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On open problems concerning Riemann-Liouville fractional integral inequality

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ABSTRACT

In this paper answers to problems proposed by Zoubir Dahmani and Hanane El Ard in [1] are given.

1. Introduction

In the paper [1], authors studied very interesting integral inequality and proved the following results :

Theorem 1. Let f, g and h be positive and continuous functions on $[0, \infty)$, such that

$$(g(\tau) - g(\rho)) \left(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0; \tau, \rho \in [0, t], t > 0$$

then we have

$$\frac{J^{\alpha}(f(t))}{J^{\alpha}(h(t))} \geq \frac{J^{\alpha}(gf(t))}{J^{\alpha}(gh(t))}$$

for all $\alpha > 0, t > 0$.

Theorem 2. Let f, g and h be positive and continuous functions on $[0, \infty)$, such that

$$(g(\tau) - g(\rho)) \left(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0; \tau, \rho \in [0, t], t > 0$$

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then for all $\alpha > 0, w, t > 0$, we have

$$\frac{J^{\alpha}(f(t)) \cdot J^{w}(gh(t)) + J^{w}(f(t)) \cdot J^{\alpha}(gh(t))}{J^{\alpha}(h(t)) \cdot J^{w}(gf(t)) + J^{w}(h(t)) \cdot J^{\alpha}(gf(t))} \geq 1$$

Theorem 3. Let f and h two positive continuous functions and $f \leq h$ on $[0, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0, \infty)$, then for any $p \geq 1, \alpha > 0, t > 0$, the inequality

$$\frac{J^{\alpha}(f(t))}{J^{\alpha}(h(t))} \geq \frac{J^{\alpha}(f^{p}(t))}{J^{\alpha}(h^{p}(t))}$$

is valid.

Theorem 4. Let f and h be two positive continuous functions and $f \leq h$ on $[0, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0, \infty)$, then for any $p \geq 1, \alpha > 0, w > 0, t > 0$, we have

$$\frac{J^{\alpha}(f(t)) \cdot J^{w}(h^{p}(t)) + J^{w}(f(t)) \cdot J^{\alpha}(h^{p}(t))}{J^{\alpha}(h(t)) \cdot J^{w}(f^{p}(t)) + J^{w}(h(t)) \cdot J^{\alpha}(f^{p}(t))} \geq 1$$

Next, they proposed the following open problems :

Open Problem 1. Under what conditions does the inequality

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\mu} \cdot f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu} \cdot f^{\gamma}(t)\right)} \tag{1}$$

hold for $\alpha, \gamma, \delta, \beta, \mu$?

Open Problem 2. Under what conditions, the inequality

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq \left(J^{\alpha}\left(t^{\alpha} f^{\beta}(t)\right)\right)^{\gamma} \tag{2}$$

hold for $\alpha, \beta, \gamma, \delta$?

Open Problem 3. Under what conditions does the inequality

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{\left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{r}}{\left(J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)\right)^{s}} \tag{3}$$

hold for $\alpha, \beta, \gamma, \delta, r, s$?

Our purpose in this paper is to establish answer to above given open problems. The paper has been organized as follows, in Section 2, we give basic concepts related to fractional calculus. In Section 3, we establish the main results. And in Section 4, we give the conclusions.

2. Preliminaries

Firstly we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. A real valued function $f(t), t > 0$ is said to be in the space $C_{\mu}, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^{p} f_{1}(t)$ where $f_{1}(t) \in C([0, \infty))$.

Definition 2. A function $f(t), t > 0$ is said to be in the space $C_{\mu}^{n}, \mu \in \mathbb{R}$, if $f^{(n)} \in C_{\mu}$.

Definition 3. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a function $f \in C_{\mu}, (\mu \geq -1)$ on $[0, \infty)$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \tau > 0$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 4. We say that two functions f and g have the same sense of variation on $[0, \infty)$ if

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0, \tau, \rho \in (0, t), t > 0$$

For more details, one can consult [2, 12].

3. Main Results

In this section we give the answer to the above open problems. Our main results are Theorem 5, Theorem 6 and Theorem 7 which will prove the open problems 1, 2 and 3, respectively. Firstly, we have

Theorem 5. Let $\alpha > 0, \mu > 0, \delta > 0, \beta > 0, \gamma > 0$ and let $f(t)$ be a positive and continuous function on $(0, \infty)$ such that

$$(\tau^{\mu} - \rho^{\mu}) \left(f^{\delta+\beta-\gamma}(\rho) - f^{\delta+\beta-\gamma}(\tau)\right) \geq 0 \tag{4}$$

$$(\tau^{\mu} - \rho^{\mu}) \left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(t)}\right) \geq 0 \tag{5}$$

$$\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right) \left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \geq 0 \tag{6}$$

for all $\tau, \rho \in (0, t]$. Then we have

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\mu} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\gamma}(t)\right)} \tag{7}$$

To prove Theorem 5, we need the following lemmas

Lemma 1. Let $\alpha > 0, \mu > 0, \gamma > 0, \delta > 0, \beta > 0, t > 0$ and let $f(t)$ be a positive and continuous function on $(0, \infty)$ such that

$$\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right) \left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \geq 0 \tag{8}$$

for all $\tau, \rho \in (0, t]$. Then we have

$$\frac{J^{\alpha}\left(t^{\mu} f^{\gamma}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\mu} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\delta+\beta}(t)\right)} \tag{9}$$

Proof. It is easy to see that

$$\left(f^{\beta-\gamma}(\tau)-f^{\beta-\gamma}(\rho)\right)\left(\frac{\rho^{\mu}f^{\gamma}(\rho)}{\rho^{\mu}f^{\delta+\gamma}(\rho)}-\frac{\tau^{\mu}f^{\gamma}(\tau)}{\tau^{\mu}f^{\delta+\gamma}(\tau)}\right)\geq 0 \tag{10}$$

By using functions

$$\overline{f}:=t^{\mu}f^{\gamma} \tag{11}$$

$$g:=f^{\beta-\gamma} \tag{12}$$

$$h:=t^{\mu}f^{\delta+\gamma} \tag{13}$$

we obtain the following inequality

$$\left(g(\tau)-g(\rho)\right)\left(\frac{\overline{f}(\rho)}{h(\rho)}-\frac{\overline{f}(\tau)}{h(\tau)}\right)\geq 0 \tag{14}$$

By using Theorem 1 we deduce that

$$\begin{aligned}
&\frac{J^{\alpha}(t^{\mu}f^{\gamma}(t))}{J^{\alpha}\left(t^{\mu}f^{\delta+\gamma}(t)\right)}=\frac{J^{\alpha}\left(\overline{f}(t)\right)}{J^{\alpha}(h(t))}\\
&\geq\frac{J^{\alpha}\left(g\cdot\overline{f}(t)\right)}{J^{\alpha}(g\cdot h(t))}\\
&=\frac{J^{\alpha}\left(f^{\beta-\gamma}(t)\cdot t^{\mu}\cdot f^{\gamma}(t)\right)}{J^{\alpha}\left(f^{\beta-\gamma}(t)\cdot t^{\mu}\cdot f^{\delta+\gamma}(t)\right)},\\
&=\frac{J^{\alpha}\left(t^{\mu}\cdot f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu}\cdot f^{\delta+\beta}(t)\right)},
\end{aligned}$$

and thus we complete the proof.

Lemma 2. Let $\alpha>0,\mu>0,\gamma>0,\delta>0,\beta>0$ and let $f(t)$ be a positive and continuous function on $(0,\infty)$ such that

$$\left(\tau^{\mu}-\rho^{\mu}\right)\left(\frac{1}{f^{\delta}(\rho)}-\frac{1}{f^{\delta}(\tau)}\right)\geq 0, \tag{15}$$

$$\left(f^{\beta-\gamma}(\tau)-f^{\beta-\gamma}(\rho)\right)\left(\frac{1}{f^{\delta}(\rho)}-\frac{1}{f^{\delta}(\tau)}\right)\geq 0, \tag{16}$$

for all $\tau,\rho\in(0,t]$. Then we have

$$\frac{J^{\alpha}\left(f^{\gamma}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)}\geq\frac{J^{\alpha}\left(t^{\mu}f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu}f^{\delta+\beta}(t)\right)} \tag{17}$$

Proof. It is easy to see that

$$\left(\tau^{\mu}-\rho^{\mu}\right)\left(\frac{f^{\gamma}(\rho)}{f^{\delta+\gamma}(\rho)}-\frac{f^{\gamma}(\tau)}{f^{\delta+\gamma}(\tau)}\right)\geq 0 \tag{18}$$

By using functions

$$\overline{f} := f^{\gamma}(t) \tag{19}$$

$$h := f^{\delta+\gamma}(t) \tag{20}$$

$$g := t^{\mu} \tag{21}$$

we obtain the following inequality

$$(g(\tau) - g(\rho))\left(\frac{\overline{f}(\rho)}{h(\rho)} - \frac{\overline{f}(\tau)}{h(\tau)}\right) \geq 0 \tag{22}$$

By using Theorem 1, we deduce that

$$\frac{J^{\alpha}\left(\overline{f}\right)}{J^{\alpha}(h)} \geq \frac{J^{\alpha}\left(g\overline{f}\right)}{J^{\alpha}(gh)} \tag{23}$$

or

$$\frac{J^{\alpha}\left(f^{\gamma}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\mu}f^{\gamma}(t)\right)}{J^{\alpha}\left(t^{\mu}f^{\delta+\gamma}(t)\right)} \tag{24}$$

The condition (16) allow us to use the Lemma 1. Therefore

$$\frac{J^{\alpha}\left(t^{\mu}f^{\gamma}(t)\right)}{J^{\alpha}\left(t^{\mu}f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\mu}f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu}f^{\delta+\beta}(t)\right)} \tag{25}$$

Combining (24) with (25) we obtain (17), and thus we complete the proof.
We now give the proof of Theorem 5
Proof. It is easy to see that inequality (4) can be rewritten as

$$\left(\tau^{\mu} - \rho^{\mu}\right)\left(\frac{f^{\delta+\beta}(\rho)}{f^{\gamma}(\rho)} - \frac{f^{\delta+\beta}(\tau)}{f^{\gamma}(\tau)}\right) \geq 0 \tag{26}$$

By using functions

$$\overline{f} := f^{\delta+\beta}(t) \tag{27}$$

$$g := t^{\mu} \tag{28}$$

$$h := f^{\gamma}(t) \tag{29}$$

We obtain the following inequality

$$(g(\tau) - g(\rho))\left(\frac{\overline{f}(\rho)}{h(\rho)} - \frac{\overline{f}(\tau)}{h(\tau)}\right) \geq 0 \tag{30}$$

The inequality (30) allow us to use Theorem 1. Therefore

$$\frac{J^{\alpha}\left(\overline{f}(t)\right)}{J^{\alpha}(h(t))} \geq \frac{J^{\alpha}\left(g \cdot \overline{f}(t)\right)}{J^{\alpha}(g \cdot h(t))} \tag{31}$$

or equivalently

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \cdot J^{\alpha}\left(t^{\mu} \cdot f^{\gamma}(t)\right) \geq J^{\alpha}\left(f^{\gamma}(t)\right) \cdot J^{\alpha}\left(t^{\mu} \cdot f^{\delta+\beta}(t)\right) \tag{32}$$

Inequalities (5)-(6) allow us to use the Lemma 2. So, we can write

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \cdot J^{\alpha}\left(t^{\mu} \cdot f^{\gamma}(t)\right) \geq J^{\alpha}\left(f^{\gamma}(t)\right) \cdot J^{\alpha}\left(t^{\mu} \cdot f^{\delta+\beta}(t)\right) \geq J^{\alpha}\left(f^{\delta+\gamma}(t)\right) \cdot J^{\alpha}\left(t^{\mu} \cdot f^{\beta}(t)\right) \tag{33}$$

From (33), the inequality (7) follows and thus we complete the proof.
Secondly we have

Theorem 6. Let $\alpha > 0, \delta > 0, \beta > 0$ and $f(t)$ be a continuous function on such $(0, \infty)$ that $f(t) \geq t$ on $(0, \infty)$.

a) If $0 < \gamma < 1$ and $J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right) \geq 1$, then we have

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq \left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{\gamma} \tag{34}$$

b) If $\gamma \geq 1$ and $0 < J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right) < 1$, then we have

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq \left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{\gamma} \tag{35}$$

Proof. We know that $f(t) \geq t$ on $(0, \infty)$. So

$$f(t) > 0 \tag{36}$$

Combining (36) with $f(t) \geq t$ we obtain

$$f^{\delta+\beta}(t) = f^{\delta}(t) \cdot f^{\beta}(t) \geq t^{\delta} \cdot f^{\beta}(t) \tag{37}$$

Using Riemann-Liouville fractional integral, we deduce that

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right) \tag{38}$$

a. Combining $0 < \gamma < 1$,

$$J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right) \geq 1 \tag{39}$$

with (38), we obtain

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right) \geq \left(J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right)\right)^{\gamma} \tag{40}$$

from (39), inequality (34) follows.
b. Combining $\gamma \geq 1, 0 < J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right) < 1$ with (38), we obtain

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right) \geq \left(J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right)\right)^{\gamma} \tag{41}$$

from (40), inequality (35) follows and thus we complete the proof.
Lastly we have
Theorem 7. Let $\alpha > 0, \delta > 0, \beta > 0, \gamma > 0, 0 < r < 1, s \geq 1$ and $f(t)$ be a continuous and positive function on $(0, \infty)$ such that

$$J^{\alpha}\left(t^{\delta} \cdot f^{\gamma}(t)\right) \geq 1, J^{\alpha}\left(t^{\delta} \cdot f^{\beta}(t)\right) \geq 1 \tag{42}$$

$$\left(\tau^{\mu} - \rho^{\mu}\right)\left(f^{\delta+\beta-\gamma}(\rho) - f^{\delta+\beta-\gamma}(\tau)\right) \geq 0 \tag{43}$$

$$\left(\tau^{\mu} - \rho^{\mu}\right)\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \geq 0 \tag{44}$$

$$\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right)\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \geq 0 \tag{45}$$

For all $\tau, \rho \in (0, t]$. Then we have

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{\left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{r}}{\left(J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)\right)^{s}} \tag{46}$$

Proof. Inequalities (42)-(44) allow us to use the Theorem 5. So, for $\mu = \delta$, we obtain

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)}$$

Combining inequalities $s \geq 1, 0 < r < 1$ with (41), yields

$$\left(J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)\right)^{s} \geq J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)$$

$$\left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{r} \leq J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)$$

by using inequalities (46)-(48), we obtain

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)} \geq \frac{\left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{r}}{\left(J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)\right)^{s}}$$

From (49), inequality (45) follows and thus we complete the proof.

4. Conclusion

With Theorems 5, 6 and 7 we have established solutions to the above open problems 1, 2 and 3, respectively.

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