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ADDENDUM TO 'A BOLTZMANN CORRECTION'

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Abstract. Some implications concerning an unobservable medium whose micro-motion (rotational motion in Faraday vortex tubes) produces magnetic phenomena after Maxwell, used in a previous article on the origin of Maxwell's equation for the magnetic induction (*JODEA*, **26** (1), 29 – 44) are clarified and explained.

Key words: the magnetic induction law, the theory of molecular vortices, Faraday vortex tubes.

2010 Mathematics Subject Classification: 74A05, 76B47, 78A02, 78A25.

1. Introduction

In our previous article [1] on the subject:

1) we showed that the well known equation [3,5] for magnetic induction $\boldsymbol{B}(\boldsymbol{x},t)$ in a moving with velocity $\boldsymbol{u}(\boldsymbol{x},t)$ continuous medium

$$\boldsymbol{B}_t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{1.1}$$

or

$$\dot{\boldsymbol{B}} \equiv \boldsymbol{B}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u}, \tag{1.2}$$

usually referred to H. Alfvén, was derived by J. C. Maxwell [7,8] using the theory of molecular vortices yet in 1861;

2) we implemented a Boltzmann correction [2] to the original Maxwell derivation of equations (1.1) and (1.2) and obtained the following equation for $\boldsymbol{B}$

$$\boldsymbol{B}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \frac{1}{2} |\boldsymbol{B}|^{-2} \left(\boldsymbol{B} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{B} \right) \boldsymbol{B}, \tag{1.3}$$

or

$$\boldsymbol{B}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \frac{1}{2} \left(\boldsymbol{\tau} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\tau} \right) \boldsymbol{B}, \tag{1.4}$$

where $\boldsymbol{\tau}(\boldsymbol{x},t) = |\boldsymbol{B}|^{-1}\boldsymbol{B}$ is the unit vector co-directed with $\boldsymbol{B}$, $\hat{\boldsymbol{S}}(\boldsymbol{x},t)$ is the stretching tensor (the symmetrical part of tensor field $\nabla \boldsymbol{u}(\boldsymbol{x},t)$).

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In the above equations $(\boldsymbol{x}, t)$ is an inertial Cartesian orthogonal frame of reference, the lower index t indicates the partial derivative with respect to t, and the dot over a symbol indicates the material (substantial, or 'total') derivative with respect to t.

Reading Maxwell [7–10] is a very attractive though not so easy business, since his way of reasoning and 'speculations' ('My object in this paper is to clear the way for speculation in this direction. . . ' [7]) was based on numerous physical analogies, metaphors and implications (in some references cited in [1] his way of reasoning is thoroughly discussed). Most essential of them we believe related to: 1) the concept of a medium responsible for electromagnetic phenomena; 2) the concept of incompressibility of the medium; 3) the concept of finite radii of molecular vortices and Faraday vortex tubes.

In what follows, we attempt to clarify and explain partially the above implications to make some conclusions of our article [1] clear.

2. Implication of a medium

Maxwell explained magnetic phenomena using the motion of an unobservable 'magnetic' medium, the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ of which he considered known.

When deriving the equation of magnetic induction (1.1), (1.2), Maxwell distinguished between two types of motion of the medium. The first type is macro motion, or transfer phenomenon of the medium, whereas the second one is micromotion, or motion in Faraday vortex tubes. This implies that Maxwell distinguished between two velocity field $\boldsymbol{u}_0(\boldsymbol{x}, t)$ and $\boldsymbol{u}'(\boldsymbol{x}, t)$, where $\boldsymbol{u}_0(\boldsymbol{x}, t)$ is the velocity of macroscopic motion, that is, the space-averaged velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, and $\boldsymbol{u}'(\boldsymbol{x}, t)$ is the velocity of microscopic motion, hence

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x}, t) + \boldsymbol{u}'(\boldsymbol{x}, t) . \tag{2.1}$$

Nevertheless, when deriving the magnetic induction equation Maxwell used the same velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ for both types of motion of the medium.

3. Implication of incompressibility

Maxwell believed that the medium whose motion produces magnetic phenomena is incompressible. This made it possible to him to describe the absence of magnetic charges, since magnetic induction field $\boldsymbol{B}$ happened to be divergent-free, that is $\nabla \cdot \boldsymbol{B} = 0$. But the velocity field of this 'magnetic' unobservable medium coincides with the directly measurable velocity field of an observed continuous medium. And it is this velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ that is present in the equation of magnetic induction (1.1), (1.2). This immediately implies that the magnetic field is 'frozen' into the moving continuous medium. If the former is compressible, Maxwell's implication fails.

4. Implication of the vortex tube radius

In this section we explain in other way than that used in our article [1], how Maxwell's definition of magnetic force $\boldsymbol{H}$ introduces the radius of a Faraday vortex tube into equations (1.3) and (1.4) for magnetic induction $\boldsymbol{B}$.

For this purpose we take a very thin vortex tube of 'magnetic' medium (see Fig. 4.1, a), and let $r_c(\boldsymbol{x}, t)$ be its radius, measured in a plane perpendicular to the axis of the tube (below the plane is referred to as plane Π) from point $O'(\boldsymbol{x}, t)$ on the axis to any point M lying at the circumference of the tube. Then we write the definition for the magnetic force after Maxwell [1, 7]

$$\boldsymbol{H} = w_c\,\boldsymbol{\tau} = r_c\,\omega\,\boldsymbol{\tau} = r_c\,\boldsymbol{\omega}\,, \tag{4.1}$$

where $\boldsymbol{\tau}$ is the unit vector co-directed with the vector of angular velocity $\boldsymbol{\omega} = \omega\,\boldsymbol{\tau}$ of the tube, w_c is 'the velocity at the circumference' [7] of the vortex tube.

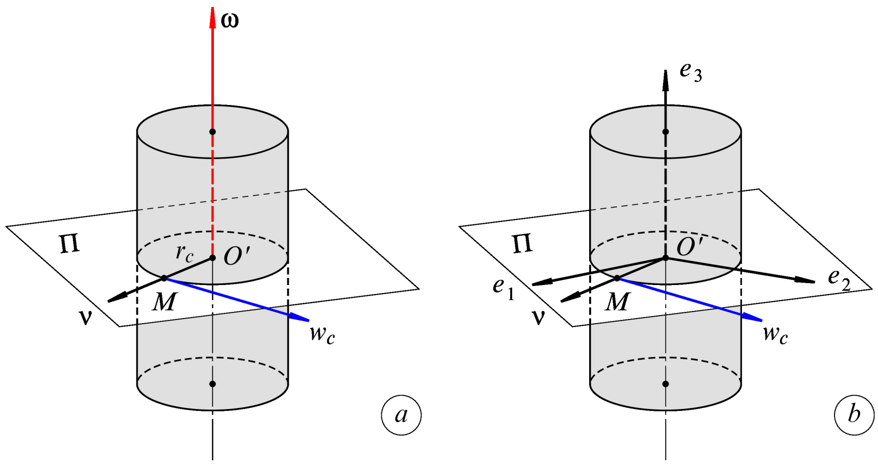

Fig. 4.1. A thin vortex tube segment and plane Π perpendicular to its axis at some point $O'(\boldsymbol{x}, t)$: any point M at the circumference of the tube is given as $\boldsymbol{x} + r_c\boldsymbol{\nu}$, r_c being the local radius of the tube, $\boldsymbol{\nu}$ being a unit vector lying in plane Π and starting at point O' (a); basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ of a local frame of reference is a set of orthonormal eigenvectors of stretching tensor $\hat{\boldsymbol{S}}$ (b)

Taking the substantial derivative of both sides of the above definition yields to

$$\boldsymbol{H}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{H} \equiv \dot{\boldsymbol{H}} = \dot{r}_c\,\boldsymbol{\omega} + r_c\,\dot{\boldsymbol{\omega}}\,. \tag{4.2}$$

We find the substantial derivative of r_c in (4.2) using the normal component of the stretching tensor in the direction of unit vector $\boldsymbol{\nu}$ [11, 13]

$$\dot{r}_c = \left(\boldsymbol{\nu}\cdot\hat{\boldsymbol{S}}\cdot\boldsymbol{\nu}\right) r_c\,, \tag{4.3}$$

where $\boldsymbol{\nu}$ lies in plane Π and starts at point O'.

To find the substantial derivative of $\dot{\boldsymbol{\omega}}$ in (4.2) we refer to the Helmholtz equation [6, 12] for vorticity $\boldsymbol{\Omega} = \nabla \times \boldsymbol{u} = 2\,\boldsymbol{\omega}$

$$\dot{\boldsymbol{\Omega}} \equiv \boldsymbol{\Omega}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u}\,,$$

hence

$$\dot{\boldsymbol{\omega}} \equiv \boldsymbol{\omega}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}\,. \tag{4.4}$$

Substituting quantities $\dot{r}_c$ (4.3) and $\dot{\boldsymbol{\omega}}$ (4.4) into (4.2) gives

$$\dot{\boldsymbol{H}} = \left(\boldsymbol{\nu} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\nu}\right) r_c\, \boldsymbol{\omega} + r_c\, \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \left(\boldsymbol{\nu} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\nu}\right) \boldsymbol{H} + \boldsymbol{H} \cdot \nabla \boldsymbol{u}\,. \tag{4.5}$$

Being symmetric the stretching tensor admits the following representation

$$\hat{\boldsymbol{S}} = \sigma_1\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_3\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\,, \tag{4.6}$$

known as the spectral decomposition [4], where σ_1, σ_2, σ_3 are the eigenvalues of $\hat{\boldsymbol{S}}$, and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are orthonormal eigenvectors of $\hat{\boldsymbol{S}}$.

From decomposition (4.6) it follows that the eigenvalues are the normal components of $\hat{\boldsymbol{S}}$ in the directions of the respective eigenvectors as follows

$$\sigma_1 = \boldsymbol{e}_1 \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{e}_1\,, \qquad \sigma_2 = \boldsymbol{e}_2 \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{e}_2\,, \qquad \sigma_3 = \boldsymbol{e}_3 \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{e}_3\,. \tag{4.7}$$

The sum of the eigenvalues is an invariant of $\hat{\boldsymbol{S}}$ and, due to incompressibility of the medium (see section 3), we have

$$\sigma_1 + \sigma_2 + \sigma_3 = \nabla \cdot \boldsymbol{u} = 0\,. \tag{4.8}$$

Now we account for the symmetry of the flow in the thin vortex tube (one should not confuse the symmetry of the flow and the symmetry of tensor $\hat{\boldsymbol{S}}$) and rearrange the eigenvectors of $\hat{\boldsymbol{S}}$ in such a way that both eigenvectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are in plane Π, whereas eigenvector $\boldsymbol{e}_3$ is directed along the axis of the tube ($\boldsymbol{e}_3 = \mp\,\boldsymbol{\tau}$). Taking the eigenvectors as the basis of a local frame of reference, we easily conclude that: 1) $\boldsymbol{\nu} = \mathtt{cos}\,\phi\; \boldsymbol{e}_1 + \mathtt{sin}\,\phi\; \boldsymbol{e}_2$, $0 \leqslant \phi < 2\pi$; 2) $\sigma_1 = \sigma_2 =: \sigma_\perp$; and 3) $\sigma_{\shortparallel} = \sigma_3 = -\,2\sigma_\perp$ (cf. Fig. 4.1, *b* and (4.8)).

Now we turn to the calculation of quantity $\boldsymbol{\nu} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\nu}$ in (4.5), which is still unknown, and find (keeping in mind spectral decomposition (4.6)) that

$$\boldsymbol{\nu} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\nu} = \sigma_1\, \mathtt{cos}^2\phi + \sigma_2\, \mathtt{sin}^2\phi = \sigma_\perp\,,$$

then equation (4.5) for the magnetic force readily reads

$$\dot{\boldsymbol{H}} = \boldsymbol{H} \cdot \nabla \boldsymbol{u} + \sigma_\perp \boldsymbol{H}\,. \tag{4.9}$$

The only thing that is remained to be done is to replace eigenvalue $\sigma_\perp$ in (4.9) with quantity $-\frac{1}{2}\,\sigma_{\shortparallel}$

$$\dot{\boldsymbol{H}} = \boldsymbol{H} \cdot \nabla \boldsymbol{u} - \frac{1}{2}\, \sigma_{\shortparallel} \boldsymbol{H}\,, \tag{4.10}$$

where eigenvalue $\sigma_{\shortparallel}$ is given in (4.7) as the normal component of tensor $\hat{\boldsymbol{S}}$ in the direction of unit vector $\boldsymbol{e}_3 = \mp\, \boldsymbol{\tau}$

$$\sigma_{\shortparallel} = \sigma_3 = \boldsymbol{e}_3 \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{e}_3 = \boldsymbol{\tau} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\tau}\,. \tag{4.11}$$

Combining (4.10) and (4.11) gives the required equation for the magnetic force

$$\dot{\boldsymbol{H}} = \boldsymbol{H} \cdot \nabla \boldsymbol{u} - \frac{1}{2} \left(\boldsymbol{\tau} \cdot \hat{\boldsymbol{S}} \cdot \boldsymbol{\tau} \right) \boldsymbol{H}\,. \tag{4.12}$$

Now we remind that the constitutive equation reads $\boldsymbol{B} = \mu \boldsymbol{H}$ (in the theory of Maxwell magnetic impermeability μ being constant is simply the density of the ‘magnetic’ medium) and from this we immediately obtain equation (1.4) for magnetic induction $\boldsymbol{B}$.

5. Conclusions

Clarifying and deciphering implications of Maxwell’s theory [7–10] allows us to better understand the nature of Maxwell’s equations and even introduce in the former the possibility of describing some new phenomena. Indeed, if the magnetic field, according to Maxwell, is generated by the surface (peripheral) velocity of the Faraday vortex tubes, then the appearance of the longitudinal velocity (along the axes of tubes) in inhomogeneous and unsteady motions of the ‘magnetic’ medium could lead to the appearance of a scalar magnetic field.

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