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PROJECTION-ITERATION REALIZATION OF A NEWTON-LIKE METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS

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Abstract. We consider the problem of existence and location of a solution of a nonlinear operator equation with a Fréchet differentiable operator in a Banach space and present the convergence results for a projection-iteration method based on a Newton-like method under the Cauchy's conditions, which generalize the results for the projection-iteration realization of the Newton-Kantorovich method. The proposed method unlike the traditional interpretation is based on the idea of whatever approximation of the original equation by a sequence of approximate operator equations defined on subspaces of the basic space with the subsequent application of the Newton-like method to their approximate solution. We prove the convergence theorem, obtain the error estimate and discuss the advantages of the proposed approach and some of its modifications.

Key words: nonlinear equation, Fréchet differentiable operator, Newton-like method, projection-iteration method, approximation, convergence, error estimate.

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1. Introduction

The fundamental tool in numerical analysis, operations research, optimization and control is Newton's method originally intended to solve algebraic equations. The basic ideas of the method, the main theoretical results of convergence, the latest developments in this area, the most up-to-date versions of the method, as well as its various applications can be found, for instance, in papers [1, 4, 5, 12–18]. Newton's method has been studied in more detail under the so-called Kantorovich conditions (the derivative of the equation operator is invertible at the initial point and satisfies the Lipschitz condition in the considered domain), under the Vertgeim conditions (the operator derivative is invertible at the initial point but satisfies only Hölder condition) and under the Mysovskih conditions (the derivative is invertible at all points in the considered domain and its inverse operator is bounded).

To solve nonlinear functional equations, other iterative methods as well as projection (approximation) type methods are also used; a survey of the relevant literature is contained, for instance, in [8]. In the same source, to solve operator

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equations of the first kind, studies have been performed for methods called the projection-iteration ones based on the following idea. An equation of the form

$$Au = f \tag{1.1}$$

with a nonlinear operator A acting on a Banach space X ($f \in X$ is a known element), is approximated by a sequence of approximate equations

$$A_n u_n = f_n, \quad n = 1,\ 2, \dots, \tag{1.2}$$

where A_n is a nonlinear operator acting on a subspace X_n of the original space ($X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X, \quad X_1 \neq \varnothing$). To solve approximate equations (1.2), some iterative method is used, at that for each of these equations only a few approximations $u_n^{(k)}$ ($k = 1,\ 2, \dots,\ k_n$) are found and the last of them $u_n^{(k_n)}$ is assumed to be equal to the initial approximation $u_{n+1}^{(0)}$ in the iterative process for the next, $(n+1)$-th approximate equation. The sequence $\{u_n^{(k_n)}\}_{n=1}^{\infty} \subset X$ is considered as a sequence of approximations to a solution u^* of equation (1.1). This approach to finding an approximate solution of the original equation naturally eliminates the difficulties that arise when solving the same equation using the conventional projection method and also makes it easier to choose the suitable initial approximation compared to solving the original equation using the iterative method.

In this paper, to solve nonlinear operator equation (1.1), the projection-iteration implementation of the Newton-like method [6] is studied under generalized Cauchy's conditions, which instead of the inverse operator to the derivative in the considered domain imply the existence of some linear operator close to it. The problems of substantiation of the projection-iteration schemes of both the basic Newton-Kantorovich method under such conditions and some of its modifications are considered.

2. Preliminaries

Let us consider equation (1.1) $Au = f$ with a nonlinear operator A which acts on a Banach space X and is Fréchet differentiable on some ball $S(u_N^{(0)}, R) = \{u \in X : \|u - u_N^{(0)}\| \leq R\}$ of this space. We approximate equation (1.1) by the sequence of approximate equations (1.2) $A_n u_n = f_n, \quad n = 1,\ 2, \dots$ with nonlinear operators A_n, each of which acts on the respective subspace $X_n \subset X$ and is Fréchet differentiable on the set $\Omega_n = X_n \cap S(u_N^{(0)}, R)$ beginning with some number $n = N \geq 1$; $f_n = P_n f$, P_n is a linear projector which maps X onto X_n ($P_n \colon X \to X_n, \quad P_n u_n = u_n$ for $u_n \in X_n$).

Assume that for each $n \geq N$ the following proximity conditions hold:

$$\|A_n u_n - P_n A u_n\| \leq \alpha_n, \quad \|A_n'(u_n) - P_n A'(u_n)\|_{X_n \to X_n} \leq \alpha_n', \quad \forall u_n \in \Omega_n; \tag{2.1}$$

$$\|P_n A u - A u\| \leq \beta_n, \quad \|P_n A'(u) - A'(u)\|_{X \to X} \leq \beta_n', \quad \forall u \in S(u_N^{(0)}, R); \tag{2.2}$$

$$\|P_n f - f\| \le \gamma_n, \quad \forall f \in X, \tag{2.3}$$

where α_n, α'_n, β_n, β'_n, $\gamma_n \to 0$ when $n \to \infty$. We will also assume that the derivative $A'_n(u_n)$ on the set Ω_n satisfies the Lipschitz condition

$$\|A'_n(u_n) - A'_n(v_n)\|_{X_n \to X_n} \le L' \, \|u_n - v_n\|, \quad \forall u_n,\, v_n \in \Omega_n, \quad n \ge N, \tag{2.4}$$

where $L' > 0$ is a Lipschitz constant.

If there exists a continuous linear operator $\Gamma_n(u_n) = [A'_n(u_n)]^{-1}$ for all $u_n \in \Omega_n \, (n \ge N)$ then one can apply the Newton-Kantorovich method [14] to each of equations (1.2) beginning from the number $n = N$, and construct a sequence of approximations to the solution u^* of equation (1.1) by the formulas

$$\begin{gathered} u_n^{(k+1)} = u_n^{(k)} - [A'_n(u_n^{(k)})]^{-1}(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \dots,\ k_n - 1; \\ u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \ge N; \quad u_N^{(0)} \in \Omega_N \subset X. \end{gathered} \tag{2.5}$$

In paper [3] the theorem is given on the existence of a solution u^* to equation (1.1), on the domain of its location, as well as on the convergence of projection-iteration process (2.5) under the Cauchy-type conditions. The following theorem is a generalization of the mentioned theorem, when instead of operators

$$\Gamma(u) = [A'(u)]^{-1}, \quad u \in S(u_N^{(0)}, R)$$

and $\Gamma_n(u_n) = [A'_n(u_n)]^{-1}$, $u_n \in \Omega_n$ $(n \ge N)$, it is required the existence only of an operator $D(u)$, $u \in S(u_N^{(0)}, R)$ in X and an operator $D_n(u_n)$, $u_n \in \Omega_n$ in X_n, which are close to $\Gamma(u)$ and $\Gamma_n(u_n)$ respectively.

Theorem 2.1. *Let the operator A be Fréchet differentiable on some ball $S(u_N^{(0)}, R) \subset X$ and let for all $n \ge N$ the operator A_n be Fréchet differentiable on the set $\Omega_n = X_n \cap S(u_N^{(0)}, R)$, at that let its derivative $A'_n(u_n)$ satisfy on Ω_n the Lipschitz condition* (2.4)*. Assume that the proximity conditions* (2.1)*–*(2.3) *hold true and there exist a linear operator $D(u)$ on X and linear operators $D_n(u_n)$ on X_n such that*

$$\|D(u)\|_{X \to X} \le b, \quad \|E - D(u)A'(u)\|_{X \to X} \le \delta < \ 1, \quad \forall u \in S(u_N^{(0)}, R); \tag{2.6}$$

$$\|E - D_n(u_n)A'_n(u_n)\|_{X_n \to X_n} \le \delta_n < \ 1, \quad \forall u_n \in \Omega_n, \quad n \ge N, \tag{2.7}$$

where $b > 0$, $\delta > 0$, $\delta_n > 0$; E is an identity operator on X. If the initial approximation $u_N^{(0)} \in \Omega_N$ satisfies the conditions

$$\|A_N u_N^{(0)} - f_N)\| \le \eta_N^{(0)}, \quad h_N^{(0)} = b_N^2 L' \eta_N^{(0)} < 2, \quad r_N = b_N \eta_N^{(0)} G_N \le R, \tag{2.8}$$

where

$$b_N = b/\left(1 - b(\alpha'_N + \beta'_N) - \delta\right),$$
$$G_N = H_N + \sum_{m=N}^{\infty} (h_N^{(0)}/2)^{2^{S_m}-1} < 2H_N,$$
$$s_m = \sum_{i=N}^{m} (k_i - 1),$$
$$H_N = \sum_{m=0}^{\infty} (h_N^{(0)}/2)^{2^{S_m}-1},$$

then equation (1.1) *has in the ball* $S(u_N^{(0)}, r_N)$ *a solution* u^* *to which the based on Newton's method process* (2.5) *converges with the error estimate*

$$\|u_n^{(k_n)} - u^*\| \le b_N \eta_N^{(0)} V_n (h_N^{(0)}/2)^{2^{S_n}-1}, \quad n \ge N, \tag{2.9}$$

where

$$V_n = \sum_{m=0}^{\infty} (h_N^{(0)}/2)^{2^{S_n}(2^m-1)} + \sum_{m=n+1}^{\infty} (h_N^{(0)}/2)^{2^{S_m}-2^{S_n}} < 2H_N.$$

The proof of Theorem 2.1 can be found in [3].

3. Proving the convergence theorem

Let us consider, to solve the operator equation (1.1), a projection-iteration process, like (2.5) with the replacement of the operator $\Gamma_n(u_n^{(k)}) = [A'_n(u_n^{(k)})]^{-1}$ by an operator $D_n(u_n^{(k)})$ close to it:

$$u_n^{(k+1)} = u_n^{(k)} - D_n(u_n^{(k)})(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \ldots,\ k_n - 1; \tag{3.1}$$
$$u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \ge N; \quad u_N^{(0)} \in \Omega_N \subset X.$$

The following theorem establishes the sufficient conditions of feasibility and convergence in the ball $S(u_N^{(0)}, R)$ of the approximations sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty} \subset X$ determined by formulas (3.1) to a solution u^* of equation (1.1).

Theorem 3.1. *Let all the conditions of Theorem 2.1 hold true and let, moreover, the derivative* $A'(u)$ *satisfy on* $S(u_N^{(0)}, R)$ *the Lipschitz condition*

$$\|A'(u) - A'(v)\|_{X \to X} \le L\, \|u - v\|, \quad \forall u,\, v \in S(u_N^{(0)}, R); \quad L > 0. \tag{3.2}$$

Assume that $bL\delta/(1-\delta) < 1$, *where* $b > 0$, $\delta > 0$ *are defined in* (2.6), *and that* $\delta_n \to 0$ *in condition* (2.7) *when* $n \to \infty$. *If the initial approximation* $u_N^{(0)} \in \Omega_N$ *satisfies the first condition* (2.8),

$$h_N^{(0)} = b_N^2 L' \eta_N^{(0)} + \frac{2 b_N L' \delta_N}{1 - \delta_N} < 2, \quad r_N = b_N \eta_N^{(0)} G_N \le R, \tag{3.3}$$

where

$$
\begin{aligned}
b_N &= b/(1 - b\rho_N),\\
\rho_N &= \alpha'_N + \beta'_N + L'\delta_N/(1-\delta_N) + L\delta/(1-\delta),\\
G_N &= H_N + \sum_{m=N}^{\infty} (h_N^{(0)}/2)^{S_m} < 2H_N,\\
s_m &= \sum_{i=N}^{m} (k_i - 1),
\end{aligned}
$$

$$H_N = 1/(1 - h_N^{(0)}/2),$$

then equation (1.1) *has in the ball* $S(u_N^{(0)}, r_N) \subset X$ *a solution* u^* *to which the projection-iteration process of approximations* (3.1) *converges with the error estimate*

$$\|u_n^{(k_n)} - u^*\| \le b_N \eta_N^{(0)} V_n (h_N^{(0)}/2)^{S_n}, \quad n \ge N, \tag{3.4}$$

where $V_n = H_N + \sum\limits_{m=n+1}^{\infty} (h_N^{(0)}/2)^{S_m - S_n} < 2H_N$.

Proof. First of all, we note that the second condition in (2.6) implies the existence of bounded inverse operator $[D(u)]^{-1}$, $u \in S(u_N^{(0)}, R)$; while taking into account (3.2) the estimate $\|[D(u)]^{-1}\|_{X\to X} \le L/(1-\delta)$ holds for all $u \in S(u_N^{(0)}, R)$. Similarly, from the conditions (2.7) and (2.4) there follows the existence of bounded inverse operators $[D_n(u_n)]^{-1}$, $u_n \in \Omega_n$ with the norm $\|[D_n(u_n)]^{-1}\|_{X_n\to X_n} \le L'/(1-\delta_n)$, $n \ge N$. Further, based on the first condition (2.6) and the proximity conditions (2.1), (2.2) the existence of operators $D_n(u_n)$ implies their boundedness, beginning with some $n = N_1 \ge N$. Indeed, since for $u_n \in \Omega_n$, $z_n \in X_n$

$$
\begin{aligned}
\|[D_n(u_n)]^{-1} z_n - [D(u_n)]^{-1} z_n\| &\le \big(\|[D_n(u_n)]^{-1} - A'_n(u_n)\|_{X_n\to X_n}\\
&\quad + \|A'_n(u_n) - P_n A'(u_n)\|_{X_n\to X_n} + \|P_n A'(u_n) - A'(u_n)\|_{X\to X}\\
&\quad + \|A'(u_n) - [D(u_n)]^{-1}\|_{X\to X}\big)\, \|z_n\| \le \rho_n \|z_n\|,
\end{aligned}
$$

where $\rho_n = L'\delta_n/(1-\delta_n) + \alpha'_n + \beta'_n + L\delta/(1-\delta)$, then

$$
\begin{aligned}
\|[D_n(u_n)]^{-1} z_n\| &\ge \|[D(u_n)]^{-1} z_n\| - \|[D_n(u_n)]^{-1} z_n - [D(u_n)]^{-1} z_n\|\\
&\ge (1 - b\rho_n)/b\, \|z_n\|,
\end{aligned}
$$

and since under the conditions of the theorem $b\rho_n < 1$ for $n \ge N_1$, then for these numbers n we will have

$$\|D_n(u_n)\|_{X_n\to X_n} \le b_n = b/(1 - b\rho_n), \quad u_n \in \Omega_n. \tag{3.5}$$

Let us prove the feasibility of process (3.1). Note that the possibility of replacing equations (1.2) by linearized equations

$$A_n u_n^{(k)} + [D_n(u_n^{(k)})]^{-1}(u_n - u_n^{(k)}) = f_n, \quad k = 0,\ 1, \dots\ ; \quad n \ge N$$

respectively follows from the existence of continuous operators $[D_n(u_n)]^{-1}$ close to $A'_n(u_n)$, $u_n \in \Omega_n$ for the specified n. We establish (by mathematical induction) that all subsequent approximations $u_n^{(0)}$ for $n > N$ have the same properties (2.8), (3.3) and that they belong to the ball $S(u_N^{(0)}, r_N) \subset X$. Based on the theorem conditions, it can be shown that for $n = N,\ N+1, \dots,\ m$

$$\|A_n u_n^{(0)} - f_n\| \le \eta_n^{(0)}, \quad h_n^{(0)} = b_n^2 L' \eta_n^{(0)} + \frac{2b_n L'\delta_n}{1-\delta_n} < 2. \tag{3.6}$$

In addition, as it follows from the proof of Theorem 2 of [6], at any fixed n $(N \le n \le m)$ the conditions

$$\|A_n u_n^{(k)} - f_n\| \le \eta_n^{(k)}, \quad h_n^{(k)} = b_n^2 L' \eta_n^{(k)} + \frac{2b_n L'\delta_n}{1-\delta_n} < 2 \tag{3.7}$$

hold for each number $k = 1,\ 2, \dots,\ k_n$. We show the feasibility of (3.6) for $n = m+1$. Insofar as

$$\|A_{m+1}u_{m+1}^{(0)} - f_{m+1}\| \le \|A_{m+1}u_{m+1}^{(0)} - A_m u_{m+1}^{(0)}\| + \|A_m u_m^{(k_m)} - f_m\| + \|f_m - f_{m+1}\|,$$

then based on the proximity conditions (2.1)–(2.3) from the relations

$$\begin{aligned} \|A_{m+1}u_{m+1}^{(0)} - A_m u_{m+1}^{(0)}\| &\le \|A_{m+1}u_{m+1}^{(0)} - P_{m+1}Au_{m+1}^{(0)}\| \\ &+ \|P_{m+1}Au_{m+1}^{(0)} - Au_{m+1}^{(0)}\| + \|Au_m^{(k_m)} - P_m A u_m^{(k_m)}\| \\ &+ \|P_m A u_m^{(k_m)} - A_m u_m^{(k_m)}\| \le \alpha_{m+1} + \beta_{m+1} + \beta_m + \alpha_m; \end{aligned}$$

$$\|f_m - f_{m+1}\| \le \|P_m f - f\| + \|f - P_{m+1} f\| \le \gamma_m + \gamma_{m+1}$$

and from the first of the conditions (3.7) we obtain:

$$\|A_{m+1}u_{m+1}^{(0)} - f_{m+1}\| \le \theta_m + \eta_m^{(k_m)} = \eta_{m+1}^{(0)}, \tag{3.8}$$

where $\theta_m = \alpha_m + \alpha_{m+1} + \beta_m + \beta_{m+1} + \gamma_m + \gamma_{m+1}$, that is, the first of the conditions (3.6) for $n = m+1$ holds true. Let us show the fulfillment of the second one.

Proof of the Theorem 2 of [6] implies that for any $k = 0,\ 1, \dots,\ k_m - 1$

$$\begin{aligned} \|A_m u_m^{(k+1)} - f_m\| &= \|A_m u_m^{(k+1)} - A_m u_m^{(k)} - [D_m(u_m^{(k)})]^{-1}(u_m^{(k+1)} - u_m^{(k)})\| \\ &\le \frac{L'}{2}\|u_m^{(k+1)} - u_m^{(k)}\|^2 + \frac{L'\delta_m}{1-\delta_m}\|u_m^{(k+1)} - u_m^{(k)}\| \\ &\le \frac{L'}{2} b_m^2 \eta_m^{(k)2} + \frac{L'\delta_m}{1-\delta_m} b_m \eta_m^{(k)} = \frac{h_m^{(k)}}{2}\eta_m^{(k)} = \eta_m^{(k+1)}, \end{aligned} \tag{3.9}$$

so

$$\eta_m^{(k_m)} = \frac{h_m^{(k_m-1)}}{2}\eta_m^{(k_m-1)} = \dots = \frac{1}{2^{k_m}} h_m^{(k_m-1)} h_m^{(k_m-2)} \dots h_m^{(0)}\, \eta_m^{(0)}.$$

Because in (3.8) $\theta_m \to 0$ when $m \to \infty$ and because by virtue of (3.7) $h_m^{(k_m-1)} < 2$, there exists a number $m = N_2 \geq N$ beginning with which

$$\eta_{m+1}^{(0)} \leq \frac{1}{2^{k_m-1}} h_m^{(k_m-2)} h_m^{(k_m-3)} \dots h_m^{(0)}\, \eta_m^{(0)}. \tag{3.10}$$

Since, obviously $b_{m+1} \leq b_m$, $\delta_{m+1} \leq \delta_m$, then taking into account (3.10) and (3.7) we have for all $m \geq N_2$:

$$\begin{aligned} h_{m+1}^{(0)} &= b_{m+1}^2 L' \eta_{m+1}^{(0)} + \frac{2b_{m+1} L' \delta_{m+1}}{1-\delta_{m+1}} \\ &\leq b_m^2 L' \frac{1}{2^{k_m-1}} h_m^{(k_m-2)} h_m^{(k_m-3)} \dots h_m^{(0)} \eta_m^{(0)} + \frac{2b_m L' \delta_m}{1-\delta_m} \\ &= b_m^2 L' \eta_m^{(k_m-1)} + \frac{2b_m L' \delta_m}{1-\delta_m} = h_m^{(k_m-1)} < 2, \end{aligned} \tag{3.11}$$

that is, the second of the conditions (3.6) for $n = m+1$ also holds true.

Let number $N := \max\{N_1,\ N_2\}$ be the initial one in formulas (3.1).

Let's show that the approximations $u_{n+1}^{(0)}$ belong to the ball $S(u_N^{(0)}, r_N) \subset X$ for all $n \geq N$. It's obvious that

$$\|u_{n+1}^{(0)} - u_N^{(0)}\| \leq \sum_{m=N}^{n} \|u_{m+1}^{(0)} - u_m^{(0)}\|, \quad n \geq N;$$

in turn, for each $m = N,\ N+1, \dots,\ n$

$$\|u_{m+1}^{(0)} - u_m^{(0)}\| = \|u_m^{(k_m)} - u_m^{(0)}\| \leq \sum_{k=0}^{k_m-1} \|u_m^{(k+1)} - u_m^{(k)}\|.$$

Based on formulas (3.1), (3.5), (3.7), (3.9) for any numbers $m = N,\ N+1, \dots,\ n$ and $k = 0,\ 1, \dots,\ k_m - 1$ we obtain:

$$\begin{aligned} \|u_m^{(k+1)} - u_m^{(k)}\| &\leq \|D_m(u_m^{(k)})\|_{X_m \to X_m} \|A_m u_m^{(k)} - f_m\| \leq b_m \eta_m^{(k)} \\ &= b_m \frac{1}{2^k} h_m^{(k-1)} h_m^{(k-2)} \dots h_m^{(0)}\, \eta_m^{(0)}, \end{aligned}$$

and because of

$$\begin{aligned} h_m^{(k+1)} &= b_m^2 L' \eta_m^{(k+1)} + \frac{2b_m L' \delta_m}{1-\delta_m} \\ &= b_m^2 L' \frac{h_m^{(k)}}{2} \eta_m^{(k)} + \frac{2b_m L' \delta_m}{1-\delta_m} \\ &< b_m^2 L' \eta_m^{(k)} + \frac{2b_m L' \delta_m}{1-\delta_m} \\ &= h_m^{(k)} < 2, \quad k = 0,\ 1, \dots,\ k_m - 1, \end{aligned} \tag{3.12}$$

we have

$$\|u_m^{(k+1)} - u_m^{(k)}\| \le b_m \big(h_m^{(0)}/2\big)^k \eta_m^{(0)}, \quad k = 0,\ 1, \dots,\ k_m - 1.$$

Let's evaluate here $\eta_m^{(0)}$ and $h_m^{(0)}$ $(N+1 \le m \le n)$ through $\eta_N^{(0)}$ and $h_N^{(0)}$. Applying (3.12) in formulas (3.10) and (3.11), we obtain the relations

$$\eta_{m+1}^{(0)} < \big(h_m^{(0)}/2\big)^{k_m-1} \eta_m^{(0)};$$
$$h_{m+1}^{(0)} \le h_m^{(k_m-1)} < h_m^{(0)} \le h_{m-1}^{(k_{m-1}-1)} < h_{m-1}^{(0)} \le \dots < h_N^{(0)}, \quad m \ge N,$$

which implies that

$$\begin{aligned}\eta_m^{(0)} &< \big(h_{m-1}^{(0)}/2\big)^{k_{m-1}-1} \eta_{m-1}^{(0)} \\ &< \big(h_{m-1}^{(0)}/2\big)^{k_{m-1}-1} \big(h_{m-2}^{(0)}/2\big)^{k_{m-2}-1} \eta_{m-2}^{(0)} < \dots \\ &< \big(h_N^{(0)}/2\big)^{S_{m-1}} \eta_N^{(0)},\end{aligned}$$

where $s_{m-1} = \sum\limits_{i=N}^{m-1} (k_i - 1),\ m = N+1,\ N+2, \dots,\ n$. With this in mind

$$\begin{gathered}\|u_m^{(k+1)} - u_m^{(k)}\| \le b_N \big(h_N^{(0)}/2\big)^{S_{m-1}+k} \eta_N^{(0)}, \\ k = 0,\ 1, \dots,\ k_m - 1; \quad m = N+1,\ N+2, \dots,\ n; \\ \|u_N^{(k+1)} - u_N^{(k)}\| \le b_N \big(h_N^{(0)}/2\big)^k \eta_N^{(0)}, \quad k = 0,\ 1, \dots,\ k_N - 1,\end{gathered} \tag{3.13}$$

so

$$\begin{aligned}\|u_{n+1}^{(0)} - u_N^{(0)}\| &\le \sum_{m=N}^{n} \sum_{k=0}^{k_m-1} \|u_m^{(k+1)} - u_m^{(k)}\| \\ &\le b_N \eta_N^{(0)} \left[\sum_{k=0}^{k_N-1} \big(h_N^{(0)}/2\big)^k + \sum_{m=N+1}^{n} \sum_{k=0}^{k_m-1} \big(h_N^{(0)}/2\big)^{S_{m-1}+k} \right] \\ &= b_N \eta_N^{(0)} \left[\sum_{k=0}^{S_n} \big(h_N^{(0)}/2\big)^k + \sum_{m=N}^{n-1} \big(h_N^{(0)}/2\big)^{S_m} \right] \\ &< b_N \eta_N^{(0)} G_N = r_N, \quad n \ge N,\end{aligned}$$

that is, each $u_{n+1}^{(0)}$ where $n \ge N$ (and also all $u_n^{(k)}$ $(k = 1,\ 2, \dots,\ k_n)$ by virtue of the Theorem 2 from [6]) belong to the ball $S(u_N^{(0)}, r_N)$. Thus, the feasibility of process (3.1) is proved.

Let's now show that the sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty}$, which is determined by formulas (3.1), converges in $S(u_N^{(0)}, r_N)$. Using (3.13) for any numbers $n \ge N$ and $p \in \mathbb{N}$

we write:

$$
\begin{aligned}
\|u_{n+p}^{(k_{n+p})} - u_n^{(k_n)}\| &\le \sum_{m=n+1}^{n+p} \|u_m^{(k_m)} - u_{m-1}^{(k_{m-1})}\| = \sum_{m=n+1}^{n+p} \|u_m^{(k_m)} - u_m^{(0)}\| \\
&\le \sum_{m=n+1}^{n+p} \sum_{k=0}^{k_m-1} \|u_m^{(k+1)} - u_m^{(k)}\| \\
&\le b_N \eta_N^{(0)} \sum_{m=n+1}^{n+p} \sum_{k=0}^{k_m-1} \big(h_N^{(0)}/2\big)^{S_{m-1}+k} \\
&= b_N \eta_N^{(0)} \left[\sum_{k=0}^{k_{n+1}-1} \big(h_N^{(0)}/2\big)^{S_n+k} + \sum_{m=n+2}^{n+p} \sum_{k=0}^{k_m-1} \big(h_N^{(0)}/2\big)^{S_{m-1}+k} \right] \\
&= b_N \eta_N^{(0)} \big(h_N^{(0)}/2\big)^{S_n} \left[\sum_{k=0}^{S_{n+p}-S_n} \big(h_N^{(0)}/2\big)^{k} + \sum_{m=n+1}^{n+p-1} \big(h_N^{(0)}/2\big)^{S_m-S_n} \right] \\
&< b_N \eta_N^{(0)} \big(h_N^{(0)}/2\big)^{S_n} 2H_N.
\end{aligned} \tag{3.14}
$$

Since $h_N^{(0)} < 2$, then $\|u_{n+p}^{(k_{n+p})} - u_n^{(k_n)}\| \to 0$ when $n \to \infty$, that means the fundamentality of the sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty} \subset S(u_N^{(0)}, r_N)$. By virtue of the completeness of the space X, there exists an element $u^* \in S(u_N^{(0)}, r_N)$ such that $u^* = \lim\limits_{n\to\infty} u_n^{(k_n)}$. Passing to the limit at $p \to \infty$ in (3.14) and denoting

$$
\begin{aligned}
V_n &= \lim_{p\to\infty} \left[\sum_{k=0}^{S_{n+p}-S_n} \big(h_N^{(0)}/2\big)^{k} + \sum_{m=n+1}^{n+p-1} \big(h_N^{(0)}/2\big)^{S_m-S_n} \right] \\
&= \sum_{k=0}^{\infty} \big(h_N^{(0)}/2\big)^{k} + \sum_{m=n+1}^{\infty} \big(h_N^{(0)}/2\big)^{S_m-S_n}, \quad n \ge N,
\end{aligned}
$$

we obtain the error estimate (3.4).

To prove that the limit u^* of the sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty}$ is a solution of equation (1.1), we consider the residual of method (3.1) on the n-th step ($n \ge N$):

$$
\begin{aligned}
\|Au_n^{(k_n)} - f\| \le \|Au_{n+1}^{(0)} - P_{n+1}Au_{n+1}^{(0)}\| + \|P_{n+1}Au_{n+1}^{(0)} - A_{n+1}u_{n+1}^{(0)}\| \\
+ \|A_{n+1}u_{n+1}^{(0)} - f_{n+1}\| + \|f_{n+1} - f\| \le \beta_{n+1} + \alpha_{n+1} + \eta_{n+1}^{(0)} + \gamma_{n+1}.
\end{aligned}
$$

Since, α_{n+1}, β_{n+1}, γ_{n+1}, $\eta_{n+1}^{(0)} \to 0$ when $n \to \infty$, and since the operator A is continuous due to Fréchet differentiability, then by tending $n \to \infty$ in the last inequality, we obtain that $Au^* = f$. The theorem is proved. □

Note that the projection-iteration implementation (3.1) of the Newton-like method generally converges more slowly than the projection-iteration process (2.5) based on the classical Newton's method. An exception is the case, when $\delta = 0$,

$\delta_n = 0$ ($n \geq N$) in formulas (2.6), (2.7), that leads to the transformation of method (3.1) into (2.5); in such a situation, the error estimate (3.4) for method (3.1) (or, equivalently, method (2.5)) is significantly overestimated, and for this case the more appropriate result is contained in Theorem 2.1.

For equation (1.1) under the Theorem 2.1 conditions, along with the projection-iteration method (2.5) based on the Newton's method, one can consider the approximation process based on the modified Newton's method:

$$u_n^{(k+1)} = u_n^{(k)} - [A_n'(u_n^{(0)})]^{-1}(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \dots,\ k_n - 1;$$
$$u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \geq N; \quad u_N^{(0)} \in \Omega_N \subset X,$$

and under the Theorem 3.1 conditions, along with the projection-iteration method (3.1), one can consider the approximation process based on the modified Newton-like method:

$$u_n^{(k+1)} = u_n^{(k)} - D_n(u_n^{(0)})(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \dots,\ k_n - 1;$$
$$u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \geq N; \quad u_N^{(0)} \in \Omega_N \subset X.$$

Such the projection-iteration processes (although they converge more slowly than the process (2.5) based on the Newton's method) are less laborious, since for each $n \geq N$ they use operators $[A_n'(u_n^{(0)})]^{-1}$ or $D_n(u_n^{(0)})$ which correspond only to the initial point $u_n^{(0)} \in \Omega_n$, and this obviously leads to a computational overhead reduction in numerical implementation.

We note, finally, that while solving nonlinear operator equations of the form (1.1), as follows from the proofs of Theorems 2.1, 3.1 on the convergence of projection-iteration methods based on the Newton's method and the Newton-like one respectively, the convergence of corresponding sequences $\{u_n^{(k_n)}\}_{n=N}^{\infty}$ (when $n \to \infty$) towards an exact solution u^* in X occurs under an arbitrary choice of numbers k_n. However, to prevent a sharp increase with increasing n of amount of computations needed to find the next approximation, we have to consider a problem of the appropriate choice of numbers k_n at each $n \geq N$. Some recommendations on this issue have been given in [2]. In particular, there has been considered a way to choose numbers k_n so that the element $u_n^{(k_n)}$ would be a good initial approximation for the $(n+1)$-th approximate equation of the form (1.2), that is, that the residual $A_{n+1}u_{n+1}^{(0)} - f_{n+1}$ would have, if possible, a small value. The idea underlying this way to choose numbers k_n also makes it possible to determine the most acceptable number $n + p$ ($p \geq 1$) of the approximate equation following the n-th one in the sequence of equations (1.2). Some other ways to choose numbers k_n in projection-iteration methods of solving nonlinear equations as well as their application in solving specific problems, can be found in [3,7,9–11].

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