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Differential Forms: A Tool for Linearizing Second Order Ordinary Differential Equations

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Abstract In this paper, we presented the differential forms method which is used in the linearization of second order non-linear differential equations. The differential forms used here is limited to 2-forms with their respective operations. After presentation of the method, an example is used to illustrate the procedure of the linearization problem.

Keywords Linearization, Differential forms, Differential Equations, Second Order

Introduction

A differential form is a quantity that can be integrated, including the differentials. In the integral $\int_a^b f(x)dx$, $f(x)dx$is a differential form. This differential form has degree one because it is integrated over a 1-dimensional region, or path. We call a differential form of degree one a one-form.

Differential forms are an approach to multivariable calculus that is independent of coordinates. They provide a unified approach to defining integrands over curves, surfaces, volumes and higher dimensions. There are various differential forms, but in this paper, we shall consider1-form and 2-forms only in solving our problem.

A smooth 1-formϕ on $\mathbb{R}^n$is a real-valued function on the set of all tangent vectors to $\mathbb{R}^n$, *i.e.*

$$\phi : T\mathbb{R}^n \to \mathbb{R}$$

with the properties that

- ϕ is linear on the tangent space $T_x\mathbb{R}^n$ for each $x \in \mathbb{R}^n$.
- For any smooth vector field $v = v(x)$, the function $\phi : T\mathbb{R}^n \to \mathbb{R}$ is smooth.

Let dx_i, dx_jbe two forms, then we define $dx_j \wedge dx_i = -dx_i \wedge dx_j$for $i \neq j$and 0 if $i = j$. The expression $dx_i \wedge dx_j$is an expression for the multiplication $dx_i dx_j$ representing a 2-form. The operator $\wedge$ is called a wedge.

Almost all important governing equations in Physics take the form of nonlinear differential equations, and, in general, are very difficult to solve explicitly. While solving problems related to nonlinear ordinary differential equations, it is often expedient to simplify equations by a suitable change of variables. One of the fundamental methods of solving upon the transformation of a given equation to another equation of standard form. The transformation may be to an equation of equal order or of greater or lesser order. In particular, the possibility that a given equation could be linearized, Berkovich [1] that is, transformed to a linear equation, was a most attractive proposition due to the special properties of linear ordinary differential equations. The reduction of a nonlinear ordinary differential equation to a linear ordinary differential equation besides simplification, allows constructing an exact solution of the original equation. Therefore, the linearization problem plays a significant role in the nonlinear problem.

Many of the classical methods for solving ordinary differential equations work by applying a change of variables to produce another equation with known solutions. The simplest form of a differential equation is a linear form. It is of interest to provide general criteria for the linearizability of nonlinear ordinary differential equations, as they can then be reduced to easily solvable equations. Linearization criteria via invertible or point trans formations for ordinary differential equations have been of great interest and have been dealt with by many authors such as [2-6] over the years.

The linearization problem presented in this paper can be stated as follows: find a change of variables such that a transformed equation becomes a linear equation. If the change of variables includes derivatives, this change is called a non-point transformation. If the change of variables only depends on the independent and dependent variables, then this change is called a point transformation. A non-point transformation that is defined by the change of the independent, dependent variables and the first-order partial derivatives is called a contact transformation. This thesis studied linearization problem by using point transformation via differential forms.

The linearization problem for a second-orderordinary differential equation was also investigated with respect to differential forms by Harrison [7]. He looked at when and how can the second order ordinary differential equation$y'' = f(x, y, y')$be linearized with differential forms.

Method

Our starting point is a second order ordinary differential equation

$$y'' = f(x, y, y'). \tag{1}$$

We assume a point transformation given by the variables

$$X = F(x, y), \qquad Y = G(x, y), \tag{2}$$

with a requirement that,

$$\frac{d^2Y}{dX^2} = 0. \tag{3}$$

We first construct, using equation (2)

$$\frac{dY}{dX} = \frac{G_x + G_y y'}{F_x + F_y y'} \tag{4}$$

where $F_x + F_y y' \neq 0$ and the subscripts x and y denote partial differentiation. The second derivative equation may be written simply in terms of a differential $d\left(\frac{dY}{dX}\right) = 0$ which becomes

$$\left(F_x + F_y y'\right)\left(dG_x + y' dG_y + G_y dy'\right) - \left(G_x + G_y y'\right)\left(dF_x + y' dF_y + F_y dy'\right) = 0. \tag{5}$$

We can expand (5) and write it as

$$T dy' + \rho y'^2 + (\lambda + \delta) y' + \sigma = 0, \tag{6}$$

where

$$T = F_x G_y - F_y G_x, \tag{7}$$

and we have the 1-forms

$$\left.\begin{aligned} \rho = F_y dG_y - G_y dF_y, \lambda = F_y dG_x - G_y dF_x, \\ \sigma = F_x dG_x - G_x dF_x, \delta = F_x dG_y - G_x dF_y. \end{aligned}\right\} \tag{8}$$

We can rewrite equation (6) as

$$dy' = \alpha + \beta y' + \gamma y'^2, \tag{9}$$

where

$$\alpha = \frac{-\sigma}{T}, \beta = \frac{-(\lambda + \delta)}{T}, \gamma = \frac{-\rho}{T}. \tag{10}$$

For integrability of equation (9) we set $ddy' = 0$, that is

$$0 = d\alpha + dy' \wedge \beta + y' d\beta + 2y' dy' \wedge \gamma + y'^2 d\gamma. \tag{11}$$

Substituting (9) into equation (11), we have:

$$0 = d\alpha + (\alpha + \beta y' + \gamma y'^2) \wedge \beta + y' d\beta + 2y'(\alpha + \beta y' + \gamma y'^2) \wedge \gamma + y'^2 d\gamma. \tag{12}$$

The y'^3term in equation (12) vanishes because $\gamma \wedge \gamma = 0$, we expand equation (12) and equate the coefficients of the other powers of y' to zero to have:

$$d\alpha = \beta \wedge \alpha, d\beta = 2\gamma \wedge \alpha, dr = \gamma \wedge \beta. \quad (13)$$

Now, we go back to equations (8) and expand the differentials, to have:

$$\rho = F_y\left(G_{xy}\,dx + G_{yy}\,dy\right) - G_y\left(F_{xy}\,dx + F_{yy}\,dy\right),$$
$$\lambda = F_y\left(G_{xx}\,dx + G_{xy}\,dy - G_y\left(F_{xx}\,dx + F_{xy}\,dy\right),$$
$$\sigma = F_x\left(G_{xx}\,dx + G_{xy}\,dy\right) - G_x\left(F_{xx}\,dx + F_{xy}\,dy\right),$$
$$\delta = F_x\left(G_{xy}\,dx + G_{yy}\,dy\right) - G_x\left(F_{xy}\,dx + F_{yy}\,dy\right),$$

which can simply be written as

$$\rho = Adx + Bdy, \lambda = Cdx + Ady, \sigma = Ddx + Edy, \delta = Edx + Hdy, \quad (14)$$

where

$$A = F_y G_{xy} - G_y F_{xy}, B = F_y G_{yy} - G_y F_{yy}$$
$$C = F_y G_{xx} - G_y F_{xx}, D = F_x G_{xx} - G_x F_{xx}$$
$$E = F_x G_{xy} - G_x F_{xy}, H = F_x G_{yy} - G_x F_{yy}.$$

Thus,

$$\alpha = \frac{-(Ddx + Edy)}{T}, \beta = \frac{-(Cdx + Edx + Ady + Hdy)}{T}, \gamma = \frac{-(Adx + Bdy)}{T}. \quad (15)$$

Substituting α, β and γ into equation (9) and dividing by dx to convert the differential forms to functions, we have:

$$y'' + f_0 + f_1 y' + f_2 y'^2 + f_3 y'^3 = 0, \quad (16)$$

where the f_k are given by

$$f_0 = \frac{D}{T}, f_1 = \frac{(C+2E)}{T}, f_2 = \frac{(H+2A)}{T}, f_3 = \frac{B}{T}. \quad (17)$$

We define K and L as

$$K = \frac{E}{T}, L = \frac{A}{T}, \quad (18)$$

and replace D, C, H and B in the 1-forms in equation (15) in favour of the f_k, K and L, obtaining

$$\alpha = -f_0 dx - Kdy, \beta = (K - f_1)dx + (L - f_2)dy, \gamma = -Ldx - f_3 dy. \quad (19)$$

We also note that

$$\frac{dT}{T} = (3K - f_1)dx + (f_2 - 3L)dy. \quad (20)$$

We see that the 1-forms α, β, γ in (19)and $\frac{dT}{T}$in equation (20) are now expressed in terms of these four known functions K and L. The first three of these 1-forms can now be substituted into equation (13) on the various functions. If we do that, the first equation for $d\alpha$, gives the equation

$$f_{0y} - K_x = -K(K - f_1) + f_0(L - f_2) \quad (21)$$

which is nonlinear in K. The other equations give the results:

$$-K_y + f_{1y} + L_x - f_{2x} = 2KL - f_0 f_3 \quad (22)$$

and

$$L_y - f_{3x} = -L(L - f_2) + f_3(K - f_1) \quad (23)$$

which are also nonlinear. However, we can simplify the situation by defining new variables:

$$T = \frac{1}{W^3},\ E = \frac{U}{W^4},\ A = \frac{V}{W^4}, \quad (24)$$

so that from (18)

$$K = \frac{U}{W},\ L = \frac{V}{W}, \quad (25)$$

and from (20)

$$3\frac{dW}{W} = (f_1 - 3K)dx + (3L - f_2)dy. \quad (26)$$

We now have this situation. The dWequation (26) gives expressions for W_x and W_y. The equation (21) gives, after substitution for W_x, an expression

$$U_x = Wf_{0y} - \frac{2}{3}Uf_1 - Vf_0 + Wf_0 f_2 \quad (27)$$

which is linear in U, $V$and W. The equation (23) gives an expression

$$V_y = Wf_{3x} + \frac{2}{3}Vf_2 + Uf_3 - Wf_1 f_3 \quad (28)$$

which is also linear. The equation (22) gives a linear expression

$$V_x - U_y = \frac{U}{3} f_2 + \frac{V}{3} f_1 - W f_{1y} + W f_{2x} - 2 f_0 f_3 W. \quad (29)$$

The integrability condition on (26) gives a linear expression

$$V_x + U_y = \frac{U}{3} f_2 + \frac{V}{3} f_1 + \frac{W}{3} f_{2x} + \frac{W}{3} f_{1y}. \quad (30)$$

Equations (29) and (30) can be solved for V_x and U_y. Thus we have expressions for all derivatives of U, V and W, all of which are linear and homogeneous in the same variables. That is

$$dU = \frac{1}{3}\left(-2Uf_1 - 3Vf_0 + W\left(3f_{0y} + 3f_0 f_2\right)\right) dx + \frac{1}{3}\left(-Uf_2 + W\left(2f_{1y} - f_{2x} + 3f_0 f_3\right)\right) dy, \quad (31)$$

$$dV = \frac{1}{3}\left(Vf_1 + W\left(2f_{2x} - f_{1y} - 3f_0 f_3\right)\right) dx + \frac{1}{3}\left(3Uf_3 + 2Vf_2 + W(3f_{3x} - 3f_1 f_3)\right) dy, \quad (32)$$

$$dW = \frac{1}{3}(-3U + Wf_1) dx + \frac{1}{3}(3V - Wf_2) dy. \quad (33)$$

We summarize all these relations in a nice matrix equation

$$dr = Mr, \quad (34)$$

where

$$r = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \text{and } M = Pdx + Qdy,$$

$$P = \left(\frac{1}{3}\right) \begin{pmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0 f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0 f_3 \\ -3 & 0 & f_1 \end{pmatrix}$$

$$Q = \left(\frac{1}{3}\right) \begin{pmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0 f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1 f_3 \\ 0 & 3 & -f_2 \end{pmatrix}.$$

For integrability of (34), $ddr = 0$ giving

$$dM = M \wedge M \quad (35)$$

which is not zero since M is a matrix. Substitution for M in terms of P and Q gives the condition

$$Q_x - P_y + QP - PQ = 0. \quad (36)$$

This matrix condition in (36) reduces to two equations:

$$f_{0yy} + f_0\left(f_{2y} - 2f_{3x}\right) + f_2 f_{0y} - f_3 f_{0x} + \left(\frac{1}{3}\right)\left(f_{2xx} - 2f_{xy} + f_1 f_{2x} - 2f_1 f_{1y}\right) = 0 \quad (37)$$

and

$$f_{3xx} + f_3\left(2f_{0y} - f_{1x}\right) + f_0 f_{3y} - f_1 f_{3x} + \left(\frac{1}{3}\right)\left(f_{1yy} - 2f_{2xy} + 2f_2 f_{2x} - f_2 f_{1y}\right) = 0. \quad (38)$$

To summarize, we note that the original differential equation is cubic in y', with the coefficients satisfying equations (37) and (38).

Now, we shall construct the point transformations proper. We will need U, V and W therefore we need to solve equations (34). Once the equations are solved, we construct K and L from equation (25).

In order to find the $F(x,y)$ and $G(x,y)$ for which we are seeking, we revert to equations (8) and solve for dF_x, dF_y, dG_x and dG_x. Solution for dF_x and dF_y gives

$$dF_x = \frac{\left(F_y \sigma - F_x \lambda\right)}{T}, \qquad dF_y = \frac{\left(F_y \delta - F_x \rho\right)}{T}.$$

Solution for dG_x and dG_y, shows that they satisfy the same equation, so we will write only equations for the derivatives of F. We note that

$\delta + \lambda = -T\beta$ and $\delta - \lambda = dT$,

so we can solve these equations for δ and λ. We can also substitute for σ and ρ in terms of α and γ. We get finally

$$dF_x = -F_y \alpha + F_x \frac{\left(\beta + \frac{dT}{T}\right)}{2}, \qquad dF_y = F_x \gamma + F_y \frac{\left(-\beta + \frac{dT}{T}\right)}{2}.$$

We substitute for α, β, γ and dT/T from equations (19) and (20) respectively in terms of the expressions obtained above, with the f_k, K and L.
We now have two equations which can be expressed in matrix form as follows;

$$dR = ZR,\ dS = ZS \qquad (39)$$

where

$$Z = \begin{pmatrix} (2K - f_1)dx - Ldy & f_0 dx + Kdy \\ -Ldx - f_3 dy & Kdx + (f_2 - 2L)dy \end{pmatrix},$$

$$R = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} G_x \\ G_y \end{pmatrix}.$$

This linear equation set can be solved for R. There will be two independent solutions, which can be taken as R and S as seen in equation (39). Integrability is guaranteed by setting $ddR = 0$. Finally, one can solve

$$dF = (dx \quad dy)R,\ dG = (dx \quad dy)S \qquad (40)$$

for F and G.
We can summarize the procedure as follows:

1. Make sure that the original differential equation is a cubic in y' as in equation (16)
2. Test the coefficients f_k to see whether they satisfy equations (37) and (38).
3. Construct the 3×3 matrix M and solve equation (34) (linear) for the three components of r —a special solution is usually sufficient and construct K and L.
4. Construct the 2×2 matrix Z and solve equation (39) (linear) for R or S.
5. Solve equation (40); the two independent solutions may be taken as F and G.

Results

Let us consider the equation

$$y'' - \frac{x}{y^2}y'^3 - \frac{1}{y}y'^2 + \frac{2}{x}y' = 0 \qquad (41)$$

as presented in Mahomed and Quadir [8]. This equation has the coefficients

$$f_0 = 0, f_1 = \frac{2}{x}, f_2 = -\frac{1}{y}, f_3 = -\frac{x}{y^2}$$

Which satisfy the linearizability conditions (37) and (38). We therefore proceed to construct the 3×3 matrix $M = Pdx + Qdy$ to have

$$M = \begin{pmatrix} -\frac{4}{3x}dx + \frac{1}{3y}dy & 0 & 0 \\ -\frac{x}{y^2}dy & \frac{2}{3x}dx - \frac{2}{3y}dy & \frac{1}{y^2}dy \\ -dx & dy & \frac{2}{3x}dx + \frac{1}{3y}dy \end{pmatrix}.$$

Now, equation (34) becomes

$$dr = \begin{pmatrix} U\left(-\frac{4}{3x}dx + \frac{1}{3y}dy\right) \\ -U\frac{x}{y^2}dy + V\left(\frac{2}{3x}dx - \frac{2}{3y}dy\right) + W\frac{1}{y^2}dy \\ -Udx + Vdy + W\left(\frac{2}{3x}dx + \frac{1}{3y}dy\right) \end{pmatrix}.$$

We now have that $dU = U\left(-\frac{4}{3x}dx + \frac{1}{3y}dy\right)$ and if $U = 0$, then $dU = 0$. In addition,

$$dV = \frac{2}{3x}Vdx + \left(\frac{W}{y^2} - \frac{2V}{3y}\right)dy \qquad (42)$$

and

$$dW = \frac{2}{x}Wdx + \left(V + \frac{W}{y}\right)dy \qquad (43)$$

so that from equation (43), $W_x = \frac{2W}{x}$ and $W_y = V + \frac{W}{y}$. We can integrate $W_x = \frac{2W}{x}$ to have

$$W = x^2 a(y) \qquad (44)$$

for some function $a(y)$. We also have that

$V = x^2 a'(y) - \frac{x^2 a(y)}{y}$. (45)

We consider the special solution $a(y) = y^2$ to have equations (44) and (45) as $V = x^2 y$ and $W = x^2 y^2$, so that $K = \frac{U}{W} = 0$ and $L = \frac{V}{W} = \frac{1}{y}$. We now construct the 2×2 matrix Z as

$$Z = \begin{pmatrix} -\frac{2}{x}dx - \frac{1}{y}dy & 0 \\ -\frac{1}{y}dx + \frac{x}{y^2}dy & -\frac{3}{y}dy \end{pmatrix}$$

and have that

$$dR = \begin{pmatrix} b\left(-\frac{2}{x}dx - \frac{1}{y}dy\right) \\ b\left(-\frac{1}{y}dx + \frac{x}{y^2}dy\right) - \frac{3}{y}cdy \end{pmatrix},$$

where $R = \binom{b}{c}$,

$db = -b\left(\frac{2}{x}dx + \frac{1}{y}dy\right)$, (46)

and

$dc = -\frac{b}{y}dx + \left(\frac{x}{y^2}b - \frac{3}{y}c\right)dy$. (47)

Integrating equation (46), we have

$b = \frac{k}{x^2 y}$ (48)

where $k = \ln j$ is a constant.

From equation (47), on substitution of (48) we have that

$c_x = -\frac{k}{x^2 y^2}$. (49)

We can integrate the above equation to have

$c = \frac{k}{xy^2} + g(y)$. (50)

Differentiating the above with respect to y, one obtain

$c_y = -\frac{2k}{xy^3} + g'(y)$. (51)

From equation (46), on substitution of the values of b and c and simplifying, we arrive at

$g'(y) + \frac{3}{y}g(y) = 0$ (52)

which we can use the integrating factor to get

$g = \frac{m}{y^3}$, (53)

where m is also a constant. Therefore equation (50) becomes

$c = \frac{k}{xy^2} + \frac{m}{y^3}$. (54)

That is $b = F_x = \frac{k}{x^2 y}$ and $c = F_y = \frac{k}{xy^2} + \frac{m}{y^3}$. Consider F_x, on integration, we have

$F = -\frac{k}{xy} + h(y)$. (55)

Differentiating equation (55) with respect to y, equating the result with equation (54) and rearranging, we have

$h'(y) = \frac{m}{y^3}$. (56)

Integrating equation (56), substituting the result into equation (55) and simplifying, we have

$F + \frac{k}{xy} + \frac{m}{2y^2} = 0$. (57)

Taking the inverse of the coefficients of the constants k and m we have

$F + k(xy) + m(2y^2) = 0$,

we therefore take

$X = F(x, y) = xy, \quad Y = G(x, y) = 2y^2$

as the linearizing point transformation of equation (41).

Conclusion

We can see in this paper how differential forms make the problem of linearization simple for understanding. Other tools for linearization such as contact transformation, reduction of order, differential substitution, and generalized Sundman transformation Thailert and Suksern [9] are not easy to go by.

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