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Some Topological Properties in M-Fuzzy Metric Space

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Abstract: This paper concerns our sustained efforts for introduction of M-fuzzy metric spaces and study their basic topological properties. As an application of this concept, we prove some convergences and continuous properties related on M-fuzzy metric spaces and introduce some related examples in support of our results.

Keywords: M-Fuzzy metric space, D-metric space, continuous t-norm, convergence.
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1. Introduction

A metric space is just a non empty set X associated with a function d of two variables enabling us to measure the distance between points. In advanced mathematics, we need to find the distance not only two between numbers and vectors, but also between more complicated objects like sequences, sets and functions. In order to find an appropriate concept of a metric space, numerous approaches exists in this sphere. Thus, new notations of distance lead to new notations of convergence and continuity. A numbers of generalization of a metric space have been discussed by many eminent mathematicians.

The concept of fuzzy sets was introduced initially by Zadesh [10] in 1965. Since then, by using this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [2] and Karmosil and Michalek [4] have introduced the concept of fuzzy topological space induced by fuzzy metric which have very important applications in quantum particle physics. many authors [3, 5, 7, 8] have proved fixed point theorem in fuzzy metric spaces. As there is a generalization in generalised metric space or D-metric space initiated by Dhage [1] in 1922. He proved some results on fixed points for a self-map satisfying the contraction for complete and bounded complete D-metric spaces.

Rhoades [6] generalised Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [9] introduced the concept of D-compatibility of maps in D- metric space and proved some fixed point theorems using a contractive condition. So far as our work is concerned, (X, D) will denote a D-metric space, N the set of natural numbers and R^+ the set of all positive real numbers.

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2. Main Results

Definition 2.1. *Let X be a non empty set. A generalised metric space (D-metric) on X is a function $D : X^3 \to R^+$ which satisfies the following conditions foe each $x, y, z, a \in X$.*

(i). $D(x, y, z) \geq 0$,

(ii). $D(x, y, z) = 0$ *if and only if* $x = y = z$,

(iii). $D(x, y, z) = D(p\{x, y, z\})$, *(symmetry) where p is a permutation function, [i.e. $D(x, y, z) = D(y, z, x) = D(z, x, y)$].*

(iv). $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

The pair (X, D) is called a generalised metric (or D-metric) space. Such functions are illustrated as

(a). $D(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\}$,

(b). $D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. *where d is the ordinary metric on X.*

(c). If $X = R^n$ then we define as

$$D(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in R^+$.

(d). If $X = R^+$ then we define

$$D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Remark 2.2. *In a D-metric space, we prove that $D(x, x, y) = D(x, y, y)$. For*

(a). $D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)$ *and similarly.*

(b). $D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)$.

Hence from above (a) and (b) we have $D(x, x, y) = D(x, y, y)$.

Definition 2.3. *Let (X, D) be a D-metric space, then we define a ball defined as $B_D(x, r)$ for $r > 0$, with centre c and radius r as*

$$B_D(c, r) = \{x \in X : D(c, x, x) < r\}$$

is called an open ball.

Example 2.4. *Let $X = R$. Let us $D(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$. Then*

$$\begin{aligned} B_D(1, 2) &= \{y \in R : D(1, y, y) < 2\} \\ &= \{y \in R : |y - 1| + |y - 1| < 2\} \\ &= \{y \in R : |y - 1| < 1\} = (0, 2). \end{aligned}$$

Definition 2.5. *Let (X, D) be a D-metric space and $A \subset X$.*

(1). The subset A of X is said to be an open suset of X If for every $x \in X$ there exists $r > 0$ such that $B_D(x, r) \subset A$.

(2). The subset A of X is said be D-bounded if there exists $r > 0$ such that $D(x, y, y) < r$ for all $x, y \in A$.

(3). A sequence $\{x_n\}$ in X is said to converges to x if and only if $D(x_n, x_n, x) = D(x, x, x_n) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all

$$n \geq n_0 \Rightarrow D(x, x, x_n) < \epsilon \qquad (*)$$

This is equivalent with, for each $\epsilon > 0$ there exists $n_0 \in N$ such that

$$m, n \geq n_0 \Rightarrow D(x, x_n, x_m) < \epsilon \qquad (**)$$

Now from $(*)$ we have

$$D(x_n, x_m, x) = D(x_n, x, x_m) \leq D(x_n, x, x) + D(x, x_m, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, by setting $m = n$ in $(**)$ we have $D(x_n, x_m, x) < \epsilon$.

(4). Sequence x_n in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $D(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$. The D-metric space (X, D) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_D(x, r) \subset A$ then τ is a topology on X induced by the D-metric D.

Theorem 2.6. *Let (X, D) be a D-metric space. If sequence x_n in X converges to x, then x is unique.*

Proof. Let $x_n \to y$ and $x \neq y$. Since x_n converges to x and y, for each $\epsilon > 0$ there exists $n_1 \in N$ such that for every $n \geq n_1 \Rightarrow D(x, x, x_n) < \frac{\epsilon}{2}$ and $n_2 \in N$ such that for every $n_1 \geq n_2 \Rightarrow D(y, y, x_n) < \frac{\epsilon}{2}$. If we suppose $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ then by triangle inequality we have $D(x, x, y) \leq D(x, x, x_n) + D(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $D(x, x, y) = 0$ which is a contradiction. So $x = y$. □

Theorem 2.7. *Let (X, D) be a D-metric space. If sequence x_n in X converges to x, then sequence$\{x_n\}$ is a cauchy sequence.*

Proof. Since $x_n \to x$ for each $\epsilon > 0$ there exists $n_1 \in N$ such that for every $n \geq n_1 \Rightarrow D(x_n, x_n, x) < \frac{\epsilon}{2}$ and $n_2 \in N$ such that for every $m \geq n_2 \Rightarrow D(x, x_m, x_m) < \frac{\epsilon}{2}$. If we assume that $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ then by triangle inequality we have $D(x_n, x_n, x_m) \leq D(x_n, x_n, x_n) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $\{x_n\}$ is a Cauchy sequence. □

Definition 2.8. *A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous triangular norm (t-norm) if for all $a, b, c \in [0, 1]$, the following conditions are satisfied*

(i). $a * 1 = a$.

(ii). $a * b = b * a$.

(iii). If $b \leq c$ then $a * b \leq a * c$.

(iv). $a * (b * c) = (a * b) * c$.

(v). $*$ is continuous.

For examples $T_p(a, b) = a.b$, $T_m(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ are the t-norms.

Definition 2.9. *A 3-tuple $(X, M, *)$ is said to be a M-fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a continuous fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for each $x, y, z, a \in X$,*

(i). $M(x, y, z, t) > 0$.

(ii). $M(x, y, z, t) = 1$ *for all* $t > 0$ *if and only if* $x = y = z$.

(iii). $M(x, y, z, t) = M(p\{x, y, z\}t)$, *(symmetric) where p is a permutation function.*

(iv). $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$ *for all* $t, s > 0$.

(v). $M(x, y, z, .) : (0, \infty) \to [0, 1]$ *is continuous.*

Remark 2.10. *Let $(X, M, *)$ be a M-fuzzy metric space. We assert that for every $t > 0$ $M(x, x, y, t) = M(x, y, y, t)$. For every $\epsilon > 0$ by triangle inequality we have*

(i). $M(x, x, y, \epsilon + t) \geq M(x, x, x, \epsilon) * M(x, y, y, t) = M(x, y, y, t)$.

(ii). $M(y, y, x, \epsilon + t) \geq M(y, y, y, \epsilon) * M(y, x, x, t) = M(y, x, x, t)$.

Now taking limit as $\epsilon \to 0$ from (i) and (ii) we obtain $M(x, x, y, t) = M(x, y, y, t)$.

Definition 2.11. *Let $(X, M, *)$ be a M-fuzzy metric space. For $t > 0$, we define an open ball $B_M(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by*

$$B_M(x, r, t) = \{y \in X : M(x, y, y, t) > 1 - r\}.$$

Definition 2.12. *A subset A of X is called open set if for each $x \in A$ there exists $t > 0$ and $0 < r < 1$ such that $B_M(x, r, t) \subseteq A$. A sequence $\{x_n\}$ in X converges to x if and only if $M(x, x, x_n, t) \to 1$ as $n \to \infty$, for each $t > 0$. Similarly it is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_n, x_m, t) > 1 - \epsilon$ for each $m, n \geq n_0$. The M-fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.*

Example 2.13. *Let X be a non empty set and D is the D-metric space on X. Let us denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and define*

$$M(x, y, z, t) = \frac{t}{t + D(x, y, z)}$$

For all $x, y, z \in X$. Then it is easy to see that $(X, M, *)$ is a M-fuzzy metric space.

Theorem 2.14. *Let $(X, M, *)$ is a fuzzy metric space. If we define $M : X^3 \times (0, \infty) \to [0, 1]$ by*

$$M(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t)$$

for every $x, y, z \in X$, then $(X, M, *)$ is a M-fuzzy metric space.

Proof.

(1). It is easy to see that for every $x, y, z \in X, M(x, y, z, t) > 0$ for all $t > 0$.

(2). $M(x, y, z, t) = 1$ if and only if $M(x, y, t) = M(y, z, t) = M(z, x, t) = 1$ if and only if $x = y = z$.

(3). $M(x, y, z, t) = M(p\{x, y, z\}, t)$, where p is a permutation function.

$$
\begin{aligned}
(4).\ M(x,y,z,t+s) &= M(x,y,t+s) * M(y,z,t+s) * M(z,x,t+s) \\
&\geq M(x,y,t) * M(y,a,t) * M(a,z,s) * M(z,a,s) * M(a,x,t) \\
&= M(x,y,a,t) * M(a,z,s) * M(z,a,s) * M(z,z,s) \\
&= M(x,y,a,t) * M(a,z,z,s)
\end{aligned}
$$

for every $s > 0$. □

Lemma 2.15. *Let $(X, M, *)$ be a M-fuzzy metric space. Then $M(x,y,z,t)$ is non decreasing with respect to t, for all $x,y,z \in X$.*

Proof. By definition of M-fuzzy metric space for each $x,y,z,a \in X$ and $t,s > 0$ we have $M(x,y,a,t) * M(a,z,z,s) \leq M(x,y,z,t+s)$, that is $M(x,y,z,t+s)$. Let us set $a = z$ we get $M(x,c,y,z,t) * M(z,z,z,s) \leq M(x,y,z,t+s)$, which gives $M(x,y,z,t+s) \geq M(x,y,z,t)$. □

Definition 2.16. *Let $(X, M, *)$ be a M-fuzzy metric space. M is said to be continuous function on $X^3 \times (0,\infty)$ if*

$$\lim_{n\to\infty} M(x_n, y_n, z_n, t_n) = M(x,y,z,t)$$

Whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0,\infty)$ converges to a point $(x,y,z,t) \in X^3 \times (0,\infty)$ i.e. $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} M(x,y,z,t_n) = M(x,y,z,t)$.

Theorem 2.17. *Let $(X, M, *)$ be a M-fuzzy metric space. Then M is continuous function on $X^3 \times (0,\infty)$.*

Proof. Let $x,y,z \in X$ and $t > 0$, and let $(x'_n, y'_n, z'_n, t'_n)n$ be sequence in $X^3 \times (0,\infty)$ that converges to (x,y,z,t). Since $(M(x'_n, y'_n, z'_n, t'_n))n$ is a sequence in $[0,1]$, there is subsequence $(x_n, y_n, z_n, t_n)n$ of sequence $(M(x'_n, y'_n, z'_n, t'_n))n$ such that sequence $(M(x_n, y_n, z_n, t_n))n$ converges to some point 0f $[0,1]$. Fix $\delta > 0$ such that $\delta < \frac{1}{2}$. Then, there is $n_0 \in N$ such that $|t - t_n| < \delta$ for every $n \geq n_0$. Hence,

$$
\begin{aligned}
M((x_n, y_n, z_n, t_n) &\geq M(x_n, y_n, z_n, , t - s) \\
&\geq M\left(x_n, y_n, z, t - \frac{4\delta}{3}\right) * M\left(z, z_n, z_n, \frac{\delta}{3}\right) \\
&\geq \left(x_n, z, y, t - \frac{5\delta}{3}\right) * \left(y, y_n, y_n, \frac{\delta}{3}\right) * M\left(z, z_n, z_n, \frac{\delta}{3}\right) \\
&\geq M(z, y, x, t - 2\delta) * M\left(x, x_n, x_n, \frac{\delta}{3}\right) * M\left(y, y_n, y_n, \frac{\delta}{3}\right) * M\left(z, z_n, z_n, \frac{\delta}{3}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
M(x,y,z,t+2\delta) &\geq M(x,y,z,t_n+\delta) \\
&\geq M\left(x, y, z_n, t_n + \frac{2\delta}{3}\right) * M\left(z_n, z, z, \frac{\delta}{3}\right) \\
&\geq M\left(x, z_n, y_n, t_n + \frac{\delta}{3}\right) * M\left(y_n, y, y, \frac{\delta}{3}\right) * M\left(z_n, z, z, \frac{\delta}{3}\right) \\
&\geq M(z_n, y_n, x_n, t) * M\left(x_n, x, x, \frac{\delta}{3}\right) * M\left(y_n, y, y, \frac{\delta}{3}\right) * M\left(z_n, z, z, \frac{\delta}{3}\right)
\end{aligned}
$$

for all $n \geq n_0$.By taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} M(x_n, y_n, z_n, t_n) \geq M(x,y,z,t-2\delta) * 1 * 1 * 1 = M(x,y,z,t-2\delta)$$

and

$$M(x,y,z,t+2\delta) \geq \lim_{n\to\infty} M(x_n,y_n,z_n,t_n)1*1*1 = \lim_{n\to\infty} M(x_n,y_n,z_n,t_n)$$

respectively. So by continuity of the function $t \to M(x,y,z,t)$, we immediately deduce that

$$\lim_{n\to\infty} M(x_n,y_n,z_n,t_n) = M(x,y,z,t).$$

Therefore M is continuous on $X^3 \times (0,\infty)$. □

Definition 2.18. *Let A and B be two self mappings of a M-fuzzy metric space $(X,M,*)$ we say that A and B satisfy a property, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} M(Ax_n,u,u,t) = \lim_{n\to\infty} M(Bx_n,u,u,t) = 1 \ldots (*)$ for some $u \in X$ and $t > 0$.*

Example 2.19. *Let $X = R$ and $M(x,y,z,t) = \dfrac{1}{t+|x-y|+|y-z|+|x-z|}$ for every $x,y,z \in X$ and $t > 0$. Let A and B be defined as $Ax = 2x+1$, $Bx = x+2$. Consider the sequence $x_n = \frac{1}{n}+1, n = 1,2,\ldots$ Thus we have*

$$\lim_{n\to\infty} M(Ax_n,3,3,t) = \lim_{n\to\infty} M(Bx_n,3,3,t) = 1$$

for every $t > o$. Then A and B satisfying the property $(*)$.

In the next example we show that there are some mappings which does not have the property $(*)$.

Example 2.20. *Let $X = R$ and $M(x,y,z,t) = \dfrac{1}{t+|x-y|+|y-z|+|x-z|}$ for every $x,y,z \in X$ and $t > 0$. Let $Ax = x+1$ and $Bx = x+2$, if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} M(Ax_n,u,u,t) = \lim_{n\to\infty} M(Bx_n,u,u,t) = 1$$

for some $u \in X$. Therefore

$$\lim_{n\to\infty} M(Ax_n,u,u,t) = \lim_{n\to\infty} M(x_n+1,u,u,t) = \lim_{n\to\infty} M(x_n,u-1,u-1,t) = 1$$

and

$$\lim_{n\to\infty} M(Bx_n,u,u,t) = \lim_{n\to\infty} M(x_n+2,u,u,t) = \lim_{n\to\infty} M(x_n,u-2,u-2,t) = 1$$

we conclude that, $x_n \to u-1$ and $x_n \to u-2$. Which is a contradiction. Hence A and B do not satisfy the property $(*)$.

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