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Research Article

On Matrix Division and Rational Matrices

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Abstract In this paper we give, for the first time, the definitions of "co-divide" and division of two matrices with the same dimension. We will see that the definitions here are analogous to the case in real numbers. Here we research their properties and give some examples. Rational matrices was define by "Matrix Division". We will see that the properties here are analogous to the case in real numbers. Here we research their properties and give some examples.

Keywords matrix, division, matrices division, co-divide, rational matrices
Subject Classification: 05B20, 15A09, 15A16, 15A24, 81U20

1. Introduction

Let us start with the definition as follows.

Definition 1.1. Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ be any two square matrices. $1 \le i \le n$, then ${}^{A}_{B}i_j$ denote the determinant of the matrix B_i obtained by putting j^{th} column of A in place of i^{th} column of B. The real number ${}^{A}_{B}i_j$ is called a co-divide of A on B.

We easily get

Lemma 1.1. Let A and B be any two square matrices of dimension n. Then the number of codivides of A on B is n^2.

Example 1.1. If $A = [1], B = [3]$.Thentheco-divides ${}^{A}_{B}1_1$ and ${}^{B}_{A}1_1$ are 1 and 3 respectively.

Theorem 1.1. Let $\mathcal{M}(\mathbb{R}) = \left\{ [a_{ij}]_{n\times n} \middle| a_{ij} \in \mathbb{R}, i,j = 1,\dots,n, n \in \mathbb{Z}^+ \right\}$, and $C_{ij}(A) = (-1)^{i+j} M_{ij}(A)$, where $M_{ij}(A)$ is the minor of a_{ij}. If $A \in \mathcal{M}(\mathbb{R})$ with $|A| \ne 0$, let I be unit matrix. Then

i. ${}^{I}_{A}i_j = C_{ij}(A)$.

ii. ${}^{A}_{I}i_j = a_{ij}$, for $j = 1,\dots,n$.

iii. ${}^{A}_{A}i_j = \begin{cases} |A|, i = j \\ 0, i \ne j \end{cases}$

Proof. *i)* Let $A, I \in \mathcal{M}(\mathbb{R})$ and $|A| \ne 0$. Then

$$ {}^{I}_{A}1_1 = \begin{vmatrix} 1 & a_{12} & \dots a_{1n} \\ 0 & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \vdots \\ 0 & a_{n2} & \dots a_{nn} \end{vmatrix} = \begin{vmatrix} a_{12} & \dots & a_{1n} \\ a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix} = C_{11}(A), $$

$$ {}^{I}_{A}1_2 = \begin{vmatrix} a_{11} & 1 & a_{13} & \dots a_{1n} \\ a_{21} & 0 & a_{23} & \dots a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & 0 & a_{n3} & \dots a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}\, a_{13} \dots a_{1(n-1)}\, a_{1n} \\ a_{21}\, a_{23} \dots a_{2(n-1)}\, a_{2n} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}\, a_{n3} \dots a_{n(n-1)}\, a_{nn} \end{vmatrix} = C_{12}(A) $$

$$ \vdots $$

$$ {}^{I}_{A}1_n = \begin{vmatrix} a_{11} \dots a_{1(n-1)} & 1 \\ a_{21} \dots a_{2(n-1)} & 0 \\ \vdots \quad \vdots \quad \vdots & \vdots \\ a_{n1} \cdots a_{n(n-1)} & 0 \end{vmatrix} = \begin{vmatrix} a_{11} \dots a_{1(n-1)} \\ a_{21} \dots a_{2(n-1)} \\ \vdots \quad \vdots \quad \vdots \\ a_{n1} \cdots a_{n(n-1)} \end{vmatrix} = C_{1n}(A), $$

$$ \vdots $$

$$ {}^{I}_{A}n_1 = \begin{vmatrix} 0 & a_{12} \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a_{22} \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n2} \dots & a_{n(n-1)} & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{12} \cdots & a_{1(n-1)} & a_{1n} \\ a_{22} \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n2} \dots & a_{n(n-1)} & a_{nn} \end{vmatrix} = C_{n1}(A) $$

$$ \vdots $$

$$ {}^{I}_{A}n_n = \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1(n-1)} & 0 \\ a_{21} & a_{22} \cdots & a_{2(n-1)} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{n(n-1)} & 1 \end{vmatrix} = \begin{vmatrix} a_{11} \dots a_{1(n-1)} \\ a_{21} \dots a_{2(n-1)} \\ \vdots \quad \vdots \quad \vdots \\ a_{n1} \cdots a_{n(n-1)} \end{vmatrix} = C_{nn}(A). $$

ii)

$$ {}^{A}_{I}1_1 = \begin{vmatrix} a_{11} & 0 \cdots 0 \\ a_{21} & 1 \cdots 0 \\ \vdots & \vdots \vdots \vdots \\ a_{n1} & 0 \cdots 1 \end{vmatrix} = a_{11}, \dots, {}^{A}_{I}1_n = \begin{vmatrix} 1 0 \cdots a_{11} \\ 0 1 \cdots a_{21} \\ \vdots \vdots \vdots \quad \vdots \\ 0 0 \cdots a_{n1} \end{vmatrix} = a_{n1}, \dots, {}^{A}_{I}n_1 = \begin{vmatrix} a_{1n} & 0 \cdots 0 \\ a_{2n} & 1 \cdots 0 \\ \vdots & \vdots \vdots \vdots \\ a_{nn} & 0 \cdots 1 \end{vmatrix} = a_{1n}, \dots, {}^{A}_{I}n_n = \begin{vmatrix} 1 0 \cdots a_{1n} \\ 0 1 \cdots a_{2n} \\ \vdots \vdots \vdots \quad \vdots \\ 0 0 \cdots a_{nn} \end{vmatrix} = a_{nn}. $$

Therefore,

$$ {}^{A}_{I}i_j = a_{ij}. $$

iii.) Let$\neq j$, then we have two equal columns, so ${}^{A}_{A}i_j$ is zero and also ${}^{A}_{A}i_j = |A|$ for $i = j$.

Note 1.1. We notice that

$$ \begin{bmatrix} {}^{I}_{A}1_1 \, {}^{I}_{A}1_2 \dots {}^{I}_{A}1_n \\ {}^{I}_{A}2_1 \, {}^{I}_{A}2_2 \cdots {}^{I}_{A}2_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ {}^{I}_{A}n_1 \, {}^{I}_{A}n_2 \cdots {}^{I}_{A}n_n \end{bmatrix} = \begin{bmatrix} C_{11}(A) \, C_{12}(A) \dots C_{1n}(A) \\ C_{21}(A) \, C_{22}(A) \cdots C_{2n}(A) \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ C_{n1}(A) \, C_{n2}(A) \cdots C_{nn}(A) \end{bmatrix}. $$

Example 1.2. Let $A = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$.Thenall of co-dividesare ${}^{B}_{A}1_1 = \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} = -6$, ${}^{B}_{A}2_1 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$, ${}^{B}_{A}1_2 = \begin{vmatrix} 6 & 5 \\ 8 & 7 \end{vmatrix} = 7$, ${}^{B}_{A}2_2 = \begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix} = -10$, ${}^{A}_{B}1_1 = \begin{bmatrix} 1 & 6 \\ 3 & 8 \end{bmatrix} = -10$, ${}^{A}_{B}2_1 = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = 2$, ${}^{A}_{B}1_2 = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = -2$, ${}^{A}_{B}2_2 = \begin{bmatrix} 2 & 5 \\ 4 & 7 \end{bmatrix} = -6$.

Matrices of co-divides of A on B (B on A resp.) and B are

$$ C = \begin{bmatrix} -6 & -2 \\ 7 & -10 \end{bmatrix} \left(D = \begin{bmatrix} -10 & 2 \\ -2 & -6 \end{bmatrix} \text{ resp.}\right). $$

Lemma 1.2. Let $A, B \in \mathcal{M}(\mathbb{R})$ and $k \in \mathbb{R} \setminus \{0\}$ then

i. ${}^{(kA)}_{B}i_j = k\left({}^{A}_{B}i_j\right)$.

ii. ${}^{A}_{(kB)}i_j = k^{(n-1)}\left({}^{A}_{B}i_j\right)$.

Proof. Let$A, B \in \mathcal{M}(\mathbb{R})$ and $k \in \mathbb{R} \setminus \{0\}$, then

i) As any entry of kA is a product of an element of A by k, we conclude that

${}^{(kA)}_{B}i_j = k\left({}^{A}_{B}i_j\right)$. And we can conclude that

$$ \text{ii) } {}^{A}_{(kB)}i_j = \begin{vmatrix} kb_{11} & \cdots & kb_{1(j-1)} & a_{1j} & kb_{1(j+1)} & \cdots & kb_{1n} \\ kb_{21} & \cdots & kb_{2(j-1)} & a_{2j} & kb_{2(j+1)} & \cdots & kb_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ kb_{n1} & \cdots & kb_{n(j-1)} & a_{nj} & kb_{n(j+1)} & \cdots & kb_{nn} \end{vmatrix} = k^{(n-1)} \sum_{i=1}^{n} a_{ij} C_{ji}(B) = k^{(n-1)}\left({}^{A}_{B}i_j\right). $$

Corollary 1.1. If $k = 0$ then ${}^{(0A)}_{B}i_j = 0\left({}^{A}_{B}i_j\right) = 0$.

Lemma 1.3. Let $A, B, C \in \mathcal{M}(\mathbb{R}), |C| \neq 0$. Then

i. $\left[{}^{A}_{B}i_j\right] = \frac{1}{|C|}\left[{}^{C}_{B}i_j\right]\left[{}^{A}_{C}i_j\right]$.

ii. $\left[{}^{A}_{B}i_j\right] = A\left[{}^{I}_{B}i_j\right]$.

We now give, the following new "Matrix Division" definition .

2. **Matrix Division**

Definition 2.1. Let A and $B \in \mathcal{M}(\mathbb{R})$ with $|B| \neq 0$. Then the matrix $\left[\frac{\left({}^{A}_{B}i_j\right)_{ji}}{|B|}\right]_{n\times n}$ is called the division of A by B and denoted by $A \div B$ or $\frac{A}{B}$.

Example2.1. Let $A = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$. Then

$$A \div B = \begin{bmatrix} \frac{\begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix}}{-8} & \frac{\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}}{-8} \\ \frac{\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}}{-8} & \frac{\begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix}}{-8} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}. \text{ And } B \div A = \begin{bmatrix} \frac{\begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix}}{-8} & \frac{\begin{vmatrix} 6 & 5 \\ 8 & 7 \end{vmatrix}}{-8} \\ \frac{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}}{-8} & \frac{\begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix}}{-8} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{bmatrix}.$$

Note2.1. In the above example we see that $B(A \div B) = A$. In fact,

$$\begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} = A.$$

Note2.2. From the above we have

$\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{5}{4} \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix} = B$. It is obvious that, in general,

$$A \div B \neq B \div A.$$

Example2.2. Let$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$.

Then$|B| = \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1$ and $|C| = \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} = -4$. So

$$(A \div B) \div C = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} \end{bmatrix} \div \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -8 \\ 7 & 11 \end{bmatrix} \div \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\begin{vmatrix} -5 & 5 \\ 7 & 1 \end{vmatrix}}{-4} & \frac{\begin{vmatrix} -8 & 5 \\ 11 & 1 \end{vmatrix}}{-4} \\ \frac{\begin{vmatrix} 1 & -5 \\ 1 & 7 \end{vmatrix}}{-4} & \frac{\begin{vmatrix} 1 & -8 \\ 1 & 11 \end{vmatrix}}{-4} \end{bmatrix}$$

$$= \begin{bmatrix} 10 & \frac{63}{4} \\ -3 & -\frac{19}{4} \end{bmatrix}. \text{ And}$$

$$A \div (B \div C) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \div \begin{bmatrix} \frac{\begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix}}{-4} & \frac{\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}}{-4} \\ \frac{\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}}{-4} & \frac{\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}}{-4} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \div \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}. \text{ Also,}$$

$(A \div B) \div C \neq A \div (B \div C)$. Consequently, the commutative and associative laws do not hold in general for division.

Example2.3. Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 0 \\ 0 & 2 & 2 \\ 3 & 3 & 0 \end{bmatrix}, C = \begin{bmatrix} -4 & -1 & 0 \\ 0 & -1 & -1 \\ -2 & -2 & 1 \end{bmatrix}$. Then

$$A \div (B + C) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ 0 & -2 & 0 \end{bmatrix},$$

$$(A \div B) + (A \div C) = \begin{bmatrix} \frac{1}{90} & -\frac{1}{45} & -\frac{2}{9} \\ -\frac{17}{45} & -\frac{11}{45} & \frac{5}{9} \\ -\frac{11}{90} & \frac{11}{45} & -\frac{1}{18} \end{bmatrix}.$$

$A \div (B + C)$ is different from $(A \div B) + (A \div C)$. Then distributive property is not hold for division either.

Theorem 2.1. Let$A \in \mathcal{M}(\mathbb{R})$ and $|A| \neq 0$. Then $\frac{A}{A} = I$.

Proof. It is clearly from theorem 1(iii.) that

$$(A \div A) = \begin{bmatrix} \frac{|A|}{|A|} & 0 \cdots 0 & 0 \\ \vdots & \ddots & \vdots \\ \frac{0}{|A|} & 0 \cdots 0 & \frac{|A|}{|A|} \end{bmatrix} = \begin{bmatrix} 1 & 0 \cdots 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 \cdots 0 & 1 \end{bmatrix} = I.$$

Theorem 2.2. Let$A \in \mathcal{M}(\mathbb{R})$ and $|A| \neq 0$. Then $\frac{I}{A} = \frac{adj\,(A)}{|A|}$.

Proof. From theorem1.1.(i) we have

$$(I \div A) = \begin{bmatrix} \frac{\frac{I}{A}1_1}{|A|} & \frac{\frac{I}{A}1_2}{|A|} & \cdots & \frac{\frac{I}{A}1_n}{|A|} \\ \frac{\frac{I}{A}2_1}{|A|} & \frac{\frac{I}{A}2_2}{|A|} & \cdots & \frac{\frac{I}{A}2_n}{|A|} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\frac{I}{A}n_1}{|A|} & \frac{\frac{I}{A}n_2}{|A|} & & \frac{\frac{I}{A}n_n}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11}(A) & C_{21}(A) & \cdots & C_{n1}(A) \\ C_{12}(A) & C_{22}(A) & \cdots & C_{n2}(A) \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n}(A) & C_{2n}(A) & \cdots & C_{nn}(A) \end{bmatrix}.$$

Corollary 2.1. Let$A \in \mathcal{M}(\mathbb{R})$ and $|A| \neq 0$. Then $\frac{I}{A} = A^{-1}$.

Example2.4. Let$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & -1 \\ 1 & 5 & 3 \end{bmatrix}$. Then $|A| = 8$.

$$I \div A = \begin{bmatrix} \frac{\begin{vmatrix} 1 & 3 & 1 \\ 0 & 3 & -1 \\ 0 & 5 & 3 \end{vmatrix}}{8} & \frac{\begin{vmatrix} 0 & 3 & 1 \\ 1 & 3 & -1 \\ 0 & 5 & 3 \end{vmatrix}}{8} & \frac{\begin{vmatrix} 0 & 3 & 1 \\ 0 & 3 & -1 \\ 1 & 5 & 3 \end{vmatrix}}{8} \\ \frac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 3 \end{vmatrix}}{8} & \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix}}{8} & \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 3 \end{vmatrix}}{8} \\ \frac{\begin{vmatrix} 1 & 3 & 1 \\ 0 & 3 & 0 \\ 1 & 5 & 0 \end{vmatrix}}{8} & \frac{\begin{vmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 1 & 5 & 0 \end{vmatrix}}{8} & \frac{\begin{vmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 5 & 1 \end{vmatrix}}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{4} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ -\frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \end{bmatrix} = A^{-1}$$

We give without proof

Corollary 2.2. Let $A \in \mathcal{M}(\mathbb{R})$ and $|A| \neq 0$. Then $A\left(\frac{I}{A}\right) = I$.

Theorem 2.3. Let$A, B \in \mathcal{M}(\mathbb{R})$ and $|A| \neq 0, |B| \neq 0$.

Then $(A \div B)(B \div A) = I$.

Proof. Let$A, B \in \mathcal{M}(\mathbb{R})$ and $|A| \neq 0, |B| \neq 0$.

$$(A \div B)(B \div A) = \frac{A}{B}\frac{B}{A} = \frac{A}{B}\frac{I}{\frac{A}{B}} = I \iff \left(\frac{A}{B}\right)^{-1} = \frac{B}{A}.$$

Example 2.5. If $A = [6]$, $B = [3]$. Then $A \div B = [2\,]$,

which is analogous to the real case$\frac{6}{3} = 2$.

Theorem2.4. Let$\forall k \in \mathbb{R} \setminus \{0\}$ and $A, B \in \mathcal{M}(\mathbb{R})$ with $|B| \neq 0$. Then,

i. $(kA \div B) = k(A \div B)$.

ii. $\left(A \div (kB)\right) = \frac{1}{k}(A \div B)$.

Proof

i. Using Lemma 1.2.(i), we have ${}^{(kA)}_{B}i_j = k\left({}^{A}_{B}i_j\right)$. Therefore

$$(kA \div B) = k(A \div B).$$

ii. Using Lemma 1.2.(ii), we have ${}_{(kB)}^{A}i_j = k^{(n-1)}\left({}^{A}_{B}i_j\right)$. Also,

$$\left(A \div (kB)\right) = \frac{k^{n-1}}{k^n}\left[\frac{{}^{A}_{B}i_j}{|B|}\right] = \frac{1}{k}(A \div B).$$

Example2.6. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Then $|B| = -1$. Therefore

$$kA \div B = \begin{bmatrix} \frac{\begin{vmatrix} k & 1 & 0 \\ k & 0 & 1 \\ -k & 1 & 1 \end{vmatrix}}{-1} & \frac{\begin{vmatrix} k & 1 & 0 \\ k & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}}{-1} & \frac{\begin{vmatrix} -k & 1 & 0 \\ k & 0 & 1 \\ k & 1 & 1 \end{vmatrix}}{-1} \\ \frac{\begin{vmatrix} 2 & k & 0 \\ 2 & k & 1 \\ 3 & -k & 1 \end{vmatrix}}{-1} & \frac{\begin{vmatrix} 2 & k & 0 \\ 2 & k & 1 \\ 3 & 0 & 1 \end{vmatrix}}{-1} & \frac{\begin{vmatrix} 2 & -k & 0 \\ 2 & k & 1 \\ 3 & k & 1 \end{vmatrix}}{-1} \\ \frac{\begin{vmatrix} 2 & 1 & k \\ 2 & 0 & k \\ 3 & 1 & -k \end{vmatrix}}{-1} & \frac{\begin{vmatrix} 2 & 1 & k \\ 2 & 0 & k \\ 3 & 1 & 0 \end{vmatrix}}{-1} & \frac{\begin{vmatrix} 2 & 1 & -k \\ 2 & 0 & k \\ 3 & 1 & k \end{vmatrix}}{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3k & 2k & 0 \\ -5k & -3k & 0 \\ -5k & -3k & k \end{bmatrix} = k(A \div B) = k\begin{bmatrix} 3 & 2 & 0 \\ -5 & -3 & 0 \\ -5 & -3 & 1 \end{bmatrix}.$$

Lemma2.1. Let$A, B, C, D \in \mathcal{M}(\mathbb{R})$be any four matrices such that $|B| \neq 0, |D| \neq 0$. If $(A \div B) = (C \div D)$ then

$$\frac{{}^{A}_{B}i_j}{{}^{C}_{D}i_j} = \frac{|B|}{|D|}.$$

Proof. If$(A \div B) = (C \div D)$ then

$$\left[\frac{{}^{A}_{B}i_j}{|B|}\right] = \left[\frac{{}^{C}_{D}i_j}{|D|}\right] \Leftrightarrow \frac{{}^{A}_{B}i_j}{|B|} = \frac{{}^{C}_{D}i_j}{|D|} \Leftrightarrow \frac{{}^{A}_{B}i_j}{{}^{C}_{D}i_j} = \frac{|B|}{|D|}.$$

Lemma 2.2. Let $A, 0 \in \mathcal{M}(\mathbb{R})$ such taht $|A| \neq 0$ is$0 \div A = 0$.

Proof. For$i, j = 1, \dots, n$ we get${}^{0}_{A}i_j = 0\left({}^{0}_{A}i_j\right) = 0$. Therefore $\frac{{}^{0}_{A}i_j}{|A|} = 0 \Leftrightarrow 0 \div A = 0 \in \mathcal{M}(\mathbb{R})$.

3. Divisibilty

For matrices A and B such that $|B| \neq 0$, we say that B divides A, or that B is a divisor (or factor) of A, or that A is a multiple of B, if there exists an integer C such that $A = BC$, and we denote this by $B|A$. Otherwise, B does not divide A, and we denote this by $B\nmid A$.

Theorem 3.1. Let $A, B, R \in \mathcal{M}(\mathbb{R})$ such taht $|B| \neq 0$. Then there exist unique , $P \in \mathcal{M}(\mathbb{R})$, $|P| \neq 0$ such that

i. $\frac{A-R}{B} = P$.

ii. $R = 0$ if $B|A$.

Proof. i. For$\forall A, B \in \mathcal{M}(\mathbb{R})$ and $|B| \neq 0$ then there is unique $P \in \mathcal{M}(\mathbb{R})$, $|P| \neq 0$ such that;

$$\frac{1}{|B|}\left[i_B^{A-R}\right] = P \Leftrightarrow \frac{A-R}{B} = P.$$

It is unique.

ii. It is clearly.

4. Rational Matrices

Let us start with the definition as follows. The term rational in reference to the set $\mathbb{Q}(\mathcal{M}) = \left\{\frac{A}{B}\middle|A,B \in \mathcal{M}(\mathbb{R}), |B| \neq 0\right\}$[1] refers to the fact that a rational matrix represents a ratio of two matrices. The set of rational matrices is include by $\mathcal{M}(\mathbb{R})$. A rational matrix is a matrix like $\frac{A}{B}$, where A, B are matrices. If $B = 0$ then this division is not defined.

$\frac{0}{A} = 0 \in \mathbb{Q}(\mathcal{M})$ and $\frac{A}{A} = I \in \mathbb{Q}(\mathcal{M})$. Then $\mathbb{Q}(\mathcal{M}) \neq \emptyset$.

Lemma 4.1. Let $A, B, C \in \mathcal{M}(\mathbb{R}), |C| \neq 0$. Then

i. $\left[\frac{A}{B}i_j\right] = \frac{1}{|C|}\left[\frac{C}{B}i_j\right]\left[\frac{A}{C}i_j\right]$.

ii. $\left[\frac{A}{B}i_j\right] = A\left[\frac{I}{B}i_j\right]$.

We give the Lemma not proof from [3].

The set of rational matrices will show with

$$\mathbb{Q}(\mathcal{M}) = \left\{\frac{A}{B}\middle|A,B \in \mathcal{M}(\mathbb{R}), |B| \neq 0\right\}.$$

We easily get

Lemma 4.2. Let $\frac{A}{B}, \frac{C}{B} \in \mathbb{Q}(\mathcal{M})$. Then $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$.

Proof. $\frac{A}{B} + \frac{C}{B} = \frac{1}{|B|}\left[\frac{A}{B}i_j\right] + \frac{1}{|B|}\left[\frac{C}{B}i_j\right] = \frac{1}{|B|}\left(\left[\frac{A}{B}i_j\right] + \left[\frac{C}{B}i_j\right]\right) = \frac{1}{|B|}\left(\left[\frac{A}{B}i_j + \frac{C}{B}i_j\right]\right) = \left[{}^{(A+C)}{}_{B}i_j\right] = \frac{A+C}{B}$.

Theorem 4. 1. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$. Then

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \frac{|D|}{|B|\left|\frac{C}{D}i_j\right|}\left[\frac{\left(\frac{A}{B}i_j\right)}{\left(\frac{C}{D}i_j\right)}i_j\right].$$

Proof. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$.

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \frac{|D|}{|B|}\frac{\left[\frac{A}{B}i_j\right]}{\left[\frac{C}{D}i_j\right]} = \frac{|D|}{|B|\left|\frac{C}{D}i_j\right|}\left[\frac{\left(\frac{A}{B}i_j\right)}{\left(\frac{C}{D}i_j\right)}i_j\right].$$

Corollary 4.1. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$. Then $\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \frac{|D|}{|B|}\left[\frac{D}{C}i_j\right]\left[\frac{A}{B}i_j\right]$.

Theorem 4. 2. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M}), |C| \neq 0$. Then

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \left(\frac{D}{C}\right)\left(\frac{A}{B}\right).$$

Proof. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M}), |C| \neq 0. \frac{\frac{A}{B}}{\frac{C}{D}} = \left[\frac{C}{D}\right]^{-1}\left[\frac{A}{B}\right] = \left[\frac{D}{C}\right]\left[\frac{A}{B}\right]$.

Example 4.1. Let $A = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}, B = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}, C = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}, D = \begin{bmatrix}5 & 7\\2 & 3\end{bmatrix}$

$$\frac{A}{B} = \begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}, \frac{C}{D} = \begin{bmatrix}-1 & -4\\1 & 3\end{bmatrix} \Rightarrow \frac{\frac{A}{B}}{\frac{C}{D}} = \begin{bmatrix}5 & 4\\-1 & -1\end{bmatrix},$$

$$\frac{C}{D} = \begin{bmatrix}-1 & -4\\1 & 3\end{bmatrix} \Rightarrow \frac{D}{C} = \begin{bmatrix}3 & 4\\-1 & -1\end{bmatrix},$$

$$\left[\frac{D}{C}\right]\left[\frac{A}{B}\right] = \begin{bmatrix}3 & 4\\-1 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix} = \begin{bmatrix}5 & 4\\-1 & -1\end{bmatrix} = \frac{\frac{A}{B}}{\frac{C}{D}}.$$

Lemma 4.3. Let $\frac{A}{B} \in \mathbb{Q}(\mathcal{M}), C \in \mathcal{M}(\mathbb{R})$. Then

a. $\left(\frac{A}{B}\right)C = \frac{AC}{B}$.

b. $\left(\frac{A}{B}\right)A = \frac{A^2}{B}$.

[1]Sequare matrix of reel numbers is denoted with $\mathcal{M}(\mathbb{R})$or $\mathcal{M}$.

c. If $|C| \neq 0$, also $\frac{A}{B} = \left[\frac{C}{B}\right].\left[\frac{A}{C}\right]$.

Proof. Let $A = [a_{ij}], B = [b_{ij}]$ and $C = [c_{ij}]$ then

i. $$\frac{AC}{B} = \begin{bmatrix} \sum_{s=1}^{n}(\sum_{k=1}^{n} a_{sk} c_{k1})C_{s1}(B) & \cdots & \sum_{s=1}^{n}(\sum_{k=1}^{n} a_{sk} a_{kn})C_{s1}(B) \\ \vdots & \vdots & \vdots \\ \sum_{s=1}^{n}(\sum_{k=1}^{n} a_{sk} c_{k1})C_{sn}(B) & \cdots & \sum_{s=1}^{n}(\sum_{k=1}^{n} a_{sk} a_{k1})C_{sn}(B) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{sk}\, C_{k1}(B) & \cdots & \sum_{k=1}^{n} a_{sk}\, C_{k1}(B) \\ \vdots & \vdots & \vdots \\ \sum_{k=1}^{n} a_{sk}\, C_{k1}(B) & \cdots & \sum_{k=1}^{n} a_{sk}\, C_{k1}(B) \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} = \left(\frac{A}{B}\right) C .$$

ii. If $C = A$ from i).

$$\left(\frac{A}{B}\right)A = \left[\frac{{}_B^A i_j}{|B|}\right]A = \frac{1}{|B|}\left[{}_B^A i_j\right]A = \frac{1}{|B|}\left[{}_B^{A^2} i_j\right] = \frac{A^2}{B}.$$

iii. If $|C| \neq 0$ from lemma 1.3.

$$\frac{A}{B} = \frac{1}{|B|}\left[{}_B^A i_j\right] = \frac{1}{|B|}\frac{1}{|C|}\left[{}_B^C i_j\right]\left[{}_C^A i_j\right] = \left[\frac{C}{B}\right].\left[\frac{A}{C}\right].$$

Note 4.1. In general, $A\left(\frac{A}{B}\right) \neq \frac{A^2}{B}$.

Theorem 4. 3. Let $\frac{A}{B}, \frac{C}{D}, \frac{E}{F} \in \mathbb{Q}(\mathcal{M})$. Then

i. $\frac{A}{B} + \frac{C}{D} = \frac{C}{D} + \frac{A}{B}$.

ii. $\frac{A}{B} + \left(\frac{C}{D} + \frac{E}{F}\right) = \left(\frac{A}{B} + \frac{C}{D}\right) + \frac{E}{F}$.

iii. $\frac{A}{B} + 0 = \frac{A}{B}, 0 + \frac{A}{B} = \frac{A}{B}$.

iv. $\left(\frac{A}{B}\right)\left(\frac{C}{D}\right)$is rational matrix.

We give the lemma without proof.

Lemma 4.4. Let$\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$. Then $\left(\frac{A}{B}\right)\left(\frac{C}{D}\right) = \frac{A\left(\frac{C}{D}\right)}{B}$.

Lemma 4.5. Let$\frac{A}{B} \in \mathbb{Q}(\mathcal{M})$ and $n \in \mathbb{Z}^+$. Then

$$\left(\frac{A}{B}\right)^n = \frac{1}{|B|^n}\left[{}_B^A i_j\right]^n.$$

Theorem 4. 4. Let$\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$ and $n, m \in \mathbb{Z}^+$. Then

i. $\left(\frac{A}{B}\right)^n = \underbrace{\left(\frac{A}{B}\right)\cdots\left(\frac{A}{B}\right)}_{n-\text{times}}$.

ii. $\left(\frac{A}{B}\right)^{-1} = \frac{B}{A} = \frac{I}{\frac{A}{B}}$.

iii. $\left(\frac{A}{B}\right)^{-n} = \left(\left(\frac{A}{B}\right)^{-1}\right)^n = \left(\frac{B}{A}\right)^n$.

iv. $\left(\frac{A}{B}\right)^n\left(\frac{A}{B}\right)^m = \left(\frac{A}{B}\right)^{n+m}$.

v. $\left(\left(\frac{A}{B}\right)^n\right)^m = \left(\frac{A}{B}\right)^{nm}$.

Proof. i. It is clearly from Lemma 2 that

$$\left(\frac{A}{B}\right)^n = \underbrace{\frac{1}{|B|}\left[{}_B^A i_j\right]\cdots\frac{1}{|B|}\left[{}_B^A i_j\right]}_{n-\text{times}} = \underbrace{\left(\frac{A}{B}\right)\cdots\left(\frac{A}{B}\right)}_{n-\text{times}}.$$

ii. This proof is the paper “Matrix Division”.
iii. It is seen clearly.

Example4.2. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $n = 2$. Then

$\frac{A}{B} = \begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \left(\frac{A}{B}\right)^2 = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$ and $\frac{A^2}{B^2} = \begin{bmatrix} 5 & 3 \\ -7 & -4 \end{bmatrix}$ then

$$\left(\frac{A}{B}\right)^2 \neq \frac{A^2}{B^2}.$$

Note 4.2. In general for rational matrices,

$$\left(\frac{A}{B}\right)^n \neq \frac{A^n}{B^n}.$$

5. Results and Discussions

The matrix division supports insistence of the way and carries to multiple systems. It increases the ability of thinking creatively. More than one different result is achieved for the same system.

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