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Fractional Supersymmetric iso(1,1)

Kesirsel Süpersimetrik iso(1,1)

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ABSTRACT: In this study, fractional supersymmetric iso(1,1) based on the permutation groups S_3, is obtained in the Hopf algebra formulation. This algebra is denoted by $U_3^2(iso(1,1))$.

Key words: Poincaré, Fractional supersymmetric, Superalgebra, Semidirect product.

Jel Classifications: C02, C60, C10

ÖZ: *Bu çalışmada, S_3 permütasyon grupları üzerine kurulmuş kesirsel süpersimetrik iso(1,1) cebri, Hopf cebri formülasyonunda elde edilmiştir. Bu cebir $U_3^2(iso(1,1))$ ile gösterilmiştir.*

Anahtar Kelimeler: *Poincaré, Kesirselsüpersimetrik, Süpercebir, Yarıdirekt çarpım.*

1. Introduction

Lie algebras, Lie groups and their representations are very important in mathematical physics and engineering literature (Wang, Han, Yu, Zheng, 2012; Vilenkin, Klimyk, 1991; De Witt, 1992; Kostant, 1997). In these studies, we can see the applications of the symmetries. Supersymmetry has been a popular work area for nearly 27 years. The supersymmetries are associated with Z_2 –graded algebra (or S_2-graded algebra) where θ is a grassmann number which satisfies $\theta = \overline{\theta}, \ \theta^2 = 0$ (De Witt, 1992; Kostant, 1997). Fractional supersymmetric algebras are generalized form of supersymmetric Lie algebras. Fractional supersymmetric algebras are associated with Z_n–graded algebra (or S_n-graded algebra) where $\theta = \overline{\theta}, \ \theta^n = 0, n = 3,4, \ldots$. There are lots of generalizations of fractional supersymmetric algebras (Raush deTraunbenberg, Slupinski, 1997; Kerner, 1992; Ahmedov, Dayi, 1999; Ahn, Bernard, Leclair, 1990; Ahmedov, Dayi, 2000; Ahmedov, Dayi, 1999; Ahmedov, Yildiz, Ucan, 2001).

In this study, using the method of Ahmedov, Yildiz and Ucan, 2001, we obtain fractional supersymmetric iso(1,1) algebra. For this purpose, after the overview of fractional supersymmetric algebra in section-2, we give fractional supersymmetric iso(1,1) algebra denoted by $U_3^2(iso(1,1))$ in section-3.

2. On Fractional Superalgebras

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Let U(g) be the universal enveloping algebra of a Lie algebra g generated by X_j, j=1,2,…,dim(g) with

$$[X_i, X_j] = \sum_{k=1}^{\dim(g)} c_{ij}^k X_k \tag{1}$$

where c_{ij}^k are the structure constants of the Lie algebra g. The Hopf algebra structure of U(g) is given by the co-multiplication $\Delta : U(g) \to U(g) \otimes U(g)$, co-unit $\varepsilon : U(g) \to C$ and antipode $S : U(g) \to U(g)$

$$\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j \ , \quad \varepsilon(X_j) = 0 \ , \quad S(X_j) = -X_j \tag{2}$$

We can extend the Hopf algebra U(g) by adding elements $Q_\alpha, \alpha = 1, ..., N$ and K with relations

$$\{Q_\alpha Q_\beta, Q_\gamma\} = b_{\alpha\beta\gamma}^j X_j \tag{3}$$

$$[Q_\alpha, X_j] = a_{\alpha\beta}^j Q_\beta \tag{4}$$

and

$$KQ_\alpha = qQ_\alpha K \ , \ q^3 = 1 \ , \ K^3 = 1. \tag{5}$$

where

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = Q_\alpha \{Q_\beta, Q_\gamma\} + Q_\beta \{Q_\alpha, Q_\gamma\} + Q_\gamma \{Q_\alpha, Q_\beta\} \tag{6}$$

is the S_3 invariant form. This algebra, which we denote by $U_3^N(g)$, can also be equipped with a Hopf algebra structure by defining

$$\Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha \ , \quad \Delta(K) = K \otimes K \tag{7}$$

$$\varepsilon(Q_j) = 0 \ , \ \varepsilon(K) = 1 \tag{8}$$

$$S(Q_j) = -K^2 Q_j \ , \ S(K) = K^2 \tag{9}$$

For structure constants $b_{\alpha\beta\gamma}^j$ and $a_{\alpha\beta}^j$, we have to derive identities involving the commutator and S_3 invariant form.

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0 \tag{10}$$

$$[A,\{B,C,D\}] + \{[B,A],C,D\} + \{B,[C,A],D\} + \{B,C,[D,A]\} = 0 \tag{11}$$

and

$$[A,\{B,C,D\}] + [B,\{A,C,D\}] + [C,\{B,A,D\}] + [D,\{B,C,A\}] = 0 \tag{12}$$

The relation (10) is the usual Jacobi identity. Inserting

$$A = X_i \qquad B = X_j \qquad C = Q_\alpha \tag{13}$$

into (10) and using (4) and (1) we get

$$\sum_{\sigma=1}^{N} \left(a_{\alpha\sigma}^i a_{\sigma\beta}^j - a_{\alpha\sigma}^j a_{\sigma\beta}^i \right) = \sum_{k=1}^{\dim(g)} c_{ij}^k a_{\alpha\beta}^k \tag{14}$$

It is obtained the following relations from (11-12) and restrictions on the structure constants as given in Ahmedov, Yildiz, Ucan, 2001.

$$\sum_{\sigma=1}^{N}\left(a^k_{\alpha\sigma}b^i_{\sigma\beta\gamma}+a^k_{\beta\sigma}b^i_{\sigma\alpha\gamma}+a^k_{\gamma\sigma}b^i_{\sigma\beta\alpha}\right)=\sum_{j=1}^{\dim(g)}c^i_{jk}b^j_{\alpha\beta\gamma} \qquad (15)$$

$$\sum_{k=1}^{\dim(g)}\left(b^k_{\alpha\beta\gamma}a^k_{\sigma\tau}+b^k_{\sigma\alpha\beta}a^k_{\gamma\tau}+b^k_{\gamma\sigma\alpha}a^k_{\beta\tau}+b^k_{\beta\gamma\sigma}a^k_{\alpha\tau}\right)=0 \qquad (16)$$

3. Fractional Supersymmetric iso(1,1)

ISO(1,1) group is given as $ISO(1,1) = SO(1,1) \rtimes T^2$ where SO(1,1) group is matrix group of transforms preserving invariant of $x_1^2 - x_2^2 = 1$ quadratic form and T^2 is translation group in $\mathbb{R}^2$ space (Vilenkin, Klimyk, 1991).

iso(1,1) algebra of commutation relations are given as follows;

$$[P_+, H] = P_+ \,,\quad [P_-, H] = -P_- \,,\quad [P_+, P_-] = 0 \qquad (17)$$

We use X_1 , X_2 and X_3 representations instead of P_+, P_- and H respectively. From the commutation relations, we have

$$[X_1, X_2] = 0 \qquad [X_1, X_3] = X_1 \qquad [X_2, X_3] = -X_2 \qquad (18)$$

For the algebra iso(1,1), we have the following structure constants

$$C^1_{13} = 1,\ C^2_{23} = -1 \qquad (19)$$

Here we consider $N = 2$ fractional super generalization of iso(1,1) at $n = 3$ (that is $q^3 = 1$)

From the relation (14) we have

$$a^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad a^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad a^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The condition (15) and (16) imply

$$b^1_{111} = 3b^1_{112} = 3b^2_{112} = -b^2_{111}$$

$$b^1_{222} = 3b^1_{122} = -3b^2_{122} = b^2_{222}$$

$$b^1_{112} = -b^1_{122}$$

$$b^2_{112} = b^2_{122}$$

with all other structure cofficients $b^j_{\alpha\beta\gamma}$ being zero. Choosing that

$b^1_{111} = 1$ we get the fractional super algebra given by

$$[Q_1, X_1] = 0 \qquad [Q_2, X_1] = 0 \qquad [Q_1, X_3] = Q_2$$

$$[Q_1, X_2] = 0 \qquad [Q_2, X_2] = 0 \qquad [Q_2, X_3] = Q_1$$

$$\{Q_1, Q_1, Q_1\} = X_1 - X_2 \qquad \{Q_1, Q_1, Q_2\} = \frac{1}{3}X_1 + \frac{1}{3}X_2$$

$$\{Q_2, Q_2, Q_2\} = -X_1 - X_2 \qquad \{Q_1, Q_2, Q_2\} = -\frac{1}{3}X_1 + \frac{1}{3}X_2$$

4. Conclusion

Two dimensional Fractional Supersymmetric iso(1,1) algebra is obtained by using the method in Ahmedov, Yildiz and Ucan, 2001. It can be seen that the results obtained here are consistent with the results of Raush deTraunbenberg, 2004 and Goze, Raush deTraunbenberg and Tanasa, 2007.

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