Asymptotical Synchronization of Coupled Time-delay Partial Differential Systems via Pinning Control and Boundary Control

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---**ABSTRACT**---
This paper focus on the asymptotic synchronization issue of coupled time-delay PDSs via pinning control and boundary control. The asymptotic synchronization of PDSs with both node-delay and coupling delay is discussed firstly. Then the pinning controller and boundary controller are also presented in order to achieve the asymptotic synchronization. Further more, synchronization criteria are established by using the Lyapunov function method and inequality techniques. Obviously, it is an efficient control technique to combine the pinning control with the boundary control for the asymptotic synchronization of the PDSs. Finally, an example of digital simulation is used to elucidate the practicability and validity of our control method and the correctness of the theorem.

Keywords-**Asymptotic synchronization, partial differential systems, pinning control, boundary control**

I. INTRODUCTION

Complex networks have become omnipresent for the irreplaceable role they play in nature and human society, and their application in engineering industry is the most noticeable. Examples of representative networks, such as Internet, power grids and interpersonal network, abound in our lives. Thus the study of complex networks not only provides great scientific value for us to further understand and adhere to the law of nature, it also shown great practical guiding significance in our life.

Synchronization is an important feature of complex networks. By taking advantage of its special property, people have successfully solved many sticky problems. Domestic and foreign scholars have proposed and studied various synchronization patterns [1-26], [28-34].

Delays are inevitable on account of the limited propagation velocity, traffic congestion in the signal transmission and other objective factors. If delays are ignored, differences between theoretical analysis and reality will emerge, which in turn, may contribute to erroneous results and immeasurable loss. Time delay exists objectively during the information transmission of complex networks. Hence, complex networks with time-delay are attracting more and more attention because of their broad prospect in application and the synchronization problem with time-delay has been studied extensively [1-8]. The delay in the complex dynamical network comprises two types, the node delay that occurs inside the system, and the coupling delays generated during the exchange of information between systems. The model discussed in this paper is the co-existence of node delay and coupling delay.

The phenomena related to time and space can be modeled by partial differential systems in the field of time and space in mathematics. Due to the existence of complex network models with PDSs in reality, the research on synchronization of complex networks with PDSs has aroused great interest of scholars and many breakthroughs have been made in this field. In [22], the author explored the pinning sampled-date synchronization of inertial neural networks. The authors studied pinning synchronization of CRDNNs and obtained the sufficient conditions to make the networks achieve synchronization in [23-24]. In [26], the author designed a novel distributed coupling protocol and proposed some criteria for the complete synchronization of CRDNNs. In [27], the author addressed the passivity-based synchronization problems of an array model of nonlinear CRDNNs. In [30], the authors considered the synchronization for the coupled PDSs via diffusion coupling. In [31], the author investigated the synchronization of coupled delay PDSs. In [34], the sufficient condition which guarantees the mean square synchronization for disturbed coupled stochastic PDSs was provided and the adaptive pinning control strategy was proposed. The synchronization control of PDSs is mainly based on the distributed control method in the documents listed above.

It is impractical to impose control on all nodes because of the complex properties and huge structure of a complex dynamical network. In order to reduce the difficulty to control and to save the cost, we naturally consider the practice of selecting some nodes in the network and taking control of them to realize synchronization. We named this control method pinning control. Owing to its low cost and convenience, pinning control is widely applied in the synchronization control of complex networks and a lot of fruitful research results have been achieved, see [9-28]. At the same time, the adaptive control method is beneficial for the network to obtain the appropriate control gain, thus the synchronization can be ensured.

We know that boundary control is an excellent control method for PDSs as it just requires a small number of controllers and is easy to use. Therefore, some authors

started their study on this topic. In [32], Kai-Ning Wu investigated the asymptotic synchronization for PDSs and presented criteria for synchronization via boundary control. In [33], the authors studied the synchronization for PDSs by using the back-stepping approach and boundary control. However, many problems related to boundary control require further study.

Obviously, it is a good control strategy to combine the pinning control with boundary control in order to realize asymptotic synchronization for the PDSs. In that the research is difficult, as far as we know, there is no relative literature. So in literature [32], the author regards this issue as an open question and looks forward to further research results.

After reviewing a large number of literature, this article, inspired by those above-mentioned analysis, is devoted to the research of the asymptotic synchronization of PDSs via pinning control and boundary control. The main innovations of this article are listed as below in comparison with other references related to boundary control.

(1) We will try to solve the pinning boundary control problem for the coupled PDSs in this article. A new adaptive pinning controller is designed to control the fraction of nodes in PDSs. The pinning controller and the boundary controller are designed to realize the asymptotic synchronization and the criteria for synchronization are established. We have imposed additional pinning control on the basis of boundary control compared with literature [32].

(2) The research scope of the asymptotic synchronization of PDSs is expanded. On the one hand, in [32], the author investigated node-delay and coupling delay of the coupled PDSs respectively and presented the synchronization criteria. In this paper, the co-existence of node-delay and coupling delay of PDSs will be studied and this situation is more complicated. On the other hand, the function in the PDSs is assumed to be a nonlinear vector valued function. From these two aspects we can see that a great improvement has been made in the model in literature [32] and the model presented in this article is more general and more practical.

(3) A new asymptotic synchronization criterion is established for the coupled delay PDSs by using the Lyapunov functional method and inequality techniques.

The major content and the structure of this article are arranged this way. The PDSs model with pinning controller and boundary controller is presented and background knowledge is introduced in section 2. The asymptotic synchronization criteria for the coupled delay PDSs are established in section 3. In section 4, an example of digital simulation is used to elucidate the practicability and validity of our control method and the correctness of the theorem. The last section summarizes the main conclusion of this paper.

II. MODEL DESCRIPTION AND PRELIMARIES

First, we consider the N-coupled PDSs with node-delay and coupling delay

$$
\begin{cases}
y_{i,t}(x,t) = \Theta y_{i,xx}(x,t) + f(y_i(x,t)) \\
+ A y_i(x,t-\tau) + \sum_{j=1}^{N} g_{ij} y_j(x,t-\tau) + v_i(x,t) \\
y_{i,x}(x,t)\big|_{x=0} = C u_i(t), y_{i,x}(x,t)\big|_{x=L} = 0 \\
y_i(x,t) = \varphi_i(x,t), t \in [-\tau,0], i = 1,2,\cdots,N
\end{cases}
\tag{1}
$$

$y_i(x,t) \triangleq [y_{i1}(x,t), \quad, y_{in}(x,t)]^T \in R^n$ is the state variable of the i-th subsystem and the subscripts t and x stand for separately the partial derivatives relative to t and x in system (1). The variables $t \in [0,\infty)$ and $x \in [0,L] \subset R$ are the time and spatial variable and $f(y_i(x,t)) \in R^n$ is a nonlinear vector valued function used to characterize the dynamic behavior of the node. Θ is required to be a symmetric positive definite matrix and $\Theta, A \in R^{n \times n}$ are real matrices. $u_i(t) \in R^m$ is the boundary control input of the i-th subsystem and $C \in R^{n \times m}$ is the control input matrix. $G \triangleq (g_{ij})_{N \times N}$ is the coupling matrix of the PDSs and g_{ij} is defined as follows: $g_{ij} \neq 0$ when there is a connection between node i and node $j(i \neq j)$; otherwise, $g_{ij} = 0(i \neq j)$ and the diagonal elements are defined by $g_{ii} = -\sum_{j \neq i} g_{ij}, i = 1,2,\cdots,N$.

Remark 1 The matrix G does not need to be symmetric or irreducible.

Let $s(x,t) \triangleq [s_1(x,t), \quad, s_n(x,t)]^T \in R^n$ be a unique solution to the isolated node of the PDSs to which all $y_i(x,t)'s$ are expected to synchronize and $s(x,t)$ satisfies the following PDSs:

$$
\begin{cases}
s_t(x,t) = \Theta s_{xx}(x,t) + f(s(x,t)) + A s(x,t-\tau), \\
s_x(x,t)\big|_{x=0} = s_x(x,t)\big|_{x=L} = 0, \\
s(x,t) = \varphi(x,t), t \in [-\tau,0].
\end{cases}
\tag{2}
$$

Define the synchronization error of the PDSs as below:

$$
e_i(x,t) = y_i(x,t) - s(x,t) . \tag{3}
$$

And $u_i(t) \in R^m$ is the state feedback controller for the i-th node of the PDSs (1) which is designed as follows:

$$
u_i(t) = \int_0^L \gamma(y_i(x,t) - s(x,t))dx = \int_0^L \gamma e_i(x,t)dx, \tag{4}
$$

where $\gamma \in R^{m \times n}$ is the undetermined control gain matrix.

$v_i(x,t), i = 1,2,\cdots,N$ are the pinning adaptive controllers which are designed as follows:

$$
\begin{cases}
v_i(x,t) = -k_i(x,t)e_i(x,t), i = 1,2,\cdots,l \\
v_i(x,t) = 0, i = l+1,\cdots N.
\end{cases}
\tag{5}
$$

We can assume that the first $l(1 \leq l \leq N)$ nodes are selected to be controlled by pinning controllers.
The pinning feedback gains are tuned by

$$k_{i,t}(x,t) = \delta_i e_i^T(x,t)e_i(x,t), i = 1,2,\cdots,l. \tag{6}$$

where δ_i is a positive constant for $1 \leq i \leq l$.
From (1) and (2), we can get the following error system of the i-th node:

$$\begin{cases} e_{i,t}(x,t) = \Theta e_{i,xx}(x,t) + f(y_i(x,t)) \\ \qquad - f(s(x,t)) + Ae_i(x,t-\tau) \\ \qquad + \sum_{j=1}^{N} g_{ij}e_j(x,t-\tau) + v_i(x,t), \\ e_{i,x}(x,t)\big|_{x=0} = Cu_i(t), e_{i,x}(x,t)\big|_{x=L} = 0, \\ e_i(x,t) = \varphi_i(x,t) - \varphi(x,t), t \in [-\tau,0]. \end{cases} \tag{7}$$

Remark 2 The model of the PDSs (1) discussed here is different from the models of PDSs aforementioned [22-24, 26-27, 30-34]. So the model of the PDSs is extended.
Denote

$$e(x,t) = [e_1^T(x,t),\cdots,e_N^T(x,t)]^T, \tag{8}$$

$$K = diag(\underbrace{k_1(x,t),\cdots,k_l(x,t)}_{l},\underbrace{0,\cdots,0}_{N-l}), \tag{9}$$

$$K^* = diag(\underbrace{k_1^*,\cdots,k_l^*}_{l},\underbrace{0,\cdots,0}_{N-l}), \tag{10}$$

$$F = [(f(y_1(x,t) - f(s(x,t)))^T, \\ \cdots,(f(y_N(x,t) - f(s(x,t)))^T]^T. \tag{11}$$

For the sake of simplicity, the variables (x,t) are omitted in the subsequent proof. I_n is the identity matrix and $A \otimes B$ stands for the Kronecker product of A and B. According to equation (7) and the above notations, the error system can be converted into the following form:

$$\begin{cases} e_t = (I_N \otimes \Theta)e_{xx} + F + (I_N \otimes A)e(x,t-\tau) \\ \qquad + (G \otimes I_n)e(x,t-\tau) - (K \otimes I_n)e, \\ e_x\big|_{x=0} = \int_0^L (I_N \otimes C\gamma)edx, e_x\big|_{x=L} = 0, \\ e = \Phi, t \in [-\tau,0]. \end{cases} \tag{12}$$

where $\Phi = [\varphi_1^T - \varphi^T,\cdots,\varphi_N^T - \varphi^T]^T$.

For a symmetric matrix M, $M < 0(M \leq 0)$ means that it is negative definite (semi-negative definite). $W^{1,2}([0,L];R^n)$ is a Sobolev space of absolutely continuous n-dimensional vector function $\omega(x):[0,L] \to R^n$ with Squire integral derivatives $\dfrac{d^l\omega(x)}{dx^l}$ of the order $l \geq 1$.

Definition 1 [32] The PDSs (1) achieve asymptotic synchronization if the error $e(x,t)$ satisfies $\lim_{t \to \infty} e(x,t) = 0$ for all $x \in [0,L]$.

Lemma 1 [35] Let $z \in W^{1,2}([0,L];R^n)$ be a vector function with $z(0) = 0$ or $z(L) = 0$. Then, for a matrix $S > 0$, we have the integral inequality:

$$\int_0^L z^T(s)Sz(s)ds \leq 4L^2\pi^{-2}\int_0^L (\frac{dz}{ds})^T S(\frac{dz}{ds})ds.$$

Assumption 1 [4] Suppose that there exists $l_i > 0$, satisfying

$$(y_i(x,t) - s(x,t))^T (f(y_i(x,t)) - f(s(x,t))) \\ \leq l_i(y_i(x,t) - s(x,t))^T (y_i(x,t) - s(x,t)).$$

III. ASYMPTOTIC SYNCHRONIZATION FOR COUPLED TIME-DELAY PDSS

In this section, we present the synchronization criteria for the coupled time-delay PDSs via pinning control and boundary control.
Theorem1. For the PDSs (1) with node-delay and coupling delay, if there exist matrices $\gamma \in R^{m \times n}$ and $K^* \in R^{N \times N}$ satisfying the following LMI:

$$\Psi = \begin{bmatrix} \Psi_{11} & (I_N \otimes \Theta C\gamma)^T \\ I_N \otimes \Theta C\gamma & -\dfrac{\pi^2}{2L^2}(I_N \otimes \Theta) \end{bmatrix} < 0, \tag{13}$$

where
$\Psi_{11} = 2I_{nN} + 2l^*I_{nN} + 2(I_N \otimes A + G \otimes I_n)(I_N \otimes A + G \otimes I_n)^T$
$-2(I_N \otimes \Theta C\gamma) - 2(K^* \otimes I_n)$, then the PDSs (1) achieve asymptotic synchronization via pinning control and boundary control.
Proof. If we want to prove that system (1) is asymptotically synchronous, we only need to prove that the system (12) is asymptotically stable.
Construct the Lyapunov function for the error system (12)

$$V(e(\cdot,t)) = \int_0^L \sum_{i=1}^{N} e_i^T(x,t)e_i(x,t)dx \\ + 2\int_0^L \int_{t-\tau}^{t} \sum_{i=1}^{N} e_i^T(x,s)e_i(x,s)dsdx \\ + \int_0^L \sum_{i=1}^{l} \frac{1}{\delta_i}(k_i(x,t) - k_i^*)^2 dx. \tag{14}$$

The time derivative of V along the solution of PDSs (7) is given by

$$\dot{V}(e(\cdot,t)) = 2\int_0^L \sum_{i=1}^{N} e_i^T e_{i,t} - 2\int_0^L \sum_{i=1}^{N} e_i^T(x,t-\tau)e_i(x,t-\tau)dx$$

$$+2\int_0^L \sum_{i=1}^N e_i^T e_i dx + 2\int_0^L \sum_{i=1}^l \frac{1}{\delta_i}(k_i - k_i^*)k_{i,t}dx$$

$$= 2\int_0^L \sum_{i=1}^N e_i^T [\Theta e_{i,xx} + f(y_i) - f(s) + Ae_i(x,t-\tau)$$

$$+ \sum_{j=1}^N g_{ij}e_j(x,t-\tau) - k_i e_i] + 2\int_0^L \sum_{i=1}^N e_i^T e_i dx$$

$$-2\int_0^L \sum_{i=1}^N e_i^T(x,t-\tau)e_i(x,t-\tau)dx + 2\int_0^L \sum_{i=1}^l (k_i - k_i^*)e_i^T e_i dx$$

$$= 2\int_0^L e^T(I_N \otimes \Theta)e_{xx}dx + 2\int_0^L e^T(I_N \otimes A)e(x,t-\tau)dx$$

$$+2\int_0^L e^T F dx + 2\int_0^L e^T(G \otimes I_n)e(x,t-\tau)dx$$

$$-2\int_0^L e^T(K \otimes I_n)edx + 2\int_0^L e^T((K-K^*)\otimes I_n)edx$$

$$+2\int_0^L e^T edx - 2\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx. \quad (15)$$

Integrating by parts and using the boundary conditions of (12), we can get

$$\int_0^L e^T(I_N \otimes \Theta)e_{xx}dx = e^T(I_N \otimes \Theta)e_x\big|_0^L - \int_0^L e_x^T(I_N \otimes \Theta)e_x dx$$

$$= -e^T(0,t)(I_N \otimes \Theta)\int_0^L (I_N \otimes C\gamma)edx$$

$$-\int_0^L e_x^T(I_N \otimes \Theta)e_x dx. \quad (16)$$

Since $\Theta > 0$, we have

$$(I_N \otimes \Theta) > 0. \quad (17)$$

Using Lemma 1 and taking (17) into account , we can obtain

$$-\int_0^L e_x^T(I_N \otimes \Theta)e_x dx \le -\frac{\pi^2}{4L^2}\int_0^L \overline{e}^T(I_N \otimes \Theta)\overline{e}dx, \quad (18)$$

where $\overline{e}(x,t) \triangleq e(x,t) - e(0,t)$.

Substituting (18) into (16), we have

$$2\int_0^L e^T(I_N \otimes \Theta)e_{xx}dx \le -2e^T(0,t)(I_N \otimes \Theta)\int_0^L (I_N \otimes C\gamma)edx$$

$$-\frac{\pi^2}{2L^2}\int_0^L \overline{e}^T(I_N \otimes \Theta)\overline{e}dx$$

$$\le -2e^T(0,t)\int_0^L (I_N \otimes \Theta C\gamma)edx$$

$$-\frac{\pi^2}{2L^2}\int_0^L \overline{e}^T(I_N \otimes \Theta)\overline{e}dx. \quad (19)$$

And by virtue of assumption 1, we can get

$$2\int_0^L e^T F dx = 2\int_0^L \sum_{i=1}^N e_i^T(f(y_i) - f(s))$$

$$\le 2\int_0^L \sum_{i=1}^N l_i e_i^T e_i.$$

Let $l^* = \max\{l_1, l_2, \cdots l_N\}$, we have

$$2\int_0^L e^T F dx \le 2l^*\int_0^L \sum_{i=1}^N e_i^T e_i dx = 2l^*\int_0^L e^T edx, \quad (20)$$

Substituting (19) and (20) into (15), we can obtain

$$\dot{V}(e(\cdot,t)) \le -2e^T(0,t)\int_0^L (I_N \otimes \Theta C\gamma)edx$$

$$-\frac{\pi^2}{2L^2}\int_0^L \overline{e}^T(I_N \otimes \Theta)\overline{e}dx + 2l^*\int_0^L e^T edx$$

$$+2\int_0^L e^T[(I_N \otimes A) + (G \otimes I_n)]e(x,t-\tau)dx$$

$$+2\int_0^L e^T edx - 2\int_0^L e^T(K^* \otimes I_n)edx$$

$$-2\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx.$$

Since

$$e^T(I_N \otimes A + G \otimes I_n)e(x,t-\tau) \le e^T(x,t-\tau)e(x,t-\tau)$$

$$+ e^T(I_N \otimes A + G \otimes I_n)(I_N \otimes A + G \otimes I_n)^T e.$$

We have

$$\dot{V}(e(\cdot,t)) \le -\frac{\pi^2}{2L^2}\int_0^L \overline{e}^T(I_N \otimes \Theta)\overline{e}dx + 2l^*\int_0^L e^T edx$$

$$-2\int_0^L (e^T - \overline{e}^T)(I_N \otimes \Theta C\gamma)edx$$

$$+2\int_0^L e^T(I_N \otimes A + G \otimes I_n)(I_N \otimes A + G \otimes I_n)^T edx$$

$$+2\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx - 2\int_0^L e^T(K^* \otimes I_n)edx$$

$$+2\int_0^L e^T edx - 2\int_0^L e^T(x,t-\tau)e(x,t-\tau)dx$$

$$\le -2\int_0^L e^T(I_N \otimes \Theta C\gamma)edx + \int_0^L \overline{e}^T(I_N \otimes \Theta C\gamma)edx$$

$$+\int_0^L e^T(I_N \otimes \Theta C\gamma)^T \overline{e}dx + 2l^*\int_0^L e^T edx$$

$$-\frac{\pi^2}{2L^2}\int_0^L \overline{e}^T(I_N \otimes \Theta)\overline{e}dx + 2\int_0^L e^T edx$$

$$-2\int_0^L e^T(K^* \otimes I_n)edx$$

$$+2\int_0^L e^T(I_N \otimes A + G \otimes I_n)(I_N \otimes A + G \otimes I_n)^T edx$$

$$\le \int_0^L e^T[2I_{nN} + 2l^*I_{nN} - 2(I_N \otimes \Theta C\gamma)$$

$$+2(I_N \otimes A + G \otimes I_n)(I_N \otimes A + G \otimes I_n)^T$$

$$-2(K^* \otimes I_n)]edx + \int_0^L \overline{e}^T(I_N \otimes \Theta C\gamma)edx$$

$$+\int_0^L e^T(I_N \otimes \Theta C\gamma)^T \overline{e}dx$$

$$-\frac{\pi^2}{2L^2}\int_0^L \overline{e}^T(I_N\otimes\Theta)\overline{e}\,dx$$

$$=\int_0^L(e^T\overline{e}^T)\Psi\left(\frac{e}{\overline{e}}\right)dx.$$

In light of (13), we can get

$$\dot{V}(e(\cdot,t))\leq 0$$

Noting $V(0)=0, \lim_{\|e\|\to\infty}V(e)=\infty,$ we can find that

$\lim_{t\to\infty}\int_0^L e^T e\,dx=0$, then $\lim_{t\to\infty}e(x,t)=0$,

This indicates that system (12) is asymptotically stable so that the proof of the theorem is complete. $\square$

Remark 3 The control method in this paper is different from the control method in literature [32] because the pinning controller and the boundary controller are designed simultaneously in system (1) to realize the asymptotic synchronization.

Remark 4 The nonlinear PDSs (1) presented in this paper is more general and applicable than the linear PDSs (2.1) in literature [32] because the function f in PDSs (1) is a nonlinear vector valued function.

Remark 5 We can see that the model (2.1) and the model (3.7) proposed in literature [32] can be considered as two special cases of the model (1) proposed in this paper because the node-delay and the coupling delay are included in model (1).

Remark 6 The model of PDSs is generalized in this paper and our control methods are different in comparison with the PDSs in literature [22-24, 26-27, 30-34]. For the first time, we combine pinning control with boundary control for synchronous control of PDSs.

Next, we discuss the coupled PDSs only with the node-delay.

$$\begin{cases} y_{i,t}(x,t)=\Theta y_{i,xx}(x,t)+f(y_i(x,t)) \\ \qquad +Ay_i(x,t-\tau)+\sum_{j=1}^N g_{ij}y_j(x,t)+v_i(x,t), \\ y_{i,x}(x,t)\big|_{x=0}=Cu_i(t),\ y_{i,x}(x,t)\big|_{x=L}=0, \\ y_i(x,t)=\varphi_i(x,t),t\in[-\tau,0],i=1,2,\cdots,N. \end{cases} \quad (21)$$

Boundary controller $u_i(t)$ and pinning controller $v_i(x,t)$ are designed according to (4) and (5).

The synchronization function $s(x,t)$ can be described by the following equation corresponding to equation (21):

$$\begin{cases} s_t=\Theta s_{xx}+f(s)+As(x,t-\tau), \\ s_x\big|_{x=0}=s_x\big|_{x=L}=0, \\ s=\varphi,t\in[-\tau,0]. \end{cases} \quad (22)$$

We can obtain the synchronization error dynamics, for $i=1,2,\cdots,N,$

$$\begin{cases} e_{i,t}=\Theta e_{i,xx}+f(y_i)-f(s) \\ \qquad +Ae_i(x,t-\tau)+\sum_{j=1}^N g_{ij}e_j+v_i, \\ e_{i,x}\big|_{x=0}=Cu_i(t),e_{i,x}\big|_{x=L}=0, \\ e_i=\varphi_i-\varphi,t\in[-\tau,0]. \end{cases} \quad (23)$$

Similarly, we can get the following error system:

$$\begin{cases} e_t=(I_N\otimes\Theta)e_{xx}+F+(I_N\otimes A)e(x,t-\tau) \\ \qquad +(G\otimes I_n)e-(K\otimes I_n)e, \\ e_x\big|_{x=0}=\int_0^L(I_N\otimes C\gamma)e\,dx,e_x\big|_{x=L}=0, \\ e=\Phi,t\in[-\tau,0]. \end{cases} \quad (24)$$

where $\Phi=[\varphi_1^T-\varphi^T,\cdots,\varphi_N^T-\varphi^T]^T$.

For system (21), we have the following corollary of asymptotic synchronization via boundary control and pinning control.

Corollary 1 For the PDSs (21), if there are matrices $\gamma\in R^{m\times n}$ and $K^*\in R^{N\times N}$ satisfying the following LMI:

$$\Psi=\begin{bmatrix} \Psi_{11} & (I_N\otimes\Theta C\gamma)^T \\ I_N\otimes\Theta C\gamma & -\dfrac{\pi^2}{2L^2}(I_N\otimes\Theta) \end{bmatrix}<0, \quad (25)$$

where $\Psi_{11}=2I_{nN}+2l^*I_{nN}+2(I_N\otimes A)(I_N\otimes A)^T$

$+2(G\otimes I_n)-2(I_N\otimes\Theta C\gamma)-2(K^*\otimes I_n).$

Then the PDSs (21) are of asymptotic synchronization. Here we omit the proof.

At the end of this section, we discuss the PDSs only with the coupling delay.

$$\begin{cases} y_{i,t}(x,t)=\Theta y_{i,xx}(x,t)+f(y_i(x,t)) \\ \qquad +\sum_{j=1}^N g_{ij}y_j(x,t-\tau)+v_i(x,t), \\ y_{i,x}(x,t)\big|_{x=0}=Cu_i(t),\ y_{i,x}(x,t)\big|_{x=L}=0, \\ y_i(x,t)=\varphi_i(x,t),t\in[-\tau,0],i=1,2,\cdots,N. \end{cases} \quad (26)$$

Boundary controller $u_i(t)$ and pinning controller $v_i(x,t)$ are designed according to (4) and (5).

Define the synchronization function $s(x,t)$ as follows:

$$\begin{cases} s_t=\Theta s_{xx}+f(s), \\ s_x\big|_{x=0}=s_x\big|_{x=L}=0, \\ s=\varphi,t\in[-\tau,0]. \end{cases} \quad (27)$$

The error system is as follows:

$$\begin{cases} e_{i,t} = \Theta e_{i,xx} + f(y_i) - f(s) \\ \qquad + \sum_{j=1}^{N} g_{ij} e_j(x, t-\tau) + v_i, \\ e_{i,x}\big|_{x=0} = Cu_i(t), e_{i,x}\big|_{x=L} = 0, \\ e_i = \varphi_i - \varphi, t \in [-\tau, 0]. \end{cases} \quad (28)$$

The error system can be converted into the following form:

$$\begin{cases} e_t = (I_N \otimes \Theta)e_{xx} + F \\ \qquad + (G \otimes I_n)e(x, t-\tau) - (K \otimes I_n)e, \\ e_x\big|_{x=0} = \int_0^L (I_N \otimes C\gamma)e\,dx, e_x\big|_{x=L} = 0, \\ e = \Phi, t \in [-\tau, 0]. \end{cases} \quad (29)$$

where $\Phi = [\varphi_1^T - \varphi^T, \cdots, \varphi_N^T - \varphi^T]^T$.

For system (26), we have the following corollary of asymptotic synchronization via boundary control and pinning control.

Corollary 2 For the coupling delay PDSs (26), if there are matrices $\gamma \in R^{m \times n}$ and $K^* \in R^{N \times N}$ satisfying the following LMI:

$$\Psi = \begin{bmatrix} \Psi_{11} & (I_N \otimes \Theta C\gamma)^T \\ I_N \otimes \Theta C\gamma & -\dfrac{\pi^2}{2L^2}(I_N \otimes \Theta) \end{bmatrix} < 0, \quad (30)$$

where

$\Psi_{11} = 2I_{nN} + 2l^* I_{nN} + 2(G \otimes I_n)(G \otimes I_n)^T - 2(I_N \otimes \Theta C\gamma)$
$-2(K^* \otimes I_n)$.

Then the PDSs (26) are of asymptotic synchronization.
We omit the proof here.

IV. EXAMPLE

An example of digital simulation is used to elucidate the practicability and validity of our control method and the correctness of the theorem in this section. We discuss the PDSs with node-delay and coupling delay

$$y_{i,t}(x, t) = 0.01 y_{i,xx}(x, t) - \sin(y_i(x, t))$$
$$+ 0.01 y_i(x, t - 0.5)$$
$$+ \sum_{j=1}^{3} g_{ij} y_j(x, t\text{-}0.5) - \frac{t}{x} e_i(x, t). \quad (31)$$

The boundary conditions and initial value are designed as below:

$$y_{i,x}\big|_{x=0} = u_i(t), y_{i,x}\big|_{x=1} = 0, y_i(x, t) = \varphi_i(x),$$
$$t \in [-0.5, 0], i = 1, 2, 3,$$

in which $u_i(t) = 4\int_0^1 e_i(x, t)dx$.

The synchronization function $s(x, t)$ satisfies

$$s_t(x, t) = 0.01 s_{xx}(x, t) - \sin(s(x, t)) + 0.01 s(x, t - 0.5).$$

The boundary conditions and initial value are designed as below:

$$s_x\big|_{x=0} = s_x\big|_{x=1} = 0, s(x, t) = \varphi(x), t \in [-0.5, 0].$$

The coupling matrix G is designed as

$$G = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Let the initial conditions be

$$y_1(x, t) = 0.4\cos \pi x,$$
$$y_2(x, t) = 0.3\sin \pi x,$$
$$y_3(x, t) = 0.3\cos 2\pi x,$$
$$s(x, t) = 0.4, t \in [-0.5, 0].$$

Let $l = 2$, it means that the first two nodes are selected to be controlled by pinning controllers.
We can verify that (13) is satisfied. Fig 1 shows that the second-order PDSs (1) can achieve asymptotic synchronization via pinning control and boundary control.
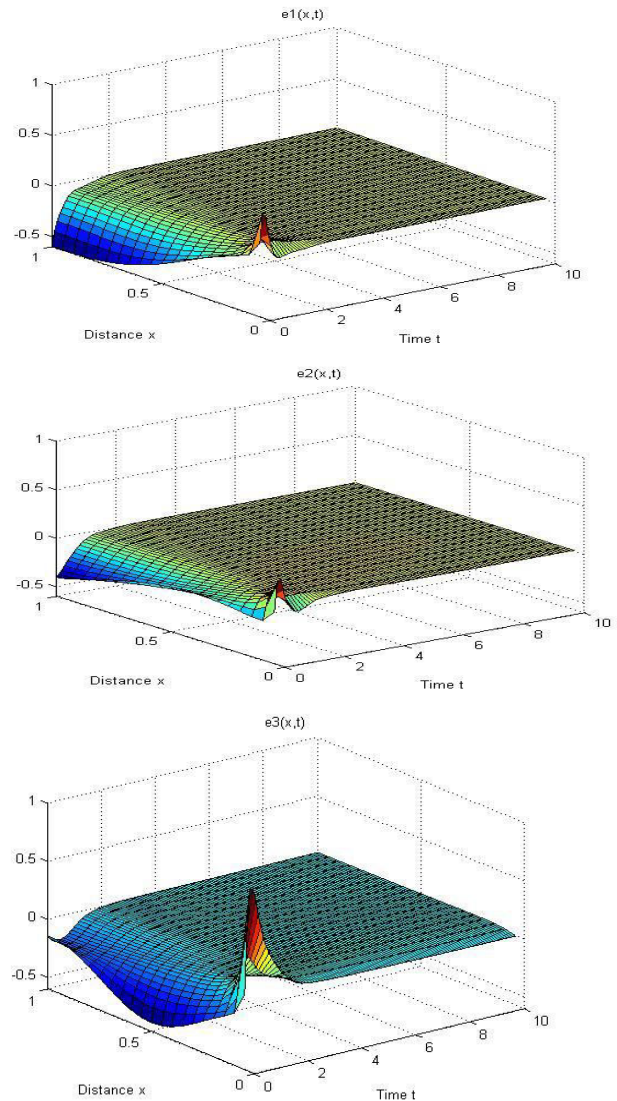

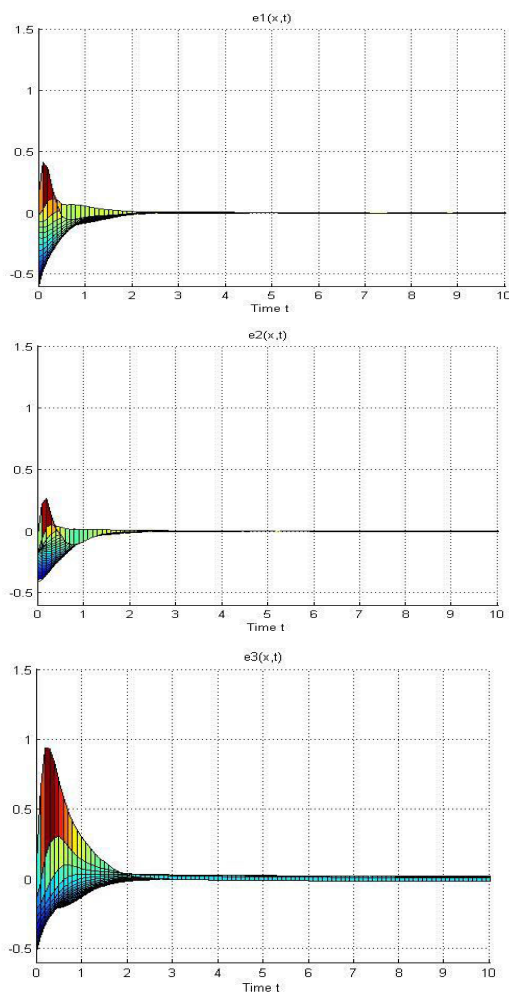Fig1 Closed-loop profiles of evolution of $e_i(x, t)$ for the system (31), i=1,2,3.

Fig2 Time evolution of the synchronization errors

$$e_i(x,t), i = 1,2,3.$$

Remark 7 From the figure 1 and the figure 2 we can see that the synchronization errors of the first two nodes which are controlled by pinning control and boundary control simultaneously can converge to zero faster than the third node which is controlled only by boundary control and the convergence effect of the synchronization errors is superior to that of the third one, which indicates combining pinning control with boundary control is a more efficient control method for the PDSs.

V. CONCLUSION

Asymptotic synchronization of PDSs via pinning control and boundary control is investigated in this paper. Boundary control is a practical control strategy for spatial-temporal systems, like the partial differential systems, which helps to implement performance gains.

On the one hand, only a small number of actuators are required for the boundary of the one-dimensional space domain. On the other hand, the pinning control only control part of the nodes, so the pinning controller and boundary controller designed by us are relatively simple and easy to implement and can ensure the asymptotic synchronization of PDSs.

Boundary controllers have unique advantages because they are different from state feedback controllers and can be designed and implemented when the system state is unknown. Therefore, the study of boundary control has important realistic significance for us.

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Biographies and Photographs
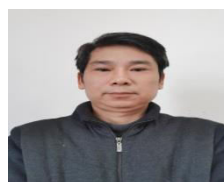
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