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Truncated Weibull Lomax Distribution and its Statistical Inference

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Abstract: A new three-parameter lifetime model, called the Truncated Weibull Lomax (TWL) distribution, is introduced. We investigate the density (pdf), distribution function (cdf), survival function (sf), hazard function (hrf), quantile function, r^{th} moment, incomplete moments and order statistics. Maximum Likelihood methods to estimate the TWL distribution parameters are studied. One real data set is applied to illustrate the flexibility of the TWL model compared with some Known models.

Keywords: Lomax distribution, Truncated Weibull-G family, Moments, Maximum likelihood estimation.

1. Introduction

In the recent years, many researchers are interested to expand generating family in order to obtain better fit for data analyzing. Some popular-known generating family are: The the beta-G by Eugene [8], gamma-G (type-1) by Zografos and Balakrishnan [21], gamma-G (type-2) studied by Ristic and Balakrishnan [18], Weibull-X family of distributions of Alzaatreh [3], Logistic-G introduced by Torabi and Montazeri [20], exponentiated Weibull-G studied by Hassan and Elgarhy [12], Kumaraswamy Weibull-G by Hassan and Elgarhy [13], Garhy-G of Elgarhy [7], Type II half logistic-G by Hassan [14], exponentiated extended-G introduced by Elgarhy [6], Najarzadegan [17] proposed a new truncated Weibull-G (TW-G), odd Frechet-G proposed by Haq and Elgarhy [10], Muth-G studied by Almarashi and Elgarhy [2], Elbatal [5] proposed a new alpha power transformation family of distributions among others. The cdf of TW-G family is given by

$$F(x;\theta,\lambda) = (1-e^{-\theta})^{-1}(1-e^{-\theta(G(x))^{\lambda}});\ x\in R,\ \theta,\lambda>0, \tag{1}$$

We interested in make $\theta=1$, then the equation (1) reduce to

$$F(x;\lambda) = A(1-e^{-(G(x))^{\lambda}});,\ x\in R,\ \lambda>0, \tag{2}$$

The corresponding pdf to (2) is

$$f(x;\lambda) = A\lambda g(x)\,(G(x))^{\lambda-1}\,e^{-(G(x))^{\lambda}};,\ x\in R,\ \lambda>0, \tag{3}$$

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where $A = (1 - e^{-1})^{-1}$, λ is the shape parameter and $G(.)$ is cdf of any baseline distribution. The Lomax (L) distribution is studied by Lomax [16] it has been widely applied in some areas, such as, analysis of income and wealth data, modeling business failure data, model firm size and queuing problems (see for example Hassan and Al-Ghamdi [11]. The cdf and pdf of the L distribution are given, respectively, by

$$G(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha};\ x > 0, \alpha, \beta > 0, \tag{4}$$

and

$$g(x) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1};\quad x > 0, \alpha, \beta > 0. \tag{5}$$

where, α and β are shape and scale parameters respectively. The aim of this article is to introduce a new three parameter life time model based on the TW-G family. The TWL model provides more flexible model. We hope that the new model will attract wider applications in some areas. This paper can be organized as follows. In the next section, the TWL distribution is introduced. Section 3 gives some mathematical properties of the TWL distribution. The maximum likelihood method is implemented to obtain the estimators of the parameters in Section 4. Application to a real data illustrating the performance of the new model is given in Section 5. Finally, Conclucions appear in the Section 6.

2. The TWL Model

A random variable is said to has the TWL model with vector parameters φ, where $\varphi = (\alpha, \beta, \lambda)$, by inserting (4), (5) in (2), (3) then, the cdf and pdf of TWL is given by

$$F(x;\varphi) = A\left(1 - e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right);\ \alpha, \beta, \lambda, x > 0. \tag{6}$$

and

$$f(x;\varphi) = \frac{A\alpha\lambda}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1}\left(1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda-1} e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}. \tag{7}$$

The sf and the hrf , reversed hrf and cumulative hrf are, respectively, given by

$$R(x;\varphi) = 1 - A\left(1 - e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right),$$

$$h_{TWFr}(x;\varphi) = \frac{\frac{A\alpha\lambda}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1}\left(1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda-1} e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}}{1 - A\left(1 - e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right)},$$

$$\tau(x;\varphi) = \frac{\frac{\alpha\lambda}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1}\left(1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda-1} e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}}{\left(1 - e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right)},$$

$$\text{and}\quad H(x;\varphi) = -\ln\left[1 - A\left(1 - e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right)\right].$$

Some descriptive pdf, cdf, sf and hrf plots of $X \sim TWL(\varphi)$ are illustrated below for specific parameter choices of φ (see Figures 1 and 2).

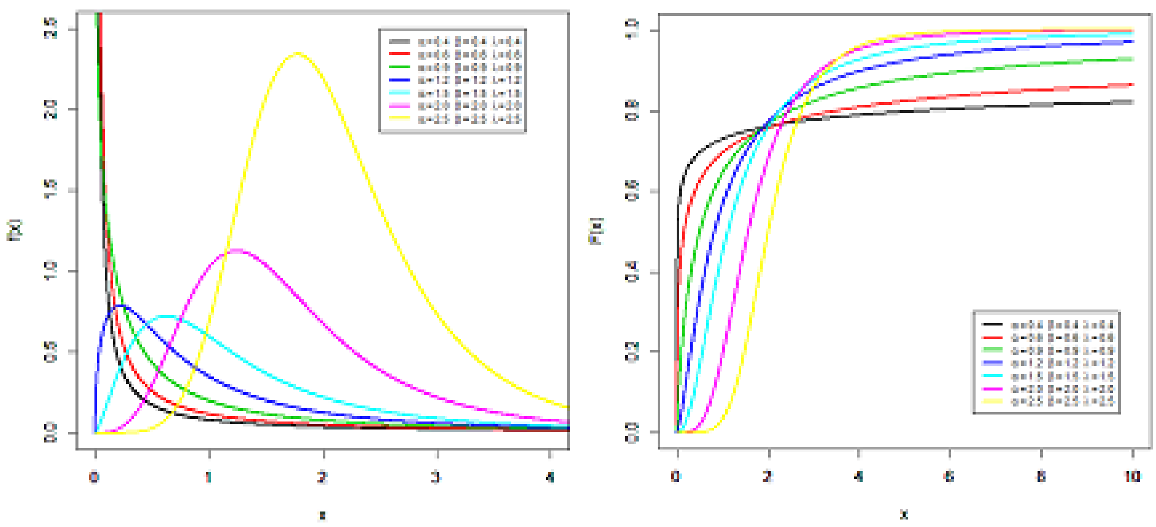

Figure 1. **Plots of the pdf and cdf of the TWL distribution for some different values of parameter.**

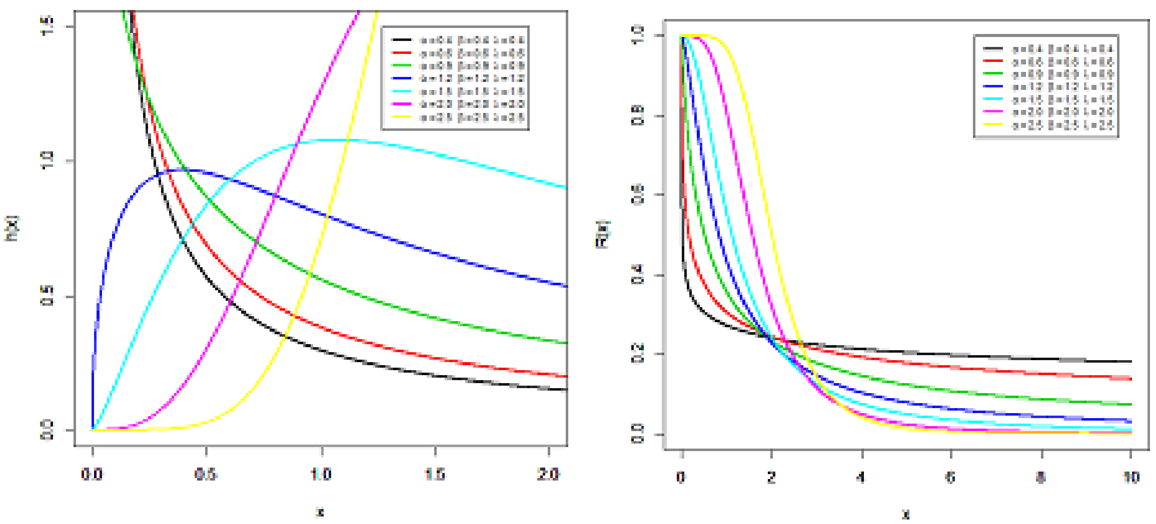

Figure 2. **Plots of the hrf and sf of the TWL distribution for some different values of parameter.**

As seen from Figure 1, of TWL distribution can be uni-model, decreasing, symmetric and right skewed. Also, the hrf of TWL distribution can be J- shaped, decreasing and unimodal as seen from Figure 2.

3. Main Mathematical Properties

This section studies some mathematical properties of TWL model.

3.1. Quantile Function

The quantile function of the TWL can be generated by inverting cdf (6) as follows

$$Q(u) = \beta \left[1 - \left(\ln \left(\frac{A}{A-u} \right) \right)^{\frac{1}{\lambda}} \right]^{\frac{-1}{\alpha}} - \beta. \tag{8}$$

Simulating the TWL random variable is straightforward. If is U a uniform variate in the unit interval $(0, 1)$, then the random variable $X = Q(u)$ follows (8).

3.2. Useful Expansion

In this subsection expansions of the pdf and cdf for TWL distribution are investigated. Using the power series for the exponential function, we obtain

$$e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda i}. \tag{9}$$

Then,

$$f(x;\varphi) = \frac{A\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda(i+1)-1}. \tag{10}$$

Using the generalized binomial theorem, for $\beta > 0$ and $|z| < 1$,

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i. \tag{11}$$

Then, by applying the binomial expansion (11) in (10), the pdf of TWL distribution becomes

$$f(x;\varphi) = \frac{\alpha}{\beta}\sum_{i=0}^{\infty} \eta_i \left(1+\frac{x}{\beta}\right)^{-\alpha(j+1)-1}. \tag{12}$$

Where $\eta_i = A\lambda \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{i!}\binom{\lambda(i+1)-1}{j}$. Using binomial theorem for $[F(x)]^h$, where h is an integer, leads to:

$$[F(x;\varphi)]^h = A^h\left(1-e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right)^h.$$

Then,

$$[F(x)]^h = \sum_{k=0}^{\infty} \eta_k \left(1+\frac{x}{\beta}\right)^{-\alpha k}. \tag{13}$$

Where,

$$\eta_k = \sum_{m=0}^{h}\sum_{u=0}^{\infty} \frac{(-1)^{m+u+k} m^u}{u!}\binom{h}{m}\binom{\lambda u}{k}\binom{ap}{q}\binom{\beta t+\kappa-1}{\kappa}\binom{\beta t+\kappa}{\ell_2}.$$

3.3. The probability Weighted Moments (PMW)

The PWMs are expectations of multiplication of two certain functions of a random variable X defined as

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx. \tag{14}$$

Inserting (12) and (13) in (14), replacing h with s, leads to:

$$\tau_{r,s} = \frac{\alpha}{\beta}\sum_{i,k=0}^{\infty} \eta_i\eta_k \int_0^{\infty} x^r \left(1+\frac{x}{\beta}\right)^{-\alpha(k+j+1)-1} dx. \tag{15}$$

Setting $y = \frac{x}{\beta}$ then,

$$\tau_{r,s} = \alpha\beta^r \sum_{i,k=0}^{\infty} \eta_i\eta_k \int_0^{\infty} y^r (1+y)^{-\alpha(k+j+1)-1} dx.$$

Again make the following transformation $y = \frac{w}{1-w}$ then,

$$\tau_{r,s} = \alpha\beta^r \sum_{i,k=0}^{\infty} \eta_i\eta_k \int_0^{1} w^r (1-w)^{\alpha(k+j+1)-r-1} dx.$$

Then the PWM of TWL becomes

$$\tau_{r,s} = \alpha\beta^r \sum_{i,k=0}^{\infty} \eta_i\eta_k B(r+1, \alpha(k+j+1)-r).$$

3.4. Moments

In this subsection, we intend to derive the moments and the moment generating function of the new model. By definition, r^{th} moment of r.v X can be achieved from

$$\mu_r' = \int_0^\infty x^r f(x;\varphi)\,dx.$$

then its rth moment of X is

$$\mu_r' = \alpha\beta^r \sum_{i=0}^{\infty} \eta_i \eta_k B(r+1, \alpha(j+1)-r).$$

The moment generating function (mgf) of the TWL distribution is

$$M_x(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} f(x;\varphi)dx = \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x;\varphi)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right)$$

$$M_x(t) = \alpha\beta^r \sum_{i,r=0}^{\infty} \frac{\eta_i \eta_k t^r}{r!} B(r+1, \alpha(j+1)-r).$$

3.5. Order Statistics

The density of the k^{th} order statistic, for $r = 1, ..., n$ from independent and identically distributed random variables $X_1, X_2, ..., X_n$ is given by

$$f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} f(x) F(x)^{k-1} [1-F(x)]^{n-k}. \tag{16}$$

Substituting (6) and (7) in (16), then

$$f_{X_{(k)}}(x) = \frac{\alpha\lambda A^k}{\beta B(k,n-k+1)} \left(1+\frac{x}{\beta}\right)^{-\alpha-1} \left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda-1} e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}$$
$$\times \left(1-e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right)^{k-1} \left[1-A\left(1-e^{-\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{\lambda}}\right)\right]^{n-k}. \tag{17}$$

Setting $r = 1$ and $r = n$, in (17), we obtain the pdf of the first and largest order statistics of the TWL distribution.

4. Maximum Likelihood Estimators

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the TWL distribution with set of parameters $\varphi = (\alpha, \beta, \lambda)$. The likelihood function based on the observed random sample of size n from density (6) is given by:

$$L(\varphi|x) = \left(\frac{\alpha\lambda A}{\beta}\right)^n \prod_{i=1}^{n} \left(1+\frac{x_i}{\beta}\right)^{-\alpha-1} \left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right)^{\lambda-1} e^{-\left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right)^{\lambda}}.$$

The partial derivatives of the log-likelihood function, say $\ln \ell$, with respect to the parameters are given by:

$$\frac{\partial \ln \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\beta}\right) + (\lambda-1)\sum_{i=1}^{n} \frac{\left(1+\frac{x_i}{\beta}\right)^{-\alpha} \ln\left(1+\frac{x_i}{\beta}\right)}{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}} - \lambda \sum_{i=1}^{n} \left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right)^{\lambda-1} \left(1+\frac{x_i}{\beta}\right)^{-\alpha} \ln\left(1+\frac{x_i}{\beta}\right)$$

$$\frac{\partial \ln \ell}{\partial \beta} = -\frac{n}{\beta} + (\alpha+1)\sum_{i=1}^{n} \frac{\frac{x_i}{\beta^2}}{\left(1+\frac{x_i}{\beta}\right)} - \alpha(\lambda-1)\sum_{i=1}^{n} \frac{\left(1+\frac{x_i}{\beta}\right)^{-\alpha-1} \frac{x_i}{\beta^2}}{1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}} + \alpha\lambda \sum_{i=1}^{n} \left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right)^{\lambda-1} \left(1+\frac{x_i}{\beta}\right)^{-\alpha-1} \frac{x_i}{\beta^2}$$

and,

$$\frac{\partial \ln \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \ln\left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right) - \sum_{i=1}^{n} \left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right)^{\lambda} \ln\left(1-\left(1+\frac{x_i}{\beta}\right)^{-\alpha}\right).$$

The ML estimators of the model parameters are determined by solving numerically the non-linear equations $\frac{\partial \ln \ell}{\partial \beta} = 0$, $\frac{\partial \ln \ell}{\partial \alpha} = 0$, and $\frac{\partial \ln \ell}{\partial \lambda} = 0$, simultaneously.

5. Data Analysis

In this section, we provide an application to a real data set to assess the flexibility of the TIIHLL model. In order to compare the TWL model with other fitted distributions. We compare the fits of the TWL model with McDonald log-logistic (McLL) (Tahir [19]), McDonald Weibull (McW) (Cordeiro [4]), new modified Weibull (NMW) (Almalki and Yuan, [1]) and beta Weibull (BW) (Lee [15]).

The data set: is taken from(Gross and Clark, [9]) on the relief times of twenty patients receiving an analgesic.

The ML estimates along with their standard error (SE) of the model parameters are provided in Tables (1) and (2). In the same tables, the analytical measures including; minus double log-likelihood (-2log L), Anderson Darling statistic (A^*), Cramér-von Mises statistic (W^*), Akaike Information Criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) are presented.

Table 1 list the MLEs of the model parameters and their corresponding standard whereas errors the values of -2LogL, AIC, CAIC, HQIC, A^* and W^* are given in Table 2.

Model	MLE and SE					
$TWL(\alpha,\beta,\lambda)$	5.248 (4.606)	0.787 (2.093)	424.916 (1882)	-	-	-
$McLL(\alpha,\beta,a,b,c)$	0.8811 (0.109)	2.0703 (3.693)	19.2254 (22.341)	32.0332 (43.077)	1.9263 (5.165)	-
$McW(\alpha,\beta,a,b,c)$	2.7738 (6.38)	0.3802 (0.188)	79.108 (119.131)	17.8976 (39.511)	3.0063 (13.968)	-
$NMW(\alpha,\beta,\gamma,\delta,\theta)$	0.1215 (0.056)	2.7837 (20.37)	8.227×10^{-5} (1.512×10^{-3})	0.0003 (0.025)	2.7871 (0.428)	-
$BW(\alpha,\beta,a,b)$	0.8314 (0.954)	0.6126 (0.34)	29.9468 (40.413)	11.6319 (21.9)	-	-

Table 1. **MLEs and their standard errors (in parentheses) for the relief times data.**

Model	**- 2log L**	**AIC**	**CAIC**	**BIC**	**HQIC**	A^*	W^*
TWL	30.797	36.797	38.297	34.7	37.38	0.15943	0.02887
McLL	33.854	43.854	48.14	40.359	44.826	0.46199	0.07904
McW	33.907	43.907	48.193	40.412	44.879	0.46927	0.08021
NMW	41.173	51.173	55.459	47.678	52.145	1.0678	0.17585
BW	34.396	42.396	45.063	39.6	43.174	0.51316	0.0873

Table 2. **Measures of goodness-of-fit statistics for the relief times data.**

Table 3 compares the fits of the TIIHLL distribution with the McLL, McW, NMW and BW and modelss. The values in Table 2 show that the TWL model has the lowest values for -2LogL, AIC, CAIC, HQIC, A^* and W^* among all fitted distributions. So, it could be chosen as the best model. The fitted pdf and pp plot for the TWL model are displayed in Figure 3. Figure 4. shows the estimated cdf and sf for the TWL model. From these plots it is evident that the new model provides close fit to the data.

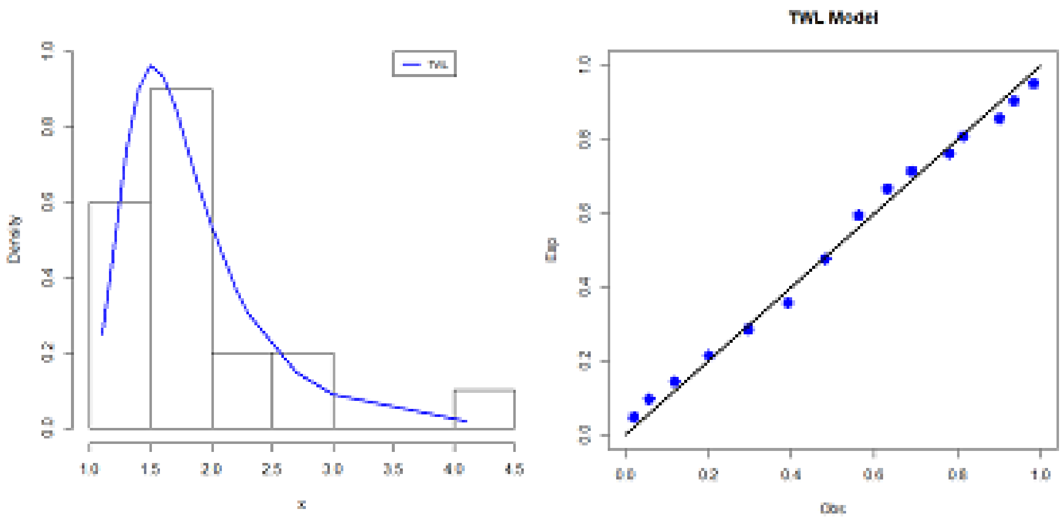

Figure 3. **The empirical pdf and pp plot of the TWL model.**

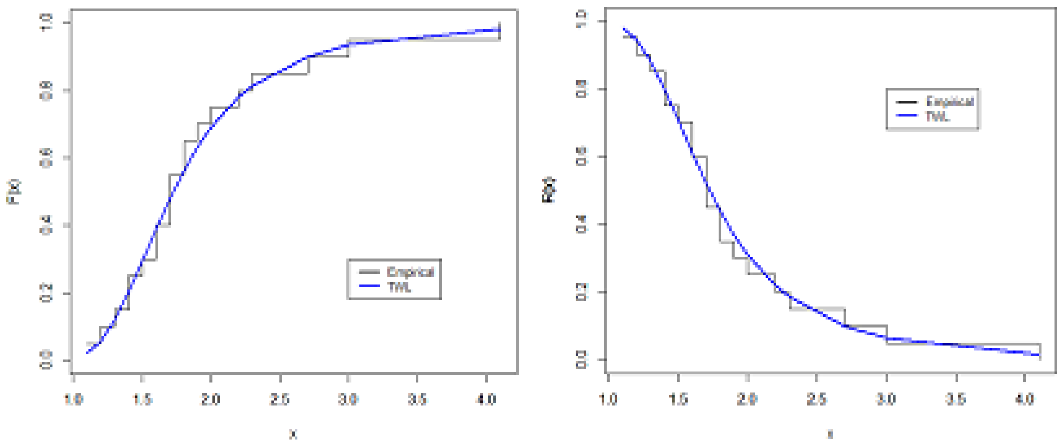

Figure 4. **The empirical cdf and Kaplan-Meir survival of the TWL model.**

6. Concluding Remarks

In this article, we propose a new three-parameter model, called the TWL distribution. We derive explicit expressions for the moments, probability weighted moments, quantile, generating functions, and order statistics. We discuss the maximum likelihood estimation of the model parameters. One application illustrate that the TWL distribution provides consistently better fit than other models.

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