The Review of Finance and Banking
Volum e 08, Issu e 2, Year 2016, Pages 055– 060
S print ISSN 2067-2713, online ISSN 2067-3825

DISCRETE PORTFOLIO ADJUSTMENT WITH FIXED TRANSACTION COSTS

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Abstract. This paper presents a closed-form solution to the portfolio adjustment problem in discrete time when an investor faces fixed transaction costs. This transaction cost model assumes a mean-variance investor who wants to adjust her holdings of a risky and risk-free asset. It is shown how this model can be calibrated to be used with a variety of risk models such as life cycle portfolio weights and value at risk (VaR) models. The decision problem can easily be inputted into and calculated in Excel. This paper finds that investors facing lower fixed transaction costs, with higher account balances, and with a greater mismatch between their desired and current allocations will be more eager to rebalance.

1. Introduction[1]

Investors often target portfolio weights, but are faced with the dilemma whether they should undertake costly trades to move their portfolios to that target. Such trading often involves fixed commissions or a fixed expenditure of time that is largely unrelated to the investor's assets under management. The fixed cost of trading assumption ignores issues of market impact endemic to large funds. Thus, this model is most appropriate for small portfolios in more liquid markets where the investor has little or no impact on market prices. This paper attempts to develop a transaction cost model for an investor who faces fixed costs of trading between a risky asset and a safe asset. The model is meant to be easily calibrated to a wide degree of risk models. The investor only needs estimates of the expected returns of the risky asset, its expected volatility, the risk-free rate, the size of her current allocation to the risky asset, her account balance, and transaction costs of trading.

Narang (2009) says that a quantitative approach to investment centers on three different models. First, an alpha model generates investment expected returns and hopefully tells the investor the precision or volatility of those estimates. Second, a risk model determines how much weight a risky asset should have in that investor's portfolio based on some risk-management principles. For example, the drawdown constraint portfolio optimization papers of Cherny and Obłój (2013) and Bonelli and Bossy (2017) represent overlaying a risk model to an alpha model. Lastly, a transaction cost model is used to determine if the costs of trading are exceeded by the benefits of trading.

This paper attempts to propose a simple transaction cost model that can be used in conjunction with a wide variety of alpha and risk models. This paper presents a closed-form solution to calibrate the model to be consistent with risk models as diverse as age based portfolio weights

Received by the editors August 26, 2015. Accepted by the editors July 21, 2017.

Keywords: adjustment costs, brokerage commissions, life cycle funds, portfolio theory, transaction costs, Value at Risk (VaR)..

JEL Classification: G11.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

[1]This is not investment advice.

or value at risk (VaR) models. Moreover, optimal portfolio rebalancing thresholds can be numerically estimated using the goal seek function in Excel.

Markowitz and van Dijk (2003) use continuous time approaches to transaction cost modeling. The Bellman equations of Bellman (1957) becomes quickly computationally impossible as one adds state variables to the mix, according to Markowitz and van Dijk (2003). In other words, a complex alpha model may make for a computationally impossible transaction cost model. Examples of Bellman equation modeling of portfolio selection with transaction costs are too numerous to justice to here. Early examples are Constanides (1979) and Zabel (1973) and a more recent example is Sun et al. (2006). Markovitz and van Dijk (2003) attempt to simplify the transaction cost problem using discrete portfolio weights of 10, 20, 30, .., 100 percent and Markov chains. That approach is much more computationally complex than the present paper, and the present paper does not need to resort to a limited set of 11 possible portfolio weights as in Markovitz and van Dijk (2003).

Overtrading is widely recognized to substantially eat up returns. Men and women were found to have lost 2.65 and 1.72 percent per annum, respectively, from excessively trading their discount brokerage accounts, according to Barber and Odean (2001). A tractable model of transaction costs, which can be put into a spreadsheet, may help many investors increase their returns.

2. The mean-variance investor's problem

Since Markovitz (1952), financial theory has focused on selecting portfolios with high returns and low variance. Von Neumann and Morgenstern (1947) introduced the concept of expected utility maximization. Freund (1956) proved that a consumer with exponential utility, facing normally distributed outcomes, will maximize expected utility with the function below:

$$V(w_t) = E\{r_{p,t}\} - \frac{1}{2}\gamma\sigma_{p,t}^2 = w_t\bar{r}_t + (1-w_t)r_f - \frac{1}{2}\gamma w_t^2\sigma_t^2 \tag{2.1}$$

$V(w_t)$ is the expected utility function of the investor based on its allocation to the risky asset, w_t, at time t. $r_{p,t}$ is the portfolio return at time t and $\sigma_{p,t}$ is the portfolio volatility at time t.

We assume that investor allocates money between the risky asset and the risk-free asset. Let $\bar{r}_t$ be the expected return to the risky asset at time t. σ_t is the volatility of that risky asset. rf is the return to the risk-free asset at time t. γ is a measure of the investor's risk-aversion. Indeed, $\gamma > 0$, it is the Arrow (1965) and Pratt (1964) measure of absolute risk aversion, $-U''(c)/U'(c)$, for this investor. (The expected utility equation (2.1) was generated in Freund (1956) from a utility function $U(c) = 1 - exp(\lambda c)$, where c is consumption.)

Optimizing this investor's expected utility leads to the following optimal weight in the risky asset:

$$w_t^* = \frac{\bar{r}_t - r_f}{\gamma\sigma_t^2} \tag{2.2}$$

Given $\bar{r}_t - r_f > 0$ we can assume that the investor will have some positive weight to the risky asset.

The investor has a risky portfolio weight at time $t-1$ of w_{t-1}. If transaction costs were zero, the investor would move to her optimal allocation at time t of w_t^*. The investor has a fixed transaction cost of adjusting its portfolio of t. The investor has assets of b at time t. Thus, reallocating the portfolio reduces the investor's returns by t/b. The transaction cost, t, is risk-free so it reduces the expected return when scaled by investment assets, but it does not affect the variance.

The mean-variance investor will only reallocate if the following is true:

$$
\begin{aligned}
&V(w_t^*) - \tfrac{\tau}{b} - V(w_{t-1}) = \\
&w_t^* \bar{r}_t + (1 - w_t^*) r_f - \tfrac{1}{2}\gamma w_t^{*2} \sigma_t^2 - \tfrac{\tau}{b} \\
&-w_{t-1}\bar{r}_t - (1 - w_{t-1}) r_f + \tfrac{1}{2}\gamma w_{t-1}^2 \sigma_t^2 \geq 0
\end{aligned} \tag{2.3}
$$

The inequality simplifies to the following with a little algebra:

$$
\begin{aligned}
&V(w_t^*) - \tfrac{\tau}{b} - V(w_{t-1}) = \\
&(w_t^* - w_{t-1})[\bar{r}_t - r_f - \tfrac{1}{2}\gamma\sigma_t^2(w_t^* + w_{t-1})] - \tfrac{\tau}{b} \geq 0
\end{aligned} \tag{2.4}
$$

If the inequality is satisfied, the investor reallocates at time t, otherwise she does not. If we substitute in w_t^*, then, with some algebra, the inequality that must be satisfied for the investor to trade is as follows:

$$
\pi \equiv V(w_t^*) - \frac{\tau}{b} - V(w_{t-1}) = \frac{1}{2\gamma\sigma_t^2}[\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2]^2 - \frac{\tau}{b} \geq 0 \tag{2.5}
$$

Proposition 1. *The mean-variance investor will be more eager to readjust her portfolio at time t if :*

1. The fixed transaction costs of trading, τ, are smaller.

2. The balance of her assets, b, is greater.

The proof of part a) also follows from differentiating the left-hand side of the constraint.

$$
\frac{d\pi}{d\tau} = -\frac{1}{b} < 0 \tag{2.6}
$$

Thus, an increase in the fixed cost of trading makes it harder to satisfy the constraint.

In a similar fashion, we can prove part b):

$$
\frac{d\pi}{db} = \frac{\tau}{b^2} > 0 \tag{2.7}
$$

When the investor has a greater account balance, adjustments become more profitable, relative to the fixed cost of trading. Q.E.D.

It is clear, that the trading constraint relaxes linearly as transaction costs decline, but the constraint is progressively less affected by larger account balances.

$$
\frac{d^2\pi}{d\tau^2} = 0,\ \frac{d^2\pi}{db^2} = -\frac{2\tau}{b^3} < 0 \tag{2.8}
$$

Proposition 2. *When the investor is increasing the allocation to the risky asset, $w_t^* > w_{t-1}$,*

1. A higher expected return to the risky asset in excess of the risk-free rate, $\bar{r}_t - r_f$, will make the investor more eager to reallocate at time t.

2. A higher parameter of risk aversion, g, decreases the investor's willingness to increase the holdings of the risky asset at time t.

3. A higher variance of the risky asset makes the investor less eager to increase the holdings of the risky asset at time t.

The quantity in the square brackets on the left-hand side of the constraint can be expressed as the following:

$$
\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2 = \gamma\sigma_t^2\left(\frac{\bar{r}_t - r_f}{\gamma\sigma_t^2} - w_{t-1}\right) = \gamma\sigma_t^2(w_t^* - w_{t-1}) \tag{2.9}
$$

When the investor wants to increase the weight of the risky asset, $\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2 > 0$. It is clear that this constraint is more easily satisfied when the gap between the expected return and the risk-free return is greater.

$$\frac{d\pi}{d(\bar{r}_t - r_f)} = \frac{1}{\gamma\sigma_t^2}[\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2] > 0, \text{ when } w_t^* > w_{t-1} \tag{2.10}$$

That proves part a) of proposition 2. Q.E.D.

The first derivative of p with respect to the coefficient of risk aversion, g, shows that a higher risk aversion parameter makes the investor less eager to trade when the optimal allocation calls for an increase in the holdings of the risky asset.

$$\frac{d\pi}{d\gamma} = -\frac{1}{2\gamma^2\sigma_t^2}[\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2]^2 - \frac{w_{t-1}}{\gamma}[\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2] < 0, \text{ when } w_t^* > w_{t-1} \tag{2.11}$$

The quantity in square brackets is positive because $\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2 = \gamma\sigma_t^2(w_t^* - w_{t-1})$, and $w_t^* > w_{t-1}$. When risk aversion is greater, the investor is less inclined to increase her holdings of the risky asset in the face of fixed transaction costs. That proves b) of the proposition 2. *Q.E.D.*

$$\frac{d\pi}{d\sigma_t^2} = -\frac{1}{2\gamma\sigma_t^4}[\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2]^2 - \frac{w_{t-1}}{\sigma_t^2}[\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2] < 0, \text{ when } w_t^* > w_{t-1} \tag{2.12}$$

The first derivative with respect to the risky assets variance proves part c) of the proposition 2. Q.E.D.

Intuitively, a greater variance of the risky asset discourages the variance-averse investor from increasing risky asset holdings.

Proposition 3. *When the investor is decreasing the allocation to the risky asset, $w_t^* < w_{t-1}$, a lower expected return to the risky asset in excess of the risk-free rate, $\bar{r}_t - r_f$, will make the investor more eager to reallocate at time t.*

When the investor wants to decrease the weight of the risky asset, $\bar{r}_t - r_f - w_{t-1}\gamma\sigma_t^2 < 0$. That means that we will have the opposite sign for the first derivative in (2.13). That proves proposition 3. Q.E.D.

Proposition 4. *The greater the squared difference between the desired allocation and the previous allocation, $(w_t^* - w_{t-1})^2$, the more the investor desires to move to a new allocation of the risky asset at time t.*

By substituting in the right hand side of equation (2.9) into equation (2.5), it is easy to show that the left hand side of the constraint can be expressed as the following:

$$\pi = \frac{\gamma\sigma_t^2}{2}[w_t^* - w_{t-1}]^2 - \frac{\tau}{b} \tag{2.13}$$

Thus,

$$\frac{d\pi}{d(w_t^* - w_{t-1})^2} = \gamma\sigma_t^2 > 0 \tag{2.14}$$

Intuitively, the investor is under more impetus to reallocate when her previous allocation to the risky asset, w_{t-1}, is more distant from her desired allocation, w_t^*. *Q.E.D.*

3. Calibrating the coefficient of risk aversion to various risk models

The biggest stumbling block to implementing such a model is determining the investor's coefficient of risk aversion. Risk models that specify a portfolio allocation can be used to determine the coefficient of risk aversion, γ, according to the utility function in equation (2.1). It is unlikely and not necessary that the risk model used will employ the simple mean-variance expected utility function in equation (2.1). (What is proposed in this paper is a tractable transaction cost model not a risk model.) In that case, w_t^* is likely to be different than is specified in equation (2.2), and that equation can be inverted to solve for a reasonable coefficient of risk aversion to evaluate the inequality in equation (2.4). If the risky asset target, w_t^*, the expected risk premium, $\bar{r}_t - r_f$, and the variance of the risky asset, σ_t^2, are estimated, we can back out a coefficient of risk aversion from equation (2.2).

$$\gamma = \frac{\bar{r}_t - r_f}{w_t^* \sigma_t^2} \tag{3.1}$$

In the examples that follow, the investors will know from their risk models and use that to solve for their coefficients of risk aversion in the transaction cost model.

3.1. **Life cycle portfolio allocations.** Large mutual fund providers often offer life cycle funds which reduce an investor's holdings of stocks as they age and increase their holdings of bonds. Many financial planners make similar recommendations to their clients.

Suppose that a financial planner tells his clients that 50-years-olds should hold 50 percent stocks and 50 percent bonds, and that 30-years-olds should hold 70 percent stocks and 30 percent bonds. What is the implied coefficient of risk aversion if $\sigma_t = 18$ percent and $\bar{r}_t - r_f =$ 7.5percent? The 50-year-old's coefficient of risk aversion is 4.630, and the 30-year-old has a coefficient of risk aversion of 3.307, according to equation (3.1).

Suppose that the stock market has been rising. The 50-year-old and the 30-year-old value their leisure at $30 per hour and it takes 20 minutes to rebalance. They are both trying to follow the advice of their financial planner when managing their retirement funds. The young man has an account balance of $100,000, and the older man has an account balance of $400,000. How high must their stock allocations rise above their target before they rebalance at a cost of $10 of leisure, assuming there are no monetary trading costs? Solving for the initial stock allocation, w_{t-1}, that makes the net benefit of adjusting the portfolio zero for these parameter values is 51.8 percent for the older man, and 73.6 percent for the younger man. The 300 percent larger account balance will make the older man rebalance more frequently with a rising stock market.

3.2. **Value at risk (VaR) type metrics.** Value at risk (VaR) models attempt to manage portfolios so that a certain dollar loss will only happen with a pre-specified probability. Suppose that a portfolio manager running $10 million only wants to lose 3 percent, $300,000, in a given month with a 5 percent probability. The risky asset is found to have a monthly volatility of 6 percent and an expected monthly return of 2 percent. The risk-free rate is 0.1 percent per month. Assuming the returns are normally distributed, then the manager will want to hold $w_t^* = 38.90$ percent of assets in the risky asset and the rest in cash based on his VaR rule. Inserting, these parameters into equation (3.1), the implied coefficient of risk aversion is 13.57. The portfolio manager faces a transaction cost of $5 including her time and trading commissions. If the allocation of the risky asset is too low, the portfolio manager will rebalance only if the risky asset allocation falls below 38.45 percent.

4. Conclusion

This paper has developed an easy-to-apply transaction cost model for an investor that faces a fixed cost of adjusting her portfolio between a risky and risk-free asset. This model is most appropriate for investors who will have little or no impact on market prices. The paper assumes mean-variance preferences to generate closed-form solutions to the transaction cost problem. The transaction cost model can be used in conjunction with a variety of risk models by using the desired portfolio weight of the risky asset, from the alpha and risk models to estimate a coefficient of absolute risk aversion for the transaction cost model. The investor just needs a forecasted mean return and variance projection for the risky asset to apply the transaction cost model in this paper.

There are several insights developed from this model. An investor will be more motivated to adjust her portfolio to her target if the fixed cost of adjustment is lower, her account balance is larger, and her current portfolio weight is distant from her target portfolio weight to the risky asset. In addition, when the investor wants to increase her weight in the risky asset, she will be more inclined to do so when the expected risk premium is larger, her coefficient of absolute risk aversion is lower, and the variance of the risky asset is lower. Finally, when she wants to decrease her holdings of the risky asset she will be more eager to do so when the expected risk premium is lower.

Appendix: Key to the Mathematical Notation

$V(w_t)$ = the expected utility function of the investor based on its allocation to the risky asset, w_t, at time t.

$r_{p,t}$ = the portfolio return at time t

$s_{p,t}$ = the portfolio volatility at time t

$\bar{r}_t$ = expected return of the risky asset at time t

r_f = risk-free asset's rate of return at time t

w_t = the weight of the risky asset at time t

γ = coefficient of absolute risk aversion

σ_t = standard deviation of the risky asset at time t

w_t^* = the optimal weight of the risky asset at time t

b = account balance

t = the fixed transaction cost of reallocating the portfolio

$w_{t \;\; 1}$ = the weight of the risky asset in the previous period

π = net benefit of trading

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