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Existence, Uniqueness and Stability Solutions of fractional Integro-Differential Equations

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AbstractThe aim of this work is to study the existence, uniqueness and stability solutions of fractional integro-differential equations by using Picard approximation method. Furthermore the study of such fractional integro-differential equations needs us improve and extend the above method and thus the fractional integro-differential equations that we have introduced in this study become more general and detailed than those introduced by Faris results.

Keywords Existence, uniqueness and stability solution,fractional order, nonlinear system, integro-differential equations, Picard approximation method

Introduction

There are many subjects in Physics and technology using Mathematical methods that depends on the fractional differential and integro-differential equations, and it becomes clear that the existence, uniqueness and stability solutions and it's algorithm structure from more important problems in the present time, where many of studies and researches [1,2,6,7,8,10,11,15] dedicates for treatment the autonomous and non-autonomous systems and specially with the fractional differential and integro-differential equations and which is dealing in the general shape with the problems about the solutions of theory and modern methods in the quality treatment for the fractional differential and integro-differential equations.

Many authors create and develop functional analysis, numerical analysis and numerical methods and schemes to investigate the solution of fractional differential and integro-differential equations describing non-linear oscillations. These methods [3,9,13,14] make extensive use of topologicalconcepts and are convenient means of qualitative investigations of the solutions for clear up the questions of existence, uniqueness and stability of a solutions, also have been especially successful investigations systems of fractional differential and integro-differential equations.

Faris [5] has been used Picard approximation methodof the solutions of integro-differential equations which were introduced by Rama [13]to study the solution of integro-differential equations which has the form

$$\left.\begin{aligned}\frac{dx}{dt} &= Ax + f\left(t, x, y, \int_{a(t)}^{b(t)} h\big(s, x(s), y(s)\big)ds\right)\\ \frac{dy}{dt} &= By + g\left(t, x, y, \int_{0}^{t} p\big(s, x(s), y(s)\big)ds\right),\end{aligned}\right\}$$

where $x, y \in D \subset R^n$, $z, w \in D_1 \subset R^m$ andD, D_1 are closed and bounded domains.

In this study we have employed the Picard approximation method of Rama [13] to investigate the existence, uniqueness and stability solution of some classes of fractional integro-differential equations. The study of such fractional integro-differential equations leads to improving and extending the above method and Butris results.

Thus fractional integro-differential equations that we have introduced in the study, become more general and detailed than those introduced by Faris[5].

Consider the following integro-differential equations which have the fractional order:-

$$\left.\begin{aligned} x^{(\alpha)}(t) &= Ax + f\left(t, x, y, \int_{a(t)}^{b(t)} (t-\tau)^{\alpha-1} h(\tau, x(\tau), y(\tau)) d\tau\right) \\ x^{(\alpha-1)}(t) &= x_0, \quad 0 < \alpha < 1 \\ y^{(\alpha)}(t) &= By + g\left(t, x, y, \int_{0}^{t} (t-\tau)^{\alpha-1} p(\tau, x(\tau), y(\tau)) d\tau\right) \\ y^{(\alpha-1)}(t) &= y_0, \quad 0 < \alpha < 1 \end{aligned}\right\} \cdots (1.1)$$

where $x, y \in G \subset R^n$, $z, w \in G_1 \subset R^m$ and G, G_1 are closed and bounded domains.

Assume thatthe vector functions $f(t, x, y, z)$, $g(t, x, y, w)$, $h(t, x, y)$ and $p(t, x, y)$ are defind on the domain

$(t, (x, y), (z, w)) \in R^1 \times G \times G_1$. $\cdots$ (1.2)

Also the abovefunctionssatisfyingthe following inequalities:-

$$\left.\begin{aligned} \|f(t,x,y,z)\| &\le N_1 \\ \|g(t,x,y,w)\| &\le N_2 \\ \|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| &\le L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| + L_3\|z_1 - z_2\| \\ \|g(t,x_1,y_1,w_1) - g(t,x_2,y_2,w_2)\| &\le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + K_3\|w_1 - w_2\| \\ \|h(t,x_1,y_1) - h(t,x_2,y_2)\| &\le Q_1\|x_1 - x_2\| + Q_2\|y_1 - y_2\| \\ \|p(t,x_1,y_1) - p(t,x_2,y_2)\| &\le H_1\|x_1 - x_2\| + H_2\|y_1 - y_2\|, \end{aligned}\right\} \cdots (1.3)$$

Forall $t \in R^1$, $x, x_1, x_2, y, y_1, y_2 \in D$ and $z, z_1, z_2, w, w_1, w_2 \in D_1$, $\|e^{A(t-\tau)}\| \le \omega_1$, $\|e^{B(t-\tau)}\| \le \omega_2$, $N_3 = \|b(t) - a(t)\|$.

where, $N_1, N_2, L_1, L_2, L_3, K_1, K_2, K_3, Q_1, Q_2, H_1, H_2, \omega_1$ and ω_1 are positive constants and A, Bare$(n \times n)$non-negative matrices, $\|.\| = \max_{t \in [o,T]}|.|$.

Define anon-empty sets as follows:-

$$\left.\begin{aligned} G_{1f} &= G_f - \left(h_1 + \frac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)}\right) \\ G_{1g} &= G_g - \left(h_2 + \frac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)}\right) \\ G_{1z} &= G_z - N_3\left[Q_1\left(h_1 + \frac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)}\right) + Q_2\left(h_2 + \frac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)}\right)\right] \\ G_{1p} &= G_p - T\left[H_1\left(h_1 + \frac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)}\right) + H_2\left(h_2 + \frac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)}\right)\right] \end{aligned}\right\} \cdots (1.4)$$

where, $h_1 = \|x_0(t, x_0)\|(\|e^{At}\| + \|E\|)$, $h_2 = \|y_0(t, x_0)\|(\|e^{Bt}\| + \|E\|)$.

Furthermore, suppose that the maximumEigen-value of the matrix $\Delta = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix}$ less thannumber one,i.e.

$$\frac{1}{2}\left((\sigma_1 + \sigma_4) + \sqrt{(\sigma_1 + \sigma_4)^2 - 4(\sigma_1\sigma_4 - \sigma_2\sigma_3)}\right) \le 1, \quad \cdots (1.5)$$

Where, $\sigma_1 = \frac{T^{\alpha+1} N_1 \omega_1 L_1}{(\alpha+1)\Gamma(\alpha)}$, $\sigma_2 = \frac{T^{\alpha+1} N_1 \omega_1 L_2}{(\alpha+1)\Gamma(\alpha)}$, $\sigma_3 = \frac{T^{\alpha+1} N_2 \omega_2 k_1}{(\alpha+1)\Gamma(\alpha)}$ and $\sigma_4 = \frac{T^{\alpha+1} N_2 \omega_2 k_2}{(\alpha+1)\Gamma(\alpha)}$.

Suppose that the sequences of vectors functions $\{x_n(t)\}_{n=0}^{\infty}$ and $\{y_n(t)\}_{n=0}^{\infty}$ are defined by the following:-

$$x_{n+1}(t) = x_0 e^{At} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{A(t-\tau)} f\left(\tau, x_n(\tau), y_n(\tau), \int_{a(\tau)}^{b(\tau)} h(\tilde{\tau}, x(\tilde{\tau}), y(\tilde{\tau})) d\tilde{\tau}\right) d\tau,$$

$x_0(0) = x_0, n = 0, 1, 2, \ldots,$ ... (1.6)

and

$$y_{n+1}(t) = y_0 e^{Bt} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{B(t-s)}\, g\left(s, x_n(\tau), y_n(\tau), \int_0^\tau p\big(\tilde{\tau}, x(\tilde{\tau}), y(\tilde{\tau})\big) d\tilde{\tau}\right) d\tau,$$

$$y_0(0) = y_0, n = 0, 1, 2, \ldots. \qquad \ldots (1.7)$$

2.Existence solution of 1. 1.

In this section, westudy theexistence solution(1.1) by the following theorem.

Theorem2.1. If the system1.1 satisfies the inequalities (1.2), (1.3)and the conditions(1.4), (1.5) has a vectorssolution$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, then the sequence of vector functions$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix}$ converges, when $m \to \infty$, uniformly into domain (1.2) to the vector functions $\begin{pmatrix} x_\infty(t) \\ y_\infty(t) \end{pmatrix}$ defined in the domain (1.2) and satisfying the integral equations

$$\left.\begin{aligned} x(t) &= x_0 e^{At} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{A(t-\tau)} f\left(\tau, x(\tau), y(\tau), \int_{a(\tau)}^{b(\tau)} h\big(\tilde{\tau}, x(\tilde{\tau}), y(\tilde{\tau})\big) d\tilde{\tau}\right) d\tau \\ y(t) &= y_0 e^{Bt} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{B(t-\tau)} g\left(\tau, x(\tau), y(\tau), \int_0^\tau p\big(\tilde{\tau}, x(\tilde{\tau}), y(\tilde{\tau})\big) d\tilde{\tau}\right) d\tau \end{aligned}\right\} \cdots (2.1)$$

Provided that

$$\begin{pmatrix} \|x_\infty(t) - x_0\| \\ \|y_\infty(t) - y_0\| \end{pmatrix} \le \begin{pmatrix} h_1 + \dfrac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)} \\ h_2 + \dfrac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)} \end{pmatrix}.$$

and

$$\begin{pmatrix} \|x_\infty(t) - x_n(t)\| \\ \|y_\infty(t) - y_n(t)\| \end{pmatrix} \le \Delta^n (E - \Delta)^{-1} \Omega_0,$$

For all $n \ge 1$and $t \in R^1$,where$\Omega_0 = \begin{pmatrix} h_1 + \frac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)} \\ h_2 + \frac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)} \end{pmatrix}$.

Proof.

I. First of all we need to show that$x_n(t) \in G_f$ and$y_n(t) \in G_g$.

By using Mathematical inductionfrom (1.6)and inequalities in (1.3)we get the value

$$\|x_n(t) - x_0(t)\| \le h_1 + \frac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)}$$

that is, $x_n(t) \in G_f$, for all $t \in [0, T], x_0 \in G_{1f}$.

Similarly, from (1.7) and using the inequality and conditions in first section we obtain

$$\|y_n(t) - y_0(t,)\| \le h_2 + \frac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)}$$

i. e. $y_n(t) \in G_g$, for all $t \in [0, T], y_0 \in G_{1g}$.

From vector sequence (1.6) and by

$$z_n(t,) = \int_{a(t)}^{b(t)} h\big(\tau, x_n(\tau), y_n(\tau)\big) d\tau, \quad n = 0, 1, 2, \ldots,$$

we obtain that,

$$\|z_n(t) - z_0\| \le \left(Q_1\left(h_1 + \frac{T^{\alpha+1} N_1 \omega_1}{(\alpha+1)\Gamma(\alpha)}\right) + Q_2\left(h_2 + \frac{T^{\alpha+1} N_2 \omega_2}{(\alpha+1)\Gamma(\alpha)}\right)\right) \|b(t) - a(t)\|$$

that is,$z_n(t) \in G_z$, for all $t \in [0, T]$, $z_0 \in G_{1z}$.

And by

$$w_n(t)=\int_0^t p(\tau,x_n(\tau),y_n(\tau))d\tau\,,n=0,1,2,\ldots,$$

we get the following

$$w_n(t)=\int_0^t p(\tau,x_n(\tau),y_n(\tau))d\tau\,,n=0,1,2,\ldots,$$

we get the following

$$\|w_n(t)-w_0\|\le\left(H_1\left(h_1+\frac{T^{\alpha+1}N_1\omega_1}{(\alpha+1)\Gamma(\alpha)}\right)+H_2\left(h_2+\frac{T^{\alpha+1}N_2\omega_2}{(\alpha+1)\Gamma(\alpha)}\right)\right)T$$

that is, $w_n(t)\in G_p$, for all $t\in[a,b]$, $w_0\in G_{1p}$.

II. In this part of theorem we prove that $\{x_n(t)\}_{n=0}^{\infty}$ and $\{y_n(t)\}_{n=0}^{\infty}$ from(1.6) and (1.7)converge uniformly on domain (1.2).

Through (1.6) and bymathematical inductionwhen,$n>1$wehave

$$\|x_{n+1}(t)-x_n(t)\|\le\sigma_1\|x_n(t)-x_{n-1}(t)\|+\sigma_2\|y_n(t)-y_{n-1}(t)\|\ldots(2.2)$$

Alsoby the same waywithin (1.7)we get

$$\|y_{n+1}(t)-y_n(t)\|\le\sigma_3\|x_n(t)-x_{n-1}(t)\|+\sigma_4\|y_n(t)-y_{n-1}(t)\|\ldots(2.3)$$

Rewrite the inequalities(2.2) and (2.3)in the form

$$\Omega_{n+1}(t)\le\Delta(t)\Omega_n(t),\qquad\ldots(2.4)$$

where,

$$\Omega_{n+1}=\begin{pmatrix}\|x_{n+1}(t,)-x_n(t,)\|\\\|y_{n+1}(t,)-y_n(t)\|\end{pmatrix},\Omega_n=\begin{pmatrix}\|x_n(t)-x_{n-1}(t)\|\\\|y_n(t)-y_{n-1}(t)\|\end{pmatrix}\text{and}\Delta(t)=\begin{pmatrix}\frac{t^{\alpha+1}N_1\omega_1L_1}{(\alpha+1)\Gamma(\alpha)}&\frac{t^{\alpha+1}N_1\omega_1L_2}{(\alpha+1)\Gamma(\alpha)}\\\frac{t^{\alpha+1}N_2\omega_2k_1}{(\alpha+1)\Gamma(\alpha)}&\frac{t^{\alpha+1}N_2\omega_2k_2}{(\alpha+1)\Gamma(\alpha)}\end{pmatrix}.$$

Taking the maximum valueof (2.4), we have

$$\Omega_{n+1}\le\Delta\Omega_n,\qquad\ldots(2.5)$$

where $\Delta=\max_{t\in[0,T]}\Delta(t)$.

Taking iterations of (2.5), we get$\Omega_{n+1}\le\Delta^n\Omega_0$and hence

$$\sum_{i=1}^{n}\Omega_i\le\sum_{i=1}^{n}\Delta^{i-1}\Omega_0.$$

Using the condition (1.5), thus the sequence of vectors functions (1.6) and (1.7) are uniformly convergent, that is

$$\lim_{n\to\infty}\sum_{i=1}^{n}\Delta^{i-1}\Omega_0=\sum_{i=1}^{\infty}\Delta^{i-1}\Omega_0=(E-\Delta)^{-1}\Omega_0.$$

Suppose that

$$\left.\begin{aligned}\lim_{n\to\infty}x_n(t)=x(t)\\\lim_{n\to\infty}y_n(t)=y(t)\end{aligned}\right\}\ldots(2.6)$$

III. In last part of theorem wewill prove that$x(t)\in G_f$and$y(t)\in G_g$,for all $t\in[0,T]$.Assume that

$\|x_n(t)-x(t)\|\le\alpha_1\|x_n(t)-x(t)\|+\alpha_2\|y_n(t)-y(t)\|$and

$\|y_n(t)-y(t)\|\le\alpha_3\|x_n(t)-x(t)\|+\alpha_4\|y_n(t)-y(t)\|$.

Rewrite the above inequalities by the vector form:-

$$\begin{pmatrix}\|x_n(t)-x(t)\|\\\|y_n(t)-y(t)\|\end{pmatrix}\le\begin{pmatrix}\sigma_1&\sigma_2\\\sigma_3&\sigma_4\end{pmatrix}\begin{pmatrix}\|x_n(t)-x(t)\|\\\|y_n(t)-y(t)\|\end{pmatrix}.$$

The sequences$\{x_n(t)\}_{n=0}^{\infty}$ and $\{y_n(t)\}_{n=0}^{\infty}$ are convergent uniformly. Fromthe propertiesof the theorem,we have$\|x_n(t)-x(t)\|\le\epsilon_1$and $\|y_n(t)-y(t)\|\le\epsilon_2$.Thus

$$\begin{pmatrix}\|x_n(t)-x(t)\|\\ \|y_n(t)-y(t)\|\end{pmatrix}\le\begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_3 & \sigma_4\end{pmatrix}\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}.$$

From the condition (1.5),we get

$\begin{pmatrix}\|x_n(t)-x(t)\|\\ \|y_n(t)-y(t)\|\end{pmatrix}\le\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}$. Therefore $\begin{pmatrix}\|x(t)\|\\ \|y(t)\|\end{pmatrix}$ is a solution of (1.1).

3. Uniqueness solution of 1. 1.

The study ofthe uniqueness solution of 1.1should be proving by thefollowing theorem.

Theorem3.1.(Uniqueness theorem). If the solutionof 1.1 satisfies all inequalities and conditions of theorem(2.1). Then the vector solution of 1.1 is a unique.

Proof.

Suppose that there exists another vector solution for 1.1,that is

$$\hat{x}(t)=x_0e^{At}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}e^{A(t-\tau)}f\left(\tau,\hat{x}(\tau),\hat{y}(\tau),\int_{a(\tau)}^{b(\tau)}h(\tau,\hat{x}(\tau),\hat{y}(\tau))d\tilde{\tau}\right)d\tau \ldots(3.1)$$

and

$$\hat{y}(t)=y_0e^{Bt}+\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\alpha-1}e^{B(t-s)}g\left(\tau,\hat{x}(\tau),\hat{y}(\tau),\int_0^{s}p(\tau,\hat{x}(\tau),\hat{y}(\tau))d\tau\right)d\tau \ldots(3.2)$$

Then

$\|x(t)-\hat{x}(t)\|\le\alpha_1\|x(t)-\hat{x}(t)\|+\alpha_2\|y(t)-\hat{y}(t)\|$ ... (3.3)

In a similar way

$\|y(t)-\hat{y}(t)\|\le\alpha_3\|x(t)-\hat{x}(t)\|+\alpha_4\|y(t)-\hat{y}(t)\|$. ... (3.4)

Rewrite the inequalities (3.3)and (3.4) in a vector form:-

$$\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}\le\Delta\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}. \quad \ldots(3.5)$$

By taking the iterations from (3.5), we have

$$\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}\le\Delta^n\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}. \quad \ldots(3.6)$$

Fromcondition, (1.5)we obtain that, $\lim_{n\to\infty}\Delta^n=0$, hence$\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}<\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}$which is contradiction.Thus$\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}\to\begin{pmatrix}0\\ 0\end{pmatrix}$. Thus $\begin{pmatrix}\|x(t)\|\\ \|y(t)\|\end{pmatrix}$is a unique solution of(1.1).

4. Stability solution of 1. 1.

In this section, we study the stability solution of (1.1) by the following theorem.

Theorem 4.1. Letthe inequalities (1.2), (1.3) and the conditions(1.4),(1.5) are satisfied and $\begin{pmatrix}\|\check{x}(t)\|\\ \|\check{y}(t)\|\end{pmatrix}$is a different solution of(1.1).Then the solution is stable if satisfy the inequality,$\begin{pmatrix}\|x(t)-\check{x}(t)\|\\ \|y(t)-\check{y}(t)\|\end{pmatrix}\le\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}$, $\epsilon_1,\epsilon_2\ge 0$.

Proof.

Taking the following:-

$$\|x(t)-\hat{x}(t)\|=\|x_0-\hat{x}_0\|\omega_1+\left(\frac{T^{\alpha+1}\omega_1L_1}{(\alpha+1)\Gamma(\alpha)}\|x(t)-\hat{x}(t)\|+\frac{T^{\alpha+1}\omega_1L_2}{(\alpha+1)\Gamma(\alpha)}\|y(t)-\hat{y}(t)\|\right)$$

$$\|y(t)-\hat{y}(t)\|=\|y_0-\hat{y}_0\|\omega_2+\left(\frac{T^{\alpha+1}\omega_2k_1}{(\alpha+1)\Gamma(\alpha)}\|x(t)-\hat{x}(t)\|+\frac{T^{\alpha+1}\omega_2k_2}{(\alpha+1)\Gamma(\alpha)}\|y(t)-\hat{y}(t)\|\right)$$

Rewrite above inequalities in a vector form:-

$$\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix}\le\begin{pmatrix}\|x_0-\hat{x}_0\|\omega_1\\ \|y_0-\hat{y}_0\|\omega_2\end{pmatrix}+\begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_3 & \sigma_4\end{pmatrix}\begin{pmatrix}\|x(t)-\hat{x}(t)\|\\ \|y(t)-\hat{y}(t)\|\end{pmatrix},$$

for $\|x_0 - \hat{x}_0\| \le \delta_1$, $\|y_0 - \hat{y}_0\| \le \delta_2$, then we have

$$\begin{pmatrix}\|x(t)-\hat{x}(t)\| \\ \|y(t)-\hat{y}(t)\|\end{pmatrix} \le \begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} + \Delta^n \begin{pmatrix}\|x(t)-\hat{x}(t)\| \\ \|y(t)-\hat{y}(t)\|\end{pmatrix}.$$

From condition (1.5) we obtain that

$\begin{pmatrix}\|x(t)-\hat{x}(t)\| \\ \|y(t)-\hat{y}(t)\|\end{pmatrix} \le \begin{pmatrix}\epsilon_1 \\ \epsilon_2\end{pmatrix}$ where $\delta_1 = \epsilon_1/\omega_1$ and $\delta_2 = \epsilon_2/\omega_2$, $\epsilon_1, \epsilon_2 \ge 0$. Then $\begin{pmatrix}\|x(t)\| \\ \|y(t)\|\end{pmatrix}$ is stable solution of (1.1).

5. Another Results

In this section, it is possible to studythe of following fractional integro-differential equations to obtain otherresults

$$\left.\begin{aligned} x^{(\alpha)}(t) &= (A + C(t))x + f\left(t, x, y, \int_{a(t)}^{b(t)} (t-\tau)^{\alpha-1} h\big(\tau, x(\tau), y(\tau)\big)d\tau\right) \\ x^{(\alpha-1)}(t) &= x_0, \quad 0<\alpha<1 \\ y^{(\alpha)}(t) &= (B + D(t))y + g\left(t, x, y, \int_0^t (t-\tau)^{\alpha-1} p\big(\tau, x(\tau), y(\tau)\big)d\tau\right) \\ y^{(\alpha-1)}(t) &= y_0, \quad 0<\alpha<1 \end{aligned}\right\} \cdots (4.1)$$

where $x, y \in G \subset R^n$, $z, w \in G_1 \subset R^m$ and G, G_1 are closed and bounded domains.

The solutions of(4.1)can be written as:

$$\left.\begin{aligned} x(t) &= x_0 e^{At} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} e^{A(t-\tau)}\left(C(\tau)x(t) + f\left(\tau, x(\tau), y(\tau), \int_{a(\tau)}^{b(\tau)} h\big(\tilde{\tau}, x(\tilde{\tau}), y(\tilde{\tau})\big)d\tilde{\tau}\right)\right)d\tau \\ y(t) &= y_0 e^{Bt} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} e^{B(t-\tau)}\left(D(\tau)y(t) + g\left(\tau, x(\tau), y(\tau), \int_0^{\tau} p\big(\tilde{\tau}, x(\tilde{\tau}), y(\tilde{\tau})\big)d\tilde{\tau}\right)\right)d\tau \end{aligned}\right\} \dots (4.2)$$

Provided that $\|C(t)\| \le h_3$ and $\|D(t)\| \le h_4$, where are a positive constants.

The condition (1.5) becomes:

$$\frac{1}{2}\left((\partial_1 + \partial_4) + \sqrt{(\partial_1 + \partial_4)^2 - 4(\partial_1 \partial_4 - \partial_2 \partial_3)}\right) \le 1, \qquad \cdots (4.3)$$

where, $\partial_1 = \frac{T^{\alpha+1} N_1 \omega_1 h_3 L_1}{(\alpha+1)\Gamma(\alpha)}$, $\partial_2 = \frac{T^{\alpha+1} N_1 \omega_1 h_3 L_2}{(\alpha+1)\Gamma(\alpha)}$, $\partial_3 = \frac{T^{\alpha+1} N_2 \omega_2 h_4 k_1}{(\alpha+1)\Gamma(\alpha)}$ and $\partial_4 = \frac{T^{\alpha+1} N_2 \omega_2 h_4 k_2}{(\alpha+1)\Gamma(\alpha)}$.

Remark. It is possible to formulate the theorems of existence, uniqueness and stability of the fractional integro-differential equations(4.1) in the same way as the theorems2.1,3.1 and4.1 but use the condition (4.3) to obtain other results that differ from the results of the fractional integro-differential equations(1.1).

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