Int. J. Math. And Appl., 9(2)(2021), 17–25
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Congruences for $(2,5)$-regular Bipartitions into Distinct Parts

M. Prasad[1], S. R. Nayaka[1,*] and K. V. Prasad[2]

1 Department of Mathematics, PES College of Engineering, Mandya, Karnataka, India.

2 Department of Mathematics, VSK University, Ballary, Karnataka, India.

Abstract: Let $B_{2,5}(n)$ denote the number of $(2,5)$-regular bipartitions of a positive integer n into distinct parts . In this paper, we establish several infinite families of congruences modulo powers of 2 for $B_{2,5}(n)$. For example,

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+1}\cdot 5^{2\beta}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 2f_1f_5^3 \pmod{2^2},$$

for $\alpha,\beta\geq 0$.

MSC: 11P83, 05A17.

Keywords: Partition identities, Theta–functions, Partition congruences, Regular partition.

1. Introduction

Throughout this paper, we let $|q|<1$. We use the standard notation

$$f_k := (q^k;q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) \;:=\; f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty, \tag{1}$$

which is special case of Ramanujan's general theta function [1]

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{2}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes,

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{3}$$

* E-mail: nayaka.abhi11@mail.com

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of its parts is divisible by ℓ. Let $b_\ell(n)$ denotes the number of ℓ-regular partitions of n with $b_\ell(0) = 1$. The generating function for $b_\ell(n)$ is

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

Recently, arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. Calkin [2] have established congruences for 5-regular partitions modulo 2 and 13-regular partitions modulo 2 and 3 using the theory of modular forms. For more details, one can see [3, 5, 6] and [7].

Suppose ℓ, $m > 0$ and $(\ell, m) = 1$. A partition is an (ℓ, m)-regular partition of the positive integer n if none of the parts is divisible by ℓ or m. Let $a_{\ell,m}(n)$ denote the number of such partitions of n into distinct parts with $a_{\ell,m}(0) = 1$. The generating function is given by

$$\sum_{n=0}^{\infty} a_{\ell,m}(n)q^n = \frac{(-q;q)_\infty(-q^{\ell m};q^{\ell m})_\infty}{(-q^\ell;q^\ell)_\infty(-q^m;q^m)_\infty}. \tag{4}$$

For example, there are 3 partitions for $a_{3,5}(11)$, namely

11, 8+2+1, 7+4.

For details, one can see [9] and [10].

Let $B_{\ell,m}(n)$ denote the number of (ℓ, m)-regular bipartitions of n into distinct parts with $B_{\ell,m}(0) = 1$ and the generating function is given by

$$\sum_{n=0}^{\infty} B_{\ell,m}(n)q^n = \frac{(-q;q)^2_\infty(-q^{\ell m};q^{\ell m})^2_\infty}{(-q^\ell;q^\ell)^2_\infty(-q^m;q^m)^2_\infty} = \frac{f_2^2 f_{2\ell m}^2 f_\ell^2 f_m^2}{f_{2\ell}^2 f_{2m}^2 f_1^2 f_{\ell m}^2}. \tag{5}$$

For example, there are 12 bipartitions for $B_{2,5}(11)$, namely

(0, 11), (11, 0), (3, 7+1), (7+1, 3), (7, 3+1), (3+1,7)
(1, 7+3), (7+3,1), (9+1, 1), (1, 9+1), (7+3+1,0), (0, 7+3+1).

2. Preliminary Results

Lemma 2.1 ([3, Theorem 2.2]). *For any prime $p \geq 5$,*

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \tag{6}$$

Furthermore, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.2. *The following 2-dissections holds*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{7}$$

and

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{8}$$

The equation (7) was proved by Hirschhorn and Sellers [5]; see also [11]. Replacing q by $-q$ in (7) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (8).

Lemma 2.1 ([8]). *We have*

$$f_1 f_5^3 = 2q^2 f_4 f_{20}^3 + f_2^3 f_{10} - 2q^3 \frac{f_4^4 f_{40}^2 f_{10}}{f_2 f_8^2} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4}, \tag{9}$$

$$f_1^3 f_5 = 2q^2 \frac{f_4^6 f_{40}^2 f_{10}}{f_2 f_8^2 f_{20}^2} + \frac{f_4 f_{10}^2 f_2^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3. \tag{10}$$

We shall prove the following Theorems:

Theorem 2.3. *For $\alpha, \beta \geq 0$,*

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+1}\cdot 5^{2\beta} n + \frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right) q^n \equiv 2 f_1 f_5^3 \pmod{2^2}, \tag{11}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta} n + \frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right) q^n \equiv 2 f_1^3 f_5 \pmod{2^2}, \tag{12}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+1}\cdot 5^{2\beta+1} n + \frac{2^{2\alpha+1}\cdot 5^{2\beta+1}-1}{3}\right) q^n \equiv 2 f_1^3 f_5 \pmod{2^2}, \tag{13}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta+1} n + \frac{2^{2\alpha+3}\cdot 5^{2\beta+1}-1}{3}\right) q^n \equiv 2 f_1 f_5^3 \pmod{2^2}. \tag{14}$$

Corollary 2.4. *For $\alpha, \beta \geq 0$, $a \in \{7, 13\}$ and $b \in \{11, 14\}$,*

$$B_{2,5}\left(2^{2\alpha+1}\cdot 5^{2\beta+1} n + \frac{b\cdot 2^{2\alpha+1}\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{2^2}, \tag{15}$$

$$B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta+1} n + \frac{a\cdot 2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{2^2}. \tag{16}$$

Theorem 2.5. *Let $r \in \{11, 17, 23\}$, $s \in \{29, 53, 77, 101\}$ and $r_1 \in \{7, 13, 19\}$. Then for all non-negative integers α, β and n, we have*

$$B_{2,5}\left(2^{2\alpha+3}\cdot 5^{2\beta} n + \frac{r\cdot 2^{2\alpha+1}\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{2^2}, \tag{17}$$

$$B_{2,5}\left(2^{2\alpha+4}\cdot 5^{2\beta+1} n + \frac{s\cdot 2^{2\alpha+1}\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{2^2}, \tag{18}$$

$$B_{2,5}\left(2^{2\alpha+4}\cdot 5^{2\beta+1} n + \frac{2^{2\alpha+1}\cdot 5^{2\beta+1}-1}{3}\right) \equiv \begin{cases} 2 \pmod{2^2} & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 \pmod{2^2} & \text{otherwise.} \end{cases} \tag{19}$$

$$B_{2,5}\left(2^{2\alpha+5}\cdot 5^{2\beta} n + \frac{r_1\cdot 2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{2^2}, \tag{20}$$

$$B_{2,5}\left(2^{2\alpha+5}\cdot 5^{2\beta} n + \frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right) \equiv \begin{cases} 2 \pmod{2^2} & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 \pmod{2^2} & \text{otherwise.} \end{cases} \tag{21}$$

Theorem 2.6. *For $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2} n + \frac{2^{2\alpha+2}-1}{3}\right) q^n \equiv 2\frac{f_4 f_{10}^2 f_5}{f_{20} f_1} \pmod{2^3}, \tag{22}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4} n + \frac{5\cdot 2^{2\alpha+3}-1}{3}\right) q^n \equiv 6\frac{f_2^3 f_{20} f_5^2}{f_4 f_{10} f_1^2} \pmod{2^3}. \tag{23}$$

Theorem 2.7. *Let $r_2 \in \{83, 107\}$ and $s_2 \in \{31, 79\}$. Then for all $\alpha, \beta \geq 0$,*

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta} n + \frac{11 \cdot 2^{2\alpha+1} \cdot 5^{2\beta} - 1}{3}\right) q^n \equiv 4 f_1 f_5^2 \pmod{2^3}, \tag{24}$$

$$B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta+1} n + \frac{r_2 \cdot 2^{2\alpha+1} \cdot 5^{2\beta} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{25}$$

$$B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta+2} n + \frac{s_2 \cdot 2^{2\alpha+1} \cdot 5^{2\beta+1} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{26}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta} n + \frac{7 \cdot 2^{2\alpha+2} \cdot 5^{2\beta} - 1}{3}\right) q^n \equiv 4 f_1^2 f_5 \pmod{2^3}, \tag{27}$$

$$B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta+2} n + \frac{s_2 \cdot 2^{2\alpha+2} \cdot 5^{2\beta} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{28}$$

$$B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta+1} n + \frac{r_2 \cdot 2^{2\alpha+2} \cdot 5^{2\beta+1} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{29}$$

Theorem 2.8. *For $\alpha, \beta, \gamma \geq 0$,*

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta} \cdot p^{2\gamma} n + \frac{11 \cdot 2^{2\alpha+1} \cdot 5^{2\beta} \cdot p^{2\gamma} - 1}{3}\right) q^n \equiv 4 f_1 f_{10} \pmod{2^3}, \tag{30}$$

$$B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta} \cdot p^{2\gamma+1}(pn+i) + \frac{11 \cdot 2^{2\alpha+1} \cdot 5^{2\beta} \cdot p^{2\gamma+2} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{31}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta+1} \cdot p^{2\gamma} n + \frac{7 \cdot 2^{2\alpha+1} \cdot 5^{2\beta+1} p^{2\gamma} - 1}{3}\right) q^n \equiv 4 f_2 f_5 \pmod{2^3}, \tag{32}$$

$$B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta+1} \cdot p^{2\gamma+1}(pn+i) + \frac{7 \cdot 2^{2\alpha+1} \cdot 5^{2\beta+1} p^{2\gamma+2} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{33}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta} \cdot p^{2\gamma} n + \frac{7 \cdot 2^{2\alpha+2} \cdot 5^{2\beta} \cdot p^{2\gamma} - 1}{3}\right) q^n \equiv 4 f_2 f_5 \pmod{2^3}, \tag{34}$$

$$B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta} \cdot p^{2\gamma+1}(pn+i) + \frac{7 \cdot 2^{2\alpha+2} \cdot 5^{2\beta} \cdot p^{2\gamma+2} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{35}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta+1} \cdot p^{2\gamma} n + \frac{11 \cdot 2^{2\alpha+2} \cdot 5^{2\beta+1} \cdot p^{2\gamma} - 1}{3}\right) q^n \equiv 4 f_1 f_{10} \pmod{2^3}, \tag{36}$$

$$B_{2,5}\left(2^{2\alpha+5} \cdot 5^{2\beta+1} \cdot p^{2\gamma+1}(pn+i) + \frac{11 \cdot 2^{2\alpha+2} \cdot 5^{2\beta+1} \cdot p^{2\gamma+2} - 1}{3}\right) \equiv 0 \pmod{2^3}, \tag{37}$$

where $i = 1, 2, 3, ...p-1$.

3. Proof of the Theorem (2.3)

From (5), we find that

$$\sum_{n=0}^{\infty} B_{2,5}(n) q^n = \frac{f_2^4 f_{20}^2}{f_4^2 f_{10}^4} \times \frac{f_5^2}{f_1^2}. \tag{38}$$

Using (7) in (38) and extracting the terms involving q^{2n+1} from both sides, we arrive

$$\sum_{n=0}^{\infty} B_{2,5}(2n+1) q^n = 2 \frac{f_2 f_{10}^3}{f_1 f_5^3}. \tag{39}$$

From binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}, \tag{40}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{2^2}, \tag{41}$$

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{2^3}. \tag{42}$$

Using (40) in (39), we find that

$$\sum_{n=0}^{\infty} B_{2,5}(2n+1)q^n = 2f_1f_5^3 \pmod{2^2}. \tag{43}$$

which is the $\alpha = \beta = 0$ case of (11).

Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2R(q^5)), \tag{44}$$

where $R(q) = \dfrac{f(-q,-q^4)}{f(-q^2,-q^3)}$. For a proof of (44), one can see [4, 12].

Let us consider the case $\alpha = 0$ and prove induction on β. Suppose that the congruence (11) holds for some integer $\beta > 0$. Employing the equation (44) in (11) with $\alpha = 0$, we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2\cdot 5^{2\beta}n + \frac{4\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 2f_5^3f_{25}(R(q^5)^{-1} - q - q^2R(q^5)) \pmod{2^2}. \tag{45}$$

Extracting the coefficients of q^{5n+1} in (45), we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2\cdot 5^{2\beta+1}n + \frac{2\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 2f_1^3f_5 \pmod{2^2}. \tag{46}$$

Again employing the equation (44) in (46), we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2\cdot 5^{2\beta+1}n + \frac{2\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 2f_5f_{25}^3(R(q^5)^{-1} - q - q^2R(q^5))^3. \tag{47}$$

Extracting the coefficients of q^{5n+3} in (47), we get

$$\sum_{n=0}^{\infty} B_{2,5}\left(2\cdot 5^{2\beta+2}n + \frac{4\cdot 5^{2\beta+2}-1}{3}\right)q^n \equiv 2f_1f_5^3 \pmod{2^2}, \tag{48}$$

which implies that (11) is true for $\beta+1$. Hence, by induction (11) is true for any non-negative integer β and $\alpha = 0$.

Now, Suppose that the congruence (11) holds for some integers $\alpha, \beta > 0$. Employing the equation (9) in the equation (11), we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+1}\cdot 5^{2\beta}n + \frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 2f_2^3f_{10} + 2qf_{40} \pmod{2^2}. \tag{49}$$

Extracting the coefficients of q^{2n} in (49), we arrive

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta}n + \frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 2f_1^3f_5 \pmod{2^2}, \tag{50}$$

which proves (12). Again, employing the equation (10), we obtain

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta}n + \frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 2f_8 + 2qf_2f_{10}^3 \pmod{2^2}. \tag{51}$$

Extracting the coefficients of q^{2n+1} in (51), we get

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+3}\cdot 5^{2\beta}n + \frac{2^{2\alpha+4}\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 2f_1f_5^3 \pmod{2^2}. \tag{52}$$

which implies that (11) is true for $\alpha+1$. Hence, by induction (11) is true for any non-negative integers α and β. This completes the proof.

Employing the equation (44) in the equations (11) and (12), we obtain (13) and (14) respectively.

4. Proof of the Corollory (2.4)

Using the equations (11) and (12) along with the equation (44), we obtain (15) and (16) respectively.

5. Proof of the Theorem (2.5)

Extracting the coefficients of q^{2n+1} in (49), we get

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta}n+\frac{2^{2\alpha+1}\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 2f_{20} \pmod{2^2}. \tag{53}$$

Again, extracting the coefficients of q^{4n+i}, we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+2}\cdot 5^{2\beta}(4n+i)+\frac{2^{2\alpha+1}\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 0 \pmod{2^2}. \tag{54}$$

On simplification, we obtain (17). Extracting the coefficients of q^{4n} in (53), we obtain

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4}\cdot 5^{2\beta}n+\frac{2^{2\alpha+1}\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 2f_{5} \pmod{2^2}. \tag{55}$$

Extracting the coefficients of q^{5n+i}, for $i=1,2,3,4$ and q^{5n} in (55), we get (18) and (19) respectively.

Extracting the coefficients of q^{2n} in (51), we get

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+3}\cdot 5^{2\beta}n+\frac{2^{2\alpha+2}\cdot 5^{2\beta}-1}{3}\right) \equiv 2f_{4} \pmod{2^2}. \tag{56}$$

Collecting the terms involving the powers of q^{4n+i} for $i=1,2,3$ and q^{2n} in (56), we obtain (20) and (21) respectively.

6. Proof of the Theorem (2.6)

Employing (41) in (39), we find that

$$\sum_{n=0}^{\infty} B_{2,5}(2n+1)q^n \equiv 2\frac{f_2f_{10}f_5}{f_1} \pmod{2^3}. \tag{57}$$

Using (7) in (57) and then extracting the coefficients of q^{2n} on both sides, we see that

$$\sum_{n=0}^{\infty} B_{2,5}(4n+1)q^n \equiv 2\frac{f_4f_{10}^2f_5}{f_{20}f_1} \pmod{2^3}. \tag{58}$$

which is the $\alpha=0$ case of (22). Suppose that the congruence (22) holds for some integer $\alpha>0$. Again using (7) in (22) and extracting the terms involving q^{2n+1} on both sides along with (41), we get

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+3}n+\frac{2^{2\alpha+4}-1}{3}\right)q^n \equiv 2\frac{f_2^2f_{20}^2f_1}{f_4f_5} \pmod{2^3}. \tag{59}$$

Using (8) in (59) and extracting the terms involoving q^{2n}, we obtain

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4}n+\frac{2^{2\alpha+4}-1}{3}\right)q^n \equiv 2\frac{f_4f_{10}^2f_5}{f_{20}f_1} \pmod{2^3}. \tag{60}$$

which implies that (22) is true for $\alpha+1$. Hence, by induction (22) is true for any non-negative integer α. Using (7) in (22) and then extracting the coefficients of q^{2n+1} on both sides, we see that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+3}n+\frac{2^{2\alpha+4}-1}{3}\right)q^n \equiv 2\frac{f_4f_{20}}{f_2^2}\times\frac{f_1}{f_5} \pmod{2^3}. \tag{61}$$

Again, using (8) in (61) and then extracting the coefficients of q^{2n+1} on both sides, we obtain (23).

7. Proof of the Theorem (2.7)

Using (7) in (39) and then extracting the coefficients of q^{2n+1} on both sides, we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(4n+3\right)q^n \equiv 2\frac{f_2^3f_{20}f_5^2}{f_4f_{10}f_1^2} \pmod{2^3}, \tag{62}$$

Using (7) in (62) and then extracting the coefficients of q^{2n+1} on both sides, we find that

$$\sum_{n=0}^{\infty} B_{2,5}\left(8n+7\right)q^n \equiv 4f_2f_{20}. \tag{63}$$

which implies

$$B_{2,5}\left(16n+15\right) \equiv 0 \pmod{2^3}, \tag{64}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(16n+7\right)q^n \equiv 4f_1f_{10}. \tag{65}$$

Using (7) in (23) and then collecting the coefficients of q^{2n+1} on both sides, we get

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+5}n+\frac{11\cdot 2^{2\alpha+3}-1}{3}\right)q^n \equiv 4f_2f_{10}^2 \pmod{2^3}, \tag{66}$$

which implies

$$B_{2,5}\left(2^{2\alpha+6}n+\frac{23\cdot 2^{2\alpha+3}-1}{3}\right) \equiv 0 \pmod{2^3}, \tag{67}$$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+6}n+\frac{11\cdot 2^{2\alpha+3}-1}{3}\right)q^n \equiv 4f_1f_5^2. \tag{68}$$

In view of (65) and (68), we see that the congruence (24) is true for $\beta=0$. Now, suppose the congruence (24) is true for $\beta>0$. Utilizing (44) in (24) and then extracting the terms involving q^{5n+1}, we deduce that

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4}\cdot 5^{2\beta+1}n+\frac{7\cdot 2^{2\alpha+1}\cdot 5^{2\beta+1}}{3}\right)q^n \equiv 4f_5f_2 \tag{69}$$

$$\equiv 4f_5f_{50}(R(q^{10})^{-1}-q^2-q^4R(q^{10})), \tag{70}$$

which implies

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4}\cdot 5^{2\beta+2}n+\frac{11\cdot 2^{2\alpha+1}\cdot 5^{2\beta+2}}{3}\right)q^n \equiv 4f_1f_{10}. \tag{71}$$

Thus, (24) is true for $\beta>0$. Hence, by mathematical induction congruence (24) holds for all $\beta\geq 0$. Congruences (25) and (26) follows from (24) and (70) respectively. Using (7) in (58) and then extracting the coefficients of q^{2n} on both sides, we find that

$$\sum_{n=0}^{\infty} B_{2,5}(8n+1)q^n \equiv 2\frac{f_2f_4f_{10}f_5^2}{f_{20}f_1^2} \pmod{2^3}. \tag{72}$$

Using (7) in (72) and then extracting the coefficients of q^{2n+1} on both sides, we get

$$\sum_{n=0}^{\infty} B_{2,5}(16n+9)q^n \equiv 4f_4f_{10} \pmod{2^3}, \tag{73}$$

which implies

$$B_{2,5}(32n + 25) \equiv 0 \pmod{2^3}, \tag{74}$$

$$\sum_{n=0}^{\infty} B_{2,5}(32n + 9)q^n \equiv 4f_1^2 f_5. \tag{75}$$

It follows from (75) and (22) that for all integers $\alpha \geq 0$

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+5}n + \frac{7 \cdot 2^{2\alpha+2} - 1}{3}\right) q^n \equiv 4f_1^2 f_5 \pmod{2^3}, \tag{76}$$

which is $\beta = 0$ case of (27). The rest of the proof by mathematical induction is similar to that of (24), so we omit the details. Congruences (28) and (29) follow immediately from the proof of (24).

8. Proof of the Theorem (2.8)

We prove the equation (30) by induction, the equation (24) is the $\gamma = 0$ case of congruence (30). Suppose that the congruence (30) holds for some integers $\gamma > 0$. For a prime $p > 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2 + k}{2} + 10 \times \frac{3m^2 + m}{2} \equiv \frac{11p^2 - 11}{24} \pmod{p}.$$

This is equivalent to $(6k + 1)^2 + 10(6m + 1)^2 \equiv 0 \pmod{p}$. Since $\left(\frac{-10}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Using (6) in (30), we obtain

$$\sum_{n=0}^{\infty} B_{2,5}\left(2^{2\alpha+4} \cdot 5^{2\beta} \cdot p^{2\gamma+1}n + \frac{11 \cdot 2^{2\alpha+1} \cdot 5^{2\beta} \cdot p^{2\gamma+2} - 1}{3}\right) q^n \equiv 4f_p f_{10p} \pmod{2^3}. \tag{77}$$

Colletcing the coefficients of q^{pn} in (77), we arrive

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+4} \cdot 5^{2\beta} \cdot p^{2\gamma+2}n + \frac{11 \cdot 2^{2\alpha+1} \cdot 5^{2\beta} \cdot p^{2\gamma+2} - 1}{3}\right) q^n \equiv 4f_1 f_{10} \pmod{2^3}, \tag{78}$$

which implies that (30) is true for $\gamma + 1$ with $\alpha, \beta > 0$. Hence by induction, (30) is true for all non-negative integers $\gamma > 0$. This proves (30). Collecting q^{pn+i} on both sides of (77), we obtain (31).

Since the proofs of (32), (33), (34), (35), (36) and (37) are similar to the proofs of (30) and (31), we omit the details.

References

[1] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, (1991).

[2] N. Calkin, N. Drake, K. James, S. Law, P. Lee, D. Penniston and J. Radder, *Divisibility properties of the 5-regular and 13-regular partition functions*, Integers, 8(2008), #A60.

[3] S. P. Cui and N. S. S. Gu, *Arithmetic properties of ℓ-regular partitions*, Adv. Appl. Math., 51(2013), 507-523.

[4] M. D. Hirschhorn, *Ramanujan's "most beautiful identity"*, Amer. Math. Monthly, 118(2011), 839-845.

[5] M. D. Hirschhorn and J. A. Sellers, *Elementary proofs of parity results for 5-regular partitions*, Bull. Aust. Math. Soc., 81(2010), 58-63.

[6] M. S. Mahadeva Naika and B. Hemanthkumar, *Arithmetic propertities of 5-regular bipartitions*, Int. J. Number Theory, 13(2)(2017), 939-956.

[7] M. S. Mahadeva Naika, B. Hemanthkumar and H. S. Sumanth Bharadwaj, *Congruences modulo* 2 *for certain partition functions*, Bull. Aust. Math. Soc., 93(3)(2016), 400-409.

[8] M. S. Mahadeva Naika, B. Hemanthkumar and H. S. Sumanth Bharadwaj, *Color partition identities arising from Ramanujan's theta functions*, Acta Math. Vietnam., 44(4)(2016), 633-660.

[9] M. Prasad and K. V. Prasad, *On* (ℓ, m)*-regular partitions with distinct parts*, Ramanujan J., 46(1)(2018), 19-27.

[10] M. Prasad and K. V. Prasad, *On* 5*-regular bipartitions into distinct parts*, Integers, 20(A94)(2020), 1-11.

[11] S. Ramanujan, *Collected papers*, Cambridge university press, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American mathematical society, RI, (2000).

[12] G. N. Watson, *Theorems stated by Ramanujan (VII): Theorems on continued fractions*, J. London Math. Soc., 4(1929), 39-48.