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On N-A-Metrically Equivalent and A-Metrically Equivalent Operators

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Abstract: The study of metric equivalence of operators was covered by Nzimbi in [3] while the study of n-metric equivalence was covered by Wanjala in [5], in this study, we extend the study of metric equivalence and n-metric equivalence of operators to Semi-Hilbertian spaces and look at some their nice algebraic properties and their relation to one another in the semi-Hilbertian spaces.

Keywords: N-A-Metrically equivalent operators,A-Metrically equivalent operators, A-normal and n-A-normal.

1. Introduction

A-metrically equivalent operators may be regarded as a generalization of metrically equivalent operators. This is realized when $A= I$ and $T^{\sharp}= T^{*}$.

Definition 1.1. *Two operators $S \in B_{\mathrm{A}}(H)$ and $T \in B_{\mathrm{A}}(H)$ are said to be:*

(1). A-metrically equivalent, denoted by $S \sim_{\text{A-m}} T$, provided $T^{\sharp_{\mathrm{A}}}T = S^{\sharp_{\mathrm{A}}}S$ equivalently; $\| T\xi \|_{\mathrm{A}} = \| S\xi \|_{\mathrm{A}} \quad \forall \quad \xi \in \mathrm{H}$. $T^{\sharp_{\mathrm{A}}} = A^{\dagger}T^{*}\mathrm{A}$, in which $A^{\dagger}$ is the Moore-penrose inverse of A.

(2). n-A-metrically equivalent, denoted by $S \sim_{\text{n-A-m}} T$, provided $T^{\sharp_{\mathrm{A}}}T^{n} = S^{\sharp_{\mathrm{A}}}S^{n}$ for a positive integer n.

Definition 1.2. *An operator $T \in B(H)$ is*

(1). A-Contraction if $\| T\xi \|_{\mathrm{A}} \leq \| \xi \|_{A}$ for every $\xi \in H \Leftrightarrow T^{*}AT \leq A$.

(2). A-Isometry if $T^{*}AT = A \Leftrightarrow \| T\xi \|_{\mathrm{A}} = \| \xi \|_{\mathrm{A}}$ for every $\xi \in H$.

(3). A-Unitary if $T^{*}AT = TAT^{*} = A \Leftrightarrow \| T^{*}\xi \|_{A} = \| T\xi \|_{A} = \| \xi \|_{\mathrm{A}}$ for every $\xi \in H$.

(4). A-Normal if $T^{*}AT = TAT^{*} \Leftrightarrow \| T\xi \|_{A} = \| T^{*}\xi \|_{A}$ for every $\xi \in H$.

(5). A-Partial isometry if $\| T\xi \|_{A} = \| \xi \|_{A}$ for every $\xi \in N(AT)^{\perp_{A}}$.

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2. Main Results

Theorem 2.1. *If T is an* A*-normal operator and $S \in B_{\mathrm{A}}(H)$ is* A*-unitarily equivalent to T, then S is* A*-normal.*

Proof. Suppose

$$
\begin{aligned}
S &= (U^{\sharp_A}T^{\sharp_A}U)(U^{\sharp_A}TU)\\
&= U^{\sharp_A}T^{\sharp_A}TU\\
&= U^{\sharp_A}TT^{\sharp_A}U\\
&= SU^{\sharp_A}T^{\sharp_A}U\\
&= SU^{\sharp_A}US^{\sharp_A}\\
&= SP_{\overline{R(A)}}S^{\sharp_A}\\
&= SS^{\sharp_A}
\end{aligned}
$$

□

Theorem 2.2. *Let S and T be* A*-metrically equivalent operators, then $S^{\sharp_{\mathrm{A}}}$ and $T^{\sharp_{\mathrm{A}}}$ are co-*A*-metrically equivalent operators.*

Proof. Since S and T are A-metrically equivalent operators;

$$
\begin{aligned}
T^{\sharp_A}T &= S^{\sharp_A}S\\
&= (T^{\sharp_A}T)^{\sharp_A}\\
&= (S^{\sharp_A}S)^{\sharp_A}\\
&= (T^{\sharp_A})^{\sharp_A}(T)^{\sharp_A}\\
&= (S^{\sharp_A})^{\sharp_A}(S)^{\sharp_A}\\
&= TT^{\sharp_A}\\
&= SS^{\sharp_A}
\end{aligned}
$$

Thus $S^{\sharp_A}$ and $T^{\sharp_A}$ are co-A-metrically equivalent operators. □

Proposition 2.3. *Let $S, T \in B_{\mathrm{A}}(H)$, then S and T are n-*A*-metrically equivalent if S^n and T^n are* A*-metrically equivalent operators for some positive definite integer n.*

Proof.

$$
\begin{aligned}
S^{\sharp_A}S^n &= T^{\sharp_A}T^n\\
&= S^n(S^{\sharp_A})^n\\
&= T^n(T^{\sharp_A})^n\\
&= S^{\sharp_A}S^n(S^{\sharp_A})^{n-1}\\
&= T^{\sharp_A}T^n(T^{\sharp_A})^{n-1}\\
&= S^{\sharp_A}S^nS^{\sharp_A}(S^{\sharp_A})^{n-2}\\
&= T^{\sharp_A}T^nT^{\sharp_A}(T^{\sharp_A})^{n-2}
\end{aligned}
$$

$$= (S^n)^{\sharp_A} S^n = (T^n)^{\sharp_A} T^n$$

Hence the proof. □

Proposition 2.4. *Let $S, T \in B_{\mathrm{A}}(H)$ be both 2-A-metrically equivalent and 3-A-metrically equivalent operators, then they are n-A-metrically equivalent for all $n \geq$.*

Proof. We first prove that the result holds when $n = 4$, then by induction we prove that the result holds for all $n > 4$. Since S and T are 2-A-metrically equivalent;

$$S^{\sharp_A} S^2 = T^{\sharp_A} T^2 \tag{1}$$

post-multiplying both sides of the equation 1 by S and T respectively, we get;

$$S^{\sharp_A} S^3 = T^{\sharp_A} T^3 \tag{2}$$

$$S^2 S^{\sharp_A} S = T^2 T^{\sharp_A} T \tag{3}$$

Pre-multiplying both sides of the equation 3 by S and T respectively, we get;

$$S^3 S^{\sharp_A} S = T^3 T^{\sharp_A} T \tag{4}$$

$$S^{\sharp_A} S^4 = T^{\sharp_A} T^4 \tag{5}$$

Suppose that $S^{\sharp_A} S^n = T^{\sharp_A} T^n$ and that the result is true for all $n > 4$, it follows that; $S^{\sharp_A} S^{n+1} = T^{\sharp_A} T^{n+1}$ and hence S and T are $(n+1)$-A-metrically equivalent and hence by induction the proof follows. □

Theorem 2.5 ([1]). *A necessary and sufficient condition that for an operator $T \in B_{\mathrm{A}}(H)$ to be A-normal is that $R(T^\sharp T) \subset \overline{R(A)}$ and $\| T^\sharp T\xi \|_A = \| TT^\sharp \xi \|_A$ for all $\xi \in H$.*

Corollary 2.6. *An operator $T \in B_{\mathrm{A}}(H)$ is A-normal if and only if T and $T^\sharp$ are A-metrically equivalent.*

Proof. Proof follows from Theorem 2.1. □

Proposition 2.7. *$(S - \lambda)$ and $(T - \lambda)$ are n-A-metrically equivalent if S and T are A-metrically equivalent.*

Proof. Let $(S - \lambda)$ and $(T - \lambda)$ be n-A-metrically equivalent; then

$$(S - \lambda)^{\sharp_A} (S - \lambda)^n = (T - \lambda)^{\sharp_A} (T - \lambda)^n$$

thus

$$(S^{\sharp_A} - \overline{\lambda}) \left(\sum_{p=1}^{n} (-1)^k \binom{n}{p} S^{n-p} \right) = (T^{\sharp_A} - \overline{\lambda}) \left(\sum_{p=1}^{n} (-1)^k \binom{n}{p} T^{n-p} \right)$$

$$\left(\sum_{p=1}^{n} (-1)^k \binom{n}{p} \right) S^{\sharp_A} S^{n-p} \lambda^k = \left(\sum_{p=1}^{n} (-1)^k \binom{n}{p} \right) T^{\sharp_A} T^{n-p} \lambda^k$$

$$\sum_{p=1}^{n} (-1)^k \binom{n}{p} (\lambda)^k S^{\sharp_A} S^{n-p} = \sum_{p=1}^{n} (-1)^k \binom{n}{p} (\lambda)^k T^{\sharp_A} T^{n-p}$$

$$\sum_{p=1}^{n} (-1)^k \binom{n}{p} (\lambda)^k (S^{\sharp_A} S^{n-p} - T^{\sharp_A} T^{n-p}) = 0$$

when $k=n$, hence

$$(-1)^{n-1}n(\lambda)^{n-1}(S^{\sharp_A}S^{n-p}-T^{\sharp_A}T^{n-p})+\sum_{p=1}^{n-2}(-1)^k\binom{n}{p}(\lambda)^k(\lambda)^k(S^{\sharp_A}S^{n-p}-T^{\sharp_A}T^{n-p})=0$$

put $\lambda=qe^{i\theta}$, $0\leq\theta\leq 2\pi$, $q>0$, we get;

$$(-1)^{n-1}n(qe^{i\theta})^{n-1}(S^{\sharp_A}S^{n-p}-T^{\sharp_A}T^{n-p})+\sum_{p=1}^{n-2}(-1)^k\binom{n}{p}(qe^{i\theta})^p(S^{\sharp_A}S^{n-p}-T^{\sharp_A}T^{n-p})$$

hence;

$$(-1)^{n-1}(S^{\sharp_A}S-T^{\sharp_A}T)+\frac{1}{n(qe^{i\theta})^{n-1}}+\left(\sum_{p=1}^{n-2}(-1)^k\binom{n}{p}(qe^{i\theta})^p(S^{\sharp_A}(S^{n-p}-T^{\sharp_A}T^{n-p}))\right)$$

taking $q\to\infty$ we obtain;

$$S^{\sharp_A}S-T^{\sharp_A}T=0 \text{ implying } S^{\sharp_A}S=T^{\sharp_A}T$$

□

Lemma 2.8. *Let S and T be linear operators on a Hilbert space H. If $S\sim_{\text{A-m}}T$, then*

(1). If T is A*-isometric, then S is also* A*-isometric.*

(2). If T is an A*-contraction, then S is also an* A*-contraction.*

(3). If T is an A*-partial isometry, then S is also an* A*-partial isometry.*

(4). If S and T are A*-positive, then $S=T$.*

Proof.

(1). The proof follows from $S^{\sharp_A}S=T^{\sharp_A}T=P_{\overline{R(A)}}$.

(2). This follows from $\| T\xi \|_A=\| S\xi \|_A\leq\| \xi \|_A$ for every $\xi\in H$.

(3). If T is an A-partial isometry, then $T^{\sharp_A}T$ is a projection. Since $S\sim_{A\text{-}m}T$, we have $S^{\sharp_A}S=T^{\sharp_A}T$. This proves that $S^{\sharp_A}S$ is a projection and hence S is an A-partial isometry.

(4). Positivity of S and T implies that S and T are A-self adjoint. Thus $S^2=SS^{\sharp_A}$ and $T^2=T^{\sharp_A}T$. By hypothesis, we have $S^2=T^2$, thus $S=T$. □

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