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Mathematical Analysis and Applications of Mathieu's Equation Revisited

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Abstract: In mathematics, Mathieu functions, sometimes called angular Mathieu functions, are solutions of Mathieu's differential equation. Émile Léonard Mathieu, modelled vibrating elliptical drumheads by Mathieu's equation. This equation has wide applications in many fields such as optics, quantum mechanics and general relativity. They occur in many problems involving periodic motion, in analysis of boundary value problems having elliptic symmetry. Many properties of the Mathieu differential equation can be deduced from the general theory of ordinary differential equations with periodic coefficients, called Floquet theory. Mathieu's differential equations appear in a wide range of contexts in engineering, physics, and applied mathematics. Many of these applications fall into one of two general categories: 1) the analysis of partial differential equations in elliptic geometries, and 2) dynamical problems which involve forces that are periodic in either space or time. Examples within both categories are discussed in this paper.

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1. Introduction

Ordinary differential equations (ODEs) arise in many contexts of mathematics as social and natural sciences. Mathematical descriptions of change or dynamically changing phenomena, evolution and variations are ordinary differential equations. Specific mathematical fields include geometry and analytical mechanics, physics include astronomy and meteorology, chemistry includes reaction rates, biology include epidemix, genetics, ecology and population models, economics include stocks, price changes etc. Many mathematicians have studied differential equations and contributed to these field, including Newton, Leibniz, the Bernoulli family, Riccati, Clairaut, d'Alembert, and Euler.

We come across many first and second order ordinary differential equations in almost all physics and chemistry problems. Though they are linear with and without constant coefficients, obtaining meaningful solutions is very difficult. Sometimes they can be solved by classic methods available in literature, but sometimes it may not be possible to apply these methods. In many of these applications we come across some famous linear second order equations with constant or variable coefficients. These equations are named after scientists, Legendres, Bessel, Hermite, Laguerre, Mathieu etc These equations are solved by using infinite series method. We are interested in solution of Mathieu's [1,2,3,4] equation which has periodic variable coefficient. It commonly occurs as homogeneous equation in nonlinear vibration problems [5] and nonhomogeneous in super conductor theory [6].

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Many properties of the Mathieu differential equation can be deduced from the general theory of ordinary differential equations with periodic coefficients, which is well explained by Floquet theory. There are relatively few analytic expressions and identities involving Mathieu functions. Moreover, unlike many other special functions, the solutions of Mathieu's equation cannot in general be expressed in terms of hypergeometric functions. This can be seen by transformation of Mathieu's equation to algebraic form, using the change of variable. Since this equation has an irregular singular point at infinity, it cannot be transformed into an equation of the hypergeometric type. Floquet Theory [1] is a branch of the theory of ordinary differential equations relating to the class of solutions to periodic linear differential equations with a piecewise continuous periodic function and also defines the state of the stability of solutions.

In this paper we revisit the details of Mathieu's equation – a classical differential equation, which has the form of a linear second-order ordinary differential equation (ODE) with Cosine type periodic forcing of the stiffness coefficient, and its different generalizations / extensions. These extensions include: the effects of linear viscous damping, geometric nonlinearity, damping nonlinearity, radio frequency quadrupole etc. The aim is to provide a systematic overview of the solution and application of classical Mathieu's equation.

2. Mathematical Theory of Mathieu Equation

The simplest form of Mathieu's equation is

$$\frac{d^2y}{dx^2} + (a + bCos2x)\,y = 0 \tag{1}$$

where a and b are constants. Though the coefficients are one valued, continuous, and periodic with period π, the general solution does not necessarily possess the period π.

2.1. Types of solution

In fact equation (1) has periodic solution of period π if $(a + bCos2x) = n^2$ and n is even. Let us consider the Mathieu equation [1] in the form

$$\frac{d^2y}{dx^2} + (a - 2\theta Cos2x)\,y = 0, \tag{2}$$

which has no finite singular points and therefore its solutions are valid for all finite values of x. If $G(x)$ is a solution which is neither even nor odd, then $G(-x)$ is a distinct solution. Then

$$\frac{1}{2}\{G(x) + G(-x)\} \neq 0 \text{ is an even solution.}$$

and

$$\frac{1}{2}\{G(x) - G(-x)\} \neq 0 \text{ is an odd solution.}$$

Thus it is sufficient to consider only even or odd solutions. Now consider two distinct even solutions satisfying initial condition

$$y(0) = 0, \quad y'(0) = 1, \tag{3}$$

then it can be observed that there cannot exist two distinct even or odd solutions. Thus one fundamental solution is even and the other is odd.

2.2. Infinite series solution

Let us consider Mathieu's equation [2] of the form

$$\frac{d^2y}{dx^2} + (a + 16bCos2x)\,y = 0, \tag{4}$$

where a and b are constants. Its general solution can be obtained by the method of integration in series as follows. Let us consider the substitution

$$\xi = cos^2x, \tag{5}$$

in equation (4) we get

$$4\xi\,(1-\xi)\,\frac{d^2y}{dx^2} + 2\,(1-2\xi)\,\frac{dy}{dx} + (a - 16b + 32b\xi)\,y = 0. \tag{6}$$

The equation (6) has non-essential singularity at $\xi = 0$, therefore we can develop a power series about it. On inserting

$$y = \sum_{n=0}^{\infty} a_n \xi^{k+n}, \tag{7}$$

in equation (6), we obtain

$$2\sum_{n=0}^{\infty}(k+n)\,(2k+2n-1)\,a_n\xi^{K+n-1} - \sum_{n=0}^{\infty}\left[4(k+n)^2 - a + 16b\right]a_n\xi^{K+n} + 32\sum_{n=0}^{\infty}a_n\xi^{K+n+1} = 0. \tag{8}$$

The equation (8) contains three different summations instead of two and will therefore lead to a three term recurrence relation between the coefficients a_n instead of the two-term relations that occurred usually. Therefore, we have to compute coefficient step by step. Equating the coefficient of lowest exponent of ξ^{k-1} to zero, we get

$$k\,(2k-1) = 0 \ \text{ such that } \ k = 0, \ \frac{1}{2}. \tag{9}$$

Next we should equate coefficient of ξ^k to zero from first and second summations, by which we get

$$a_1 = \frac{(4k^2 - a + 16b)}{2(k+1)(2k+1)}a_0, \tag{10}$$

now equating coefficients of ξ^{k+1} to zero which is there in all three summations, we get

$$2\,(k+2)\,(2k+3)\,a_2 - \left[4(k+1)^2 - a + 16b\right]a_1 + 32ba_0 = 0 \tag{11}$$

In this way we can calculate odd and even coefficients of power series one for $k = 0$ and the other for $k = \frac{1}{2}$, linear combination of which gives general solution of (6) as

$$y = \sum_{n=0}^{\infty} a_n\xi^n + \xi^{\frac{1}{2}}\sum_{n=0}^{\infty} b_k\xi^n \tag{12}$$

and hence that of (5) and it is convergent for $|\xi| < 1$. But this general solution is not useful for physics and chemistry because it is not periodic in x because in most of Mathieu's equation, x represents an angle so x and $x+2n\pi$ are not distinct for an integer n. The solution sought must satisfy the property that

$$y\,(x + 2n\pi) = y(x).$$

Qualitative analysis of solution explains that the failure of obtaining periodic solution is due to non-convergence of solution at $\xi = \pm 1$. We have excluded the values at $x = n\pi$ because there exist a branch point at $\xi = 0$ which arises due to the factor $\xi^{\frac{1}{2}}$. Thus, it is impossible to obtain periodic solution of Mathieu's equation without any conditions on constant a and b.

3. Application of Floquet Theory to Mathieu's Equation

Floquet theorem is an important theorem concerned with general solution of Mathieu's equation. This theory is indeed applicable to all linear differential equations with periodic coefficients which are one valued functions of independent variable. In this section we will establish Floquet theory of Mathieu's equation which has periodic coefficients. Suppose

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{13}$$

be the general solution of equation (4), then $y_1(x+2\pi)$ and $y_2(x+2\pi)$ will also be solutions as substitution of $x+2\pi$ in place of x will not change the equation. But $y_1(x+2\pi) \neq y_1(x)$ and $y_2(x+2\pi) \neq y_2(x)$, thus

$$y_1(x+2\pi) = c_{11} y_1(x) + c_{12} y_2(x) \tag{14}$$

$$y_2(x+2\ \pi) = c_{21} y_1(x) + c_{22} y_2(x) \tag{15}$$

By equation (13) we have

$$y(x+2\pi) = c_1 y_1(x+2\pi) + c_2 y_2(x+2\pi) \tag{16}$$

Using (14) and (15) in (16) we get

$$y\,(x+2\pi) = c_1\,[c_{11} y_1\,(x) + c_{12} y_2\,(x)] + c_2\,[c_{21} y_1\,(x) + c_{22} y_2\,(x)] \tag{17}$$

$$= (c_1 c_{11} + c_2 c_{21})\, y_1\,(x) + (c_1 c_{12} + c_2 c_{22}) y_2\,(x)\,. \tag{18}$$

It can be observed that c_1 and c_2 are arbitrary constants and $c_{11}, c_{12}, c_{21}, c_{22}$ are fixed by the choice of $y_1\,(x)$ and $y_2\,(x)$. It is possible to choose

$$c_1 c_{11} + c_2 c_{21} = k c_1 \quad \text{and} \quad c_1 c_{12} + c_2\ c_{22} = k c_2, \tag{19}$$

Where choice of k is subject to

$$\begin{vmatrix} c_{11} - k & c_{21} \\ c_{12} & c_{22} - k \end{vmatrix} = 0. \tag{20}$$

The equation (20) in (18) implies

$$y\,(x+2\pi) = k y(x), \tag{21}$$

This means there exists a particular solution $y(x)$ such that, when x is increased by 2π, then the solution is multiple of k, if $k = 1$ then the solution should be periodic. The another way of expressing this result is that, put $k = e^{2\pi\mu}$ and $y\,(x) = e^{\mu x} P(x)$ in equation (21) and we obtain

$$e^{\mu(x+2\pi)} P\,(x+2\pi) = e^{2\pi\mu + \mu x} P(x), \tag{22}$$

By (22), we get $P\,(x) = P\,(x+2\pi)$, thus $P\,(x)$ is periodic and there exists a particular solution of Mathieu's equation of the form

$$y\,(x) = e^{\mu x} P(x). \tag{23}$$

The equation (4) will not change if we replace x by $-x$, hence $e^{-\mu x} P(-x)$ must also be a solution and is independent because it is not the constant multiple of (23). Therefore, we can write the linear combination of these two as complete solution of (4) which is

$$y\,(x) = A\ e^{\mu x} P\,(x) + B e^{-\mu x} P(-x). \tag{24}$$

This result is known as Floquet Theorem.

Stability of Solution of Mathieu's equation

If $Re\mu > 0$, then the solution $y(x) = e^{\mu x} P(x)$ grows exponentially so solution is unstable.

If $Re\mu \leq 0$, then the solution $y(x) = e^{\mu x} P(x)$ decays so solution is stable.

If $Re\mu > 0$, then the solution $y(x) = e^{-\mu x} P(-x)$ decays exponentially so solution is stable.

If $Re\mu \leq 0$, then the solution $y(x) = e^{-\mu x} P(-x)$ grows exponentially so solution is unstable.

But both the solutions are stable if $Re\mu = 0$.

4. Applications of Mathieu's Equation

The Mathieu's equation has applications in astronomy, quantum theory of metals.

Schrodinger Equation

$$\frac{d^2\psi}{dx^2} + [A + V(x)]\psi = 0 \tag{25}$$

has solution $\psi = e^{ikx} v(x)$ where A and k are constants, $V(x)$ and $v(x)$ are periodic (Holch's theorem) [3].

Pendulum Equation [8]

The governing equation of Pendulum whose support is periodically forced in vertical direction is

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L} - \frac{A\omega^2}{L} cos\omega t\right) sinx = 0, \tag{26}$$

where L is length of pendulum, g is gravity force, $A\cos\omega t$ and x is angle of deflection. By linearising this equation, we get Mathieu equation of the form $\frac{d^2y}{dt^2} + (\delta + \varepsilon cost)\,x = 0$, where $\sin\ x \sim x$, $\delta = \frac{g}{L}$ and $\varepsilon = -\frac{A\omega^2}{L}$.

Nonlinear Mathieu's Equation [8]

The vertically driven pendulum equation (26) gives us nonlinear Mathieu's equation by the following substitutions: $\tau = \omega t$, $\delta = \frac{g}{L\omega^2}$, $\varepsilon = \frac{A}{L}$ and using Taylor series expansion of $\sin\ x$ we get

$$\frac{d^2x}{d\tau^2} + (\delta - \varepsilon cos\tau)\,(x - \frac{x^3}{6} + \ldots\cdots = 0, \tag{27}$$

Further, scaling $x = \sqrt{\varepsilon}\ y$ and neglecting $O(\varepsilon^2)$ terms we get nonlinear Mathieu's equation

$$\frac{d^2y}{d\tau^2} + (\delta - \varepsilon cos\tau)\,y - \varepsilon\frac{\delta y^3}{6} = 0. \tag{28}$$

Damped Mathieu's Equation [8]

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (\delta + \varepsilon cost)\,x = 0. \tag{29}$$

Radio Frequency Quadrupole [12]

In case of radio frequency of electric field $E = E_0\sin(\omega t)$ the force on a singly charged positive ion of mass m, in x direction, is

$$m\frac{d^2x}{dt^2} = \left(\frac{\varepsilon E_0 x}{d}\right)\sin(\omega t). \tag{30}$$

The equation (30) reduces to Mathieu's equation by using transformation

$$Y = x,\ 2X = \frac{\pi}{2} - \omega t,\ 2q = \frac{4eE_0}{m\omega^2 d}. \tag{31}$$

5. Conclusion

Many physical and chemical models can be reduced to Mathieu's equation with some transformation. In turn we can solve almost all physics and chemistry problems by using well developed theory of Mathieu's equation. This classic equation has many more applications apart from physics and chemistry and thus it needs lot of attention and elaborate study in different conditions.

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