Int. J. Math. And Appl., 9(2)(2021), 91–95
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Common Fixed Point Theorem for Weakly Compatible Mappings in Dislocated Metric Space

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Abstract: In this paper we have to prove a common fixed point theorem for two pairs of compatible maps in dislocated metric space which generalizes and improves similar fixed point results.

MSC: 47H10, 54H25.

Keywords: Dislocated metric space, weakly compatible maps, common fixed point.

1. Introduction

In 1922, S. Banach established a fixed point theorem for contraction mapping in metric space. After that many fixed point theorems have been established by different authors. In 2000, P. Hitzler and A. K. Seda [8] generalized the notion of dislocated metric space in which self distance of a point need not be equal to zero. They also introduced the famous Banach contraction principle in this space. The study of common fixed points of mappings in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous re-search activity. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. C. T. Aage and J. N. Salunke [2], A. Isufati [1] generalized some important fixed point theorems in single and pair of mappings in dislocated metric space. The purpose of this paper is to prove a common fixed point theorem for two pairs of weakly compatible mappings in dislocated metric space. Our result generalizes and improves the similar results of fixed points.

2. Preliminaries

The following definitions, lemmas and theorems will be help to prove the main result.

Definition 2.1 ([4]). *Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

(1). $d(x, y) = d(y, x)$

(2). $d(x, y) = d(y, x) = 0$ *implies* $x = y$.

(3). $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

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Then d is called dislocated metric(or simply d-metric) on X.

Definition 2.2 ([8]). *A sequence $\{x_n\}$ in a d-metric space (X,d) is called a Cauchy sequence if for given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $m,n \geq n_0$, we have $d(x_m, x_n) < \varepsilon$.*

Definition 2.3 ([8]). *A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n x) \to 0$ as $n \to \infty$. In this case, x is called limit of $\{x_n\}$ in d and we write $x_n \to x$.*

Definition 2.4 ([8]). *A d-metric space (X,d) is called complete if every Cauchy sequence in it is convergent with respect to d.*

Definition 2.5 ([8]). *Let (X,d) be a d-metric space. A map $T : X \to X$ is called contraction if there exists a number k with $0 \leq k < 1$ such that $d(Tx, Ty) \leq kd(x,y)$.*

We state the following lemmas without proof.

Lemma 2.6. *Let (X,d) be a d-metric space. If $T : X \to X$ is a contraction function, then $\{T^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.*

Lemma 2.7 ([8]). *Limits in a d-metric space are unique.*

Definition 2.8 ([5]). *Let A and S be mappings from a metric space (X,d) into itself. Then, A and S are said to be weakly compatible if they commute at their coincident point; that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.*

Theorem 2.9 ([8]). *Let (X,d) be a complete d-metric space and let $T : X \to X$ be a contraction mapping, then T has a unique fixed point.*

3. Main Results

Theorem 3.1. *Let (X,d) be a complete d-metric space. Let $P,Q,S,T : X \to X$ be Continuous mappings satisfying the conditions:*

(1). $S(X) \subset Q(X)$ and $T(X) \subset P(X)$.

(2). The pairs (S,P) and (T,Q) and weakly compatible and

(3). $$d(Sx,Ty) \leq \alpha \frac{[d(Px,Sx)]^3 + [d(Qy,Ty)]^3}{[d(Px,Sx)]^2 + [d(Qy,Ty)]^2} + \beta \frac{[d(Px,Ty)]^2 + [d(Qy,Sx)]^2}{d(Px,Ty) + d(Qy,Sx)} + \gamma [d(Px,Qy)].$$

for all $x,y \in X$ where $\alpha,\beta,\gamma \geq 0$, $0 \leq \alpha+\beta+\gamma \leq \frac{1}{2}$. Then P,Q,S and T have a unique common fixed point.

Proof. Using condition (1), we define sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Qx_{2n+1} = Sx_{2n}$ and $y_{2n+1} = Px_{2n+2} = Tx_{2n+1}$, $n = 1,2,\ldots$. If $y_{2n} = y_{2n+1}$ for some n, them $Qx_{2n+1} = Tx_{2n+1}$. Therefore x_{2n+1} is a coincidence point of Q and T. Also if $y_{2n+1} = y_{2n+2}$ for some n, then $Px_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of S and A. Assume that $y_{2n} \neq y_{2n+1}$ for all n, then we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha \frac{[d(Px_{2n}, Sx_{2n})]^3 + [d(Qx_{2n+1}, Tx_{2n+1})]^3}{[d(Px_{2n}, Sx_{2n})]^2 + [d(Qx_{2n+1}, Tx_{2n+1})]^2} \\ &\leq \beta \frac{[d(Px_{2n}, Tx_{2n+1})]^2 + [d(Qx_{2n+1} + Sx_{2n})]^2}{d(Px_{2n} + Tx_{2n+1}) + d(Qx_{2n+1} + Sx_{2n})} + \gamma [d(Px_{2n} + Qx_{2n+1})] \end{aligned}$$

$$< \alpha\left[d\left(Px_{2n}, Sx_{2n}\right)+d\left(Qx_{2n+1}, Tx_{2n+1}\right)\right]+\beta\left[d\left(Px_{2n}, Tx_{2n+1}\right)+d\left(Qx_{2n+1}, Sx_{2n}\right)\right]+\gamma\left[d\left(Px_{2n}, Qx_{2n+1}\right)\right]$$

$$< \alpha\left[d\left(y_{2n-1}, y_{2n}\right)+d\left(y_{2n}, y_{2n+1}\right)\right]+\beta\left[d\left(y_{2n-1}, y_{2n+1}\right)+d\left(y_{2n}, y_{2n}\right)\right]+\gamma\left[d\left(y_{2n-1}, y_{2n}\right)\right]$$

$$< \alpha\left[d\left(y_{2n-1}, y_{2n}\right)+d\left(y_{2n}, y_{2n+1}\right)\right]+\beta\left[d\left(y_{2n-1}, y_{2n}\right)+d\left(y_{2n}, y_{2n+1}\right)\right]+\gamma\left[d\left(y_{2n-1}, y_{2n}\right)\right]$$

$$\leq (\alpha+\beta)\, d\left(y_{2n}, y_{2n+1}\right)+(\alpha+\beta+\gamma)\, d\left(y_{2n-1}, y_{2n}\right)$$

Therefore,

$$\begin{aligned} d\left(y_{2n}, y_{2n+1}\right) &< \frac{\alpha+\beta+\gamma}{1-\alpha-\beta} d\left(y_{2n-1}, y_{2n}\right) \\ &= h d\left(y_{2n-1}, y_{2n}\right) \end{aligned}$$

Where, $h=\frac{\alpha+\beta+\gamma}{1-\alpha-\beta}<1$. This shows that

$$d\left(y_n, y_{n+1}\right)<h d\left(y_{n-1}, y_n\right)<\cdots<h^n d(y_0, y_1)$$

Thus for every integer $k>0$, we have

$$\begin{aligned} d\left(y_n, y_{n+k}\right) &\leq d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)+d\left(y_{n+2}, y_{n+3}\right)+\cdots+d\left(y_{n+k-1}, y_{n+k}\right) \\ &\leq \left(1+h+h^2+\cdots+h^{k-1}\right) d\left(y_n, y_{n+1}\right) \\ &\leq \frac{h^n}{1-h} d\left(y_0, y_1\right) \end{aligned}$$

Since $0<h<1$, $h^n \rightarrow 0$ as $n \rightarrow \infty$. So, we get $d\left(y_n, y_{n+k}\right) \rightarrow 0$, This implies that $\{y_n\}$ is a Cauchy sequence in a complete d-metric space (X, d). So there exists a point $z \in X$ such that $\lim_{n \rightarrow \infty} y_n=z$. Therefore $\lim_{n \rightarrow \infty} Sx_{2n}=z$, $\lim_{n \rightarrow \infty} Qx_{2n+1}=z$, $\lim_{n \rightarrow \infty} Tx_{2n+1}=z$ and $\lim_{n \rightarrow \infty} Px_{2n+2}=z$. Since $T(X) \subset P(X)$, there exists a point $u \in X$ such that $z=Pu$. So from condition (3), we have

$$\begin{aligned} d(Su, z) &= d\left(Su, Tx_{2n+1}\right) \\ &\leq \alpha \frac{\left[d(Pu, Su)\right]^3+\left[d\left(Qx_{2n+1}, Tx_{2n+1}\right)\right]^3}{\left[d(Pu, Su)\right]^2+\left[d\left(Qx_{2n+1}, Tx_{2n+1}\right)\right]^2} \\ &< \beta \frac{\left[d\left(Pu, Tx_{2n+1}\right)\right]^2+\left[d\left(Qx_{2n+1}, Su\right)\right]^2}{d\left(Pu, Tx_{2n+1}\right)+d\left(Qx_{2n+1}, Su\right)}+\gamma\left[d\left(Pu, Qx_{2n+1}\right)\right] \\ &< \alpha\left[d(Pu, Su)+d\left(Qx_{2n+1}, Tx_{2n+1}\right)\right]+\beta\left[d\left(Pu, Tx_{2n+1}\right)+d\left(Qx_{2n+1}, Su\right)\right]+\gamma\left[d\left(Pu, Qx_{2n+1}\right)\right] \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned} d(Su, z) &< \alpha[d(z, Su)+d(z, z)]+\beta[d(z, z)+d(z, Su)]+\gamma[d(z, z)] \\ &= \alpha d(z, Su)+\beta d(z, Su) \\ &= (\alpha+\beta)\, d(z, Su) \end{aligned}$$

Hence $d(Su, z)<(\alpha+\beta)\, d(z, Su)$. Which is a contradiction. Therefore we have $Su=Pu=z$. Again, Since $S(X) \subset Q(X)$, there exist a point $\nu \in X$ such that $z=Qv$. From condition (3), we have

$$d(z, Tv)=d(Su, Tv)$$

$$
\begin{aligned}
&\leq \alpha\frac{[d(Pu,Su)]^3+[d(Qv,Tv)]^3}{[d(Pu,Su)]^2+[d(Qv,Tv)]^2}+\beta\frac{[d(Pu,Tv)]^2+[d(Qv,Su)]^2}{d(Pu,Tv)+d(Qv,Su)}+\gamma\,[d(Pu,Qv)]\\
&< \alpha\,[d(Pu,Su)+d(Qv,Tv)]+\beta\,[d(Pu,Tv)+d(Q\nu,Su)]+\gamma\,[d(Pu,Qv)]\\
&= \alpha\,[d(z,z)+d(z,Tv)]+\beta\,[d(z,Tv)+d(z,z)]+\gamma\,[d(z,z)]\\
&= (\alpha+\beta)\,d(z,Tv)
\end{aligned}
$$

Hence, $d(z,Tv) < (\alpha+\beta)\,d(z,Tv)$. Which is a contradiction. So, we get $z = Tv$. Therefore, we have $Su = Pu = Tv = Qv = z$.

Since the pair (S,P) are weakly compatible, so by definition, $SPu = PSu$ implies that $Sz = Pz$.

Now we have to prove that z is a fixed point of S. From condition(3), we have

$$
\begin{aligned}
d(Sz,z) &= d(Sz,Tv)\\
&\leq \alpha\frac{[d(Pz,Sz)]^3+[d(Qv,Tv)]^3}{[d(Pz,Sz)]^2+[d(Qv,Tv)]^2}+\beta\frac{[d(Pz,Tv)]^2+[d(Qv,Sz)]^2}{d(Pz,Tv)+d(Qv,Sz)}+\gamma\,[d(Pz,Qv)]\\
&< \alpha\,[d(Pz,Sz)+d(Qv,Tv)]+\beta\,[d(Pz,Tv)+d(Q\nu,Sz)]+\gamma\,[d(Pz,Qv)]\\
&= \alpha\,[d(Sz,Sz)+d(z,z)]+\beta\,[d(Sz,z)+d(z,Sz)]+\gamma\,[d(Sz,z)]\\
&= (2\beta+\gamma)\,d(Sz,z)\\
d(Sz,z) &\leq (2\beta+\gamma)\,d(Sz,z)
\end{aligned}
$$

Which is a contradiction. So, we have $Sz = z$. This implies that $Pz = Sz = z$. Again the pair (T,Q) are weakly compatible, so by definition $TQv = QTv$ implies that $Tz = Qz$. Now we show that z is fixed point of T. For condition (3) we have

$$
\begin{aligned}
d(z,Tz) &\leq d(Sz,Tz)\\
&\leq \alpha\frac{[d(Pz,Sz)]^3+[d(Qz,Tz)]^3}{[d(Pz,Sz)]^2+[d(Qz,Tz)]^2}+\beta\frac{[d(Pz,Tz)]^2+[d(Qz,Sz)]^2}{d(Pz,Tz)+d(Qz,Sz)}+\gamma\,[d(Pz,Qz)]\\
&\leq \alpha\,[d(Pz,Sz)+d(Qz,Tz)]+\beta\,[d(Pz,Tz)+d(Qz,Sz)]+\gamma\,[d(Pz,Qz)]\\
&= \alpha\,[d(z,z)+d(Tz,Tz)]+\beta\,[d(z,Tz)+d(Tz,z)]+\gamma\,[d(z,Tz)]\\
&= (2\beta+\gamma)\,d(z,Tz)\\
d(z,Tz) &\leq (2\beta+\gamma)\,d(z,Tz)
\end{aligned}
$$

Which is a contradiction. This implies that $z = Tz$. Hence, we have $Pz = Qz = Sz = Tz = z$. This shown that z is a common fixed point of the self mappings P,Q,S, and T.

To prove that uniqueness of z, let z and $w, z \neq w$ are common fixed points of P,Q, S and T. From condition (3), we have

$$
\begin{aligned}
d(z,w) &= d(Sz,Tw)\\
&\leq \alpha\frac{[d(PzSz)]^3+[d(Qw,Tw)]^3}{[d(Pz,Sz)]^2+[d(Qw,Tw)]^2}+\beta\frac{[d(Pz,Tw)]^2+[d(Qw,Sz)]^2}{d(Pz,Tw)+d(Qw,Sz)}+\gamma\,[d(Pz,Qw)]\\
&\leq \alpha\,[d(Pz,Sz)+d(Qw,Tw)]+\beta\,[d(Pz,Tw)+d(Qw,Sz)]+\gamma\,[d(Pz,Qw)]\\
&= \alpha\,[d(z,z)+d(w,w)]+\beta\,[d(z,w)+d(w,z)]+\gamma\,[d(z,w)]\\
&= (2\beta+\gamma)\,d(z,w)\\
d(z,w) &\leq (2\beta+\gamma)\,d(z,w)
\end{aligned}
$$

Which in a contradiction. This shows that $d(z, w) = 0$. Since (X, d) is a dislocated metric space, so we have $z = w$. Therefore z is a unique common fixed point of P,Q,S and T. $\square$

From above theorem we can obtain the following corollaries.

Corollary 3.2. *Let (X, d) be a complete d-metric space. Let $P, Q, S, T : X \to X$ be continuous mappings satisfying*

$$d(Sx, Ty) \leq \alpha \frac{[d(Qy, Sx)]^3 + [d(Px, Ty)]^3}{[d(Qy, Sx)]^2 + [d(Px, Ty)]^2} + \beta \frac{[d(Qy, Ty)]^2 + [d(Px, Sx)]^2}{d(Qy, Ty) + d(Px, Sx)} + \gamma [d(Px, Qy)]$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$, $0 \leq \alpha + \beta + \gamma \leq \frac{1}{2}$. Then P, Q, S and T have a unique common fixed point.

Corollary 3.3. *Let (X, d) be a complete d metric space. Let $S, T : X \to X$ be continuous mappings satisfying*

$$d(Sx, Ty) \leq \alpha \frac{[d(x, Sx)]^3 + [d(y, Ty)]^3}{[d(x, Sx)]^2 + [d(y, Ty)]^2} + \beta \frac{[d(x, Ty)]^2 + [d(y, Sx)]^2}{d(x, Ty) + d(y, Sx)} + \gamma [d(x, y)]$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$, $0 \leq \alpha + \beta + \gamma \leq \frac{1}{2}$. Then S and T have a unique common fixed point.

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