After simplification, we get $c^2 = a^2(1 + r^n)^{\frac{2}{n}}$. Substituting this in equation (2), and putting $b/a = r$, we get,

$$\cos\theta = \frac{(a^2(1 + r^2) - a^2(1 + r^n)^{\frac{2}{n}}}{2 * a^2 * r} \tag{3}$$

Dividing numerator and denominator by a^2, we get

$$\cos\theta = \frac{(1 + r^2) - (1 + r^n)^{\frac{2}{n}}}{2r} \tag{4}$$

This is the equation for finding 'n' for any triangle with given value of 'r'. The theorem is proved by constructional verifications by considering the following examples.

Example 1.2. *Now let us find 'n' for the given triangle with $r = 1$ (when $r = 1$, $a = b$), $a = b = 10$ units and $c = 12$ units. This triangle can be constructed by scale and compass.*

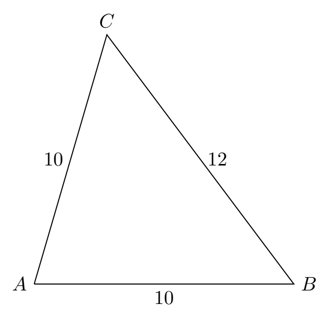

Figure 2. **Finding 'n' for the given triangle**

When $r = 1$, the basic equation reduces to

$$\cos\theta = \frac{(2) - (2)^{\frac{2}{n}}}{2} \tag{5}$$

From cosine rule, for $a = b = 10$ and $c = 12$, we get $\cos\theta = 0.28$. So, $0.28 = \frac{(2)-(2)^{\frac{2}{n}}}{2}$, or,

$$2 * 0.28 - 2 = -2^{\frac{2}{n}}$$

$$-1.44 = -2^{\frac{2}{n}}$$

$$n = 2\left(\frac{\log 2}{\log 1.44}\right)$$

Solving, we get $n = 3.0802$ *(approx). Also,* $\cos\theta = 0.28$*. Hence* $\theta = 73.74^0$*.*

Verification by construction: A triangle is constructed with sides $a = 10$, $b = 10$, $c = 12$. On measuring the angle, we see that angle $\theta = 73.74^0$. This step proves the angle relationship with given sides. Now, putting $a = 10$, $b = 10$ and $n = 3.802$ in equation (1), we get $10^{3.802}$+$10^{3.802}$=6338.7 +6338.7 =12677.4 (LHS of the equation (1)). In this triangle, $c = 12$. So, $12^{3.802}$ =12677.9 (RHS of the equation (1)) (the decimal error is due to calculation approximation). So, equation of the theorem $10^{3.802}$+$10^{3.802} = 12^{3.802}$ is satisfied. Hence the proof of the theorem. □

Remark 1.3. *According to Fermat's Last Theorem,* $a^n + b^n = c^n$ *will not exist for* $n > 2$ *when* a, b, c *and* n *are integers. But here, n is 3.802, which is not an integer. So, this example makes a new way of understanding the Fermat's Last Theorem*

Corollary 1.4. *It is always possible to construct a triangle for* $a^n + b^n = c^n$ *with 'c' being longest side and n being a number varying from 1 to infinity.*

Proof. let $a = 2$ units, $b = 3$ units and $n = 1.5$. So, $r = b/a = 1.5$. Substituting this in equation (4), we get,

$$\cos\theta = \frac{(1 + 1.5^2) - (1 + 1.5^{1.5})^{\frac{2}{1.5}}}{2 * 1.5} = -0.2554.$$

Hence $\theta = 104.8^0$

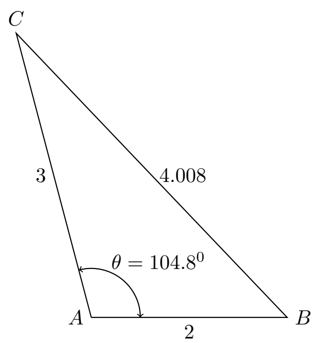

Figure 3. **Constructing the triangle for the given values**

Construction is shown in Figure 3. By measuring, we get the side 'c' as 4.008 units.

Mathematical verification: Substituting the values for $a = 2$, $b = 3$ and $n = 1.5$, we get $2^{1.5} + 3^{1.5} = 8.025$. Here, $c = 4.008$. So, $4.008^{1.5} = 8.024$. Hence the proof of the corollary. □

2. Main Result

In this section, results are discussed.

Special cases

Case 1: When $n = 2$

In a right angled triangle, $a^2 + b^2 = c^2$ or $a^2 + b^2 - c^2 = 0$. Substituting this in equation (2), we get, $\cos\theta = 0/2 * a * b$ =0 or $\theta = 90^0$. So, it leads to Pythagoras theorem, for $n = 2$.

Case 2: When $n = 1$

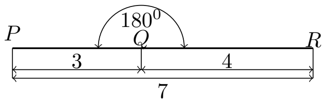

Figure 4. **Triangle becoming a straight line**

Putting $n = 1$ in basic equation $a^n + b^n = c^n$, we get $a + b = c$ or $c^2 = (a + b)^2$. Substituting this in equation (2), we get, $\cos\theta = a^2 + b^2 - (a + b)^2/2 * a * b = 0$ or $\cos\theta = -1$. So, $\theta = 180^0$. This is shown in Figure 3.

2.1. Tables and Graphs

S.No	Angle (in degrees, for $r = 1$)	n
1	180	1
2	90	2
3	78.03	3
4	72.96	4
5	70.11	5
6	68.28	6
7	67.01	7
8	66.08	8
9	65.37	9
10	64.8	10

Table 1. **Variation of angle for n**

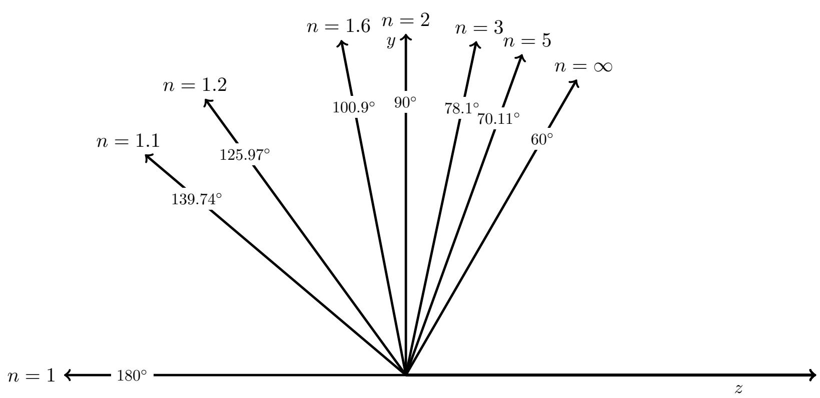

Figure 5. **Variation of angle with ‘n’, pictorial representation**

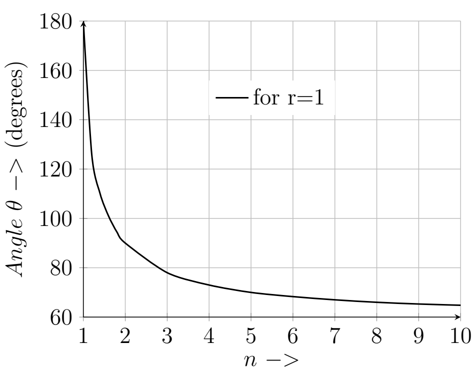

Figure 6. **Variation of ‘n’ for r =1,graphical representation.**

2.2. For large values of 'n'

The value of 'n' varies from 1 to 2 for the angles between 180 to 90 degrees but varies from 2 to infinity for angles between 90 to 60 degrees. As 'n' $\rightarrow \infty$, angle $\rightarrow$ to 60^0. This can be mathematically illustrated as below:
By putting $r = 1$, equation (4) becomes, $\cos\theta = (2 - 2^{2/n})/2$. For large values of n, $\cos\theta = (2 - 2^{2/n})/2 = (2 - 1)/2 = 1/2$. So, $\theta = 60^0$.

S.No	Angle for $r = 2$	Angle for $r = 3$	Angle for $r = 4$	n
1	180	180	180	1
2	90	90	90	2
3	80.31	82.54	84	3
4	77.34	80.94	83	4
5	76.25	80.54	82.86	5
6	75.83	80.44	82.83	6
7	75.65	80.41	82.821	7
8	75.58	80.409	82.8196	8
9	75.55	80.407	82.8193	9
10	75.53	80.40	82.8192	10

Table 2. **Table showing "n" for $r = 2, 3$ and 4**

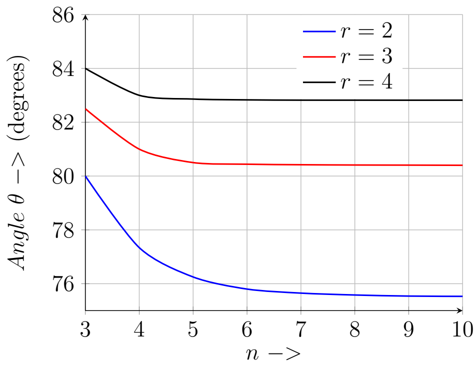

Figure 7. **Variation of 'n' for different values of 'r'**

As 'r' increases, rate of variation on 'n' decreases. This can be visualised from the graph in the Figure 7.

2.3. Finding the angle for given value of 'n' from the graph

Let the given value of 'n' be 5.

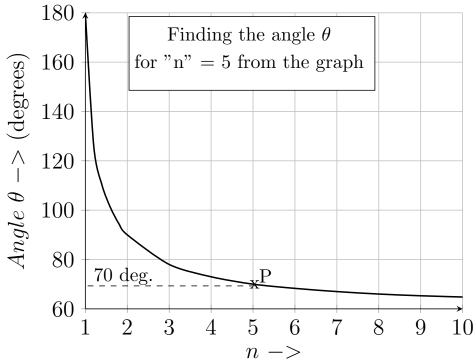

Figure 8. **Finding the angle for $n = 5$**

Vertical line drawn from $n = 5$ intersects the graph at the point 'P'. The angle for that point is found to be 70 degrees.

3. Conclusions

3.1. Establishing a basic equation

This paper establishes a Fundamental equation $a^n + b^n = c^n$ for any triangle with sides a, b,and c, where 'c' is the longest side and 'n' is any number between 1 to ∞

3.2. Pythagoras Theorem

This paper shows that the Pythagoras theorem is a particular case of the basic equation for $n = 2$.

3.3. Finding the largest angle

This paper arrives at generalized equation for finding the largest angle θ opposite to the longest side 'c', for any given values of 'r' and 'n', where 'r' = b/a. Since 'a' is taken as the smallest side, 'r' varies from 1 to ∞

3.4. About Fermat's Last theorem

This paper shows with an example that the basic equation $a^n + b^n = c^n$ is satisfied when 'a', 'b' and 'c' are integers but 'n' is not an integer, even for values of $n > 2$. This has been illustrated in Article 1.3 of this paper, providing a new perspective of looking at Fermat's last theorem. In this connection, the paper concludes that the fundamental equation of the triangle is very likely to become a stepping stone in finding a simpler proof for Fermat's last theorem.

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