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An Improved Morgan-Voyce Collocation Method for Numerical Solution of Generalized Pantograph Equations

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Abstract In this paper, we propose an improved collocation method based on the Morgan-Voyce polynomials for the approximates solution of generalized pantograph equations. The method is based on the improvement of Morgan-Voyce the polynomial solutions with the aid of the residual error function. As a beginning, the Morgan-Voyce collocation method is applied to the generalized pantograph equations and then Morgan-Voyce polynomial solutions are obtained. Next, an error problem is constructed by means of the residual error function and this error problem is solved by using the Morgan-Voyce collocation method. By summing the Morgan-Voyce polynomial solutions of the original problem and the error problem, we have the improved Morgan-Voyce polynomial solutions. When the exact solution of problem is not known, the absolute error can be approximately computed by the Morgan-Voyce polynomial solution of the error problem. We give numerical examples. We have applied all of the numerical computations on computer using a program written in MATLAB (R2013a).

Keywords Morgan-Voyce polynomials; Pantograph equations; approximate methods, residual error function

1. Introduction

These equations arise in many applications such as population studies, number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics and cell growth, among others. In particular, it was used by Ockendon and Tayler [1] to study how the electric current is collected by the pantograph of an electric locomotive. The name pantograph originated from this work. Properties of the analytic solution of these equations with variable coefficients are treated in [2-4]. In recent years, there has been a growing interest in the numerical treatment of pantograph equations of the retarded and the advanced type. A special feature of this type is the existence of compactly supported solutions [5]. This phenomenon was studied in [6] and has direct applications to approximation theory and to wavelets.

Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many ODEs based model fails. These equations arise in industrial applications [1,7,8] and in studies based on biology, economy, control and electrodynamics [9,10].

The Taylor method has been used to find the approximate solutions of differential, difference, integral and integro-differential-difference, multi-pantograph and generalized pantograph equations [11-20]. The Morgan-Voyce method has been used to find the approximate solutions of differential, integral and integro-differential equations [21]. The basic motivation of this work is to apply the the Morgan-Voyce method to the nonhomogenous and the homogenous generalized pantograph equations with variable coefficients, which is extented of the multi-pantograph equations given in [5, 22].

In this study, we consider generalized pantograph equations of type [5,23,24]

$$L[y(t)]=y^{(m)}(t)-\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}(t)y^{(k)}(\lambda_{jk}t+\mu_{jk})=g(t)\ \ 0\le t\le b \tag{1}$$

in with initial conditions

$$\sum_{k=0}^{m-1}c_{ik}y^{(k)}(0)=\lambda_i,\quad i=0,1,...,m-1. \tag{2}$$

Here $P_{jk}(t)$ and $g(t)$ are continuous functions defined in the interval $0\le t\le b; c_{ik},\lambda_i,\lambda_{jk}$, and μ_{jk} are real or complex constants.

In this paper, by improving the Morgan-Voyce collocation method with the aid of residual error function used in [25-28], we obtain an approximate solution of (1) expressed in the truncated Morgan-Voyce series form

$$y_{N,M}(t)=y_N(t)+e_{N,M}(t) \tag{3}$$

where

$$y_N(t)=\sum_{n=0}^{N}a_nB_n(t) \tag{4}$$

is the Morgan-Voyce solution and

$$e_{N,M}(t)=\sum_{n=0}^{M}a_n^*B_n(t)$$

is the Morgan-Voyce polynomial solution of the error problem obtained with the aid of the residual error function. Here $a_n,a_n^*,n=0,1,2,...,N$ are the unknown Morgan-Voyce coefficients. N and M are any chosen positive integers such that $M\ge N\ge 2$; and $B_n(t),n=0,1,2,....N$ are the Morgan-Voyce polynomials defined by

$$B_n(t)=\sum_{k=0}^{n}\binom{n+k+1}{n-k}t^k,\quad n\in\mathbb{N}.$$

2. Fundamental Matrix Relations

Firstly, the Morgan-Voyce polynomials $B_n(t)$ can be written in the matrix form as follows,

$$\boldsymbol{B}^T(t)=\boldsymbol{R}\boldsymbol{T}^T(t)\ \Leftrightarrow\ \boldsymbol{B}(t)=\boldsymbol{T}(t)\boldsymbol{R}^T \tag{5}$$

where

$$\boldsymbol{B}(t)=[B_0(t)\ \ B_1(t)\ \ ...\ \ B_N(t)],\ \boldsymbol{T}(t)=[1\ \ t^1\ \ t^2\ \ ...\ \ t^N];$$

$$\boldsymbol{R}=\begin{bmatrix}\binom{1}{0} & 0 & 0 & \cdots & 0\\ \binom{2}{1} & \binom{3}{0} & 0 & \cdots & 0\\ \binom{3}{2} & \binom{4}{1} & \binom{5}{0} & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \binom{n+1}{n} & \binom{n+2}{n-1} & \binom{n+3}{n-2} & \cdots & \binom{2n+1}{0}\end{bmatrix}_{(N+1)\times(N+1)}$$

We consider the desired solution $y(t)$ of Eq. (1) defined by the truncated Morgan-Voyce series (4). So the finite series (4) can be written in the matrix form

$$y(t)=\boldsymbol{B}(t)\boldsymbol{A}\,;\quad \boldsymbol{A}=[a_0\ \ a_1\ \ \dots\ \ a_N]^T$$

or from Eq. (5)

$$y(t)=\boldsymbol{T}(t)\boldsymbol{R}^T\boldsymbol{A}. \tag{6}$$

On the other hand, from [29,30] the relation between the matrix $\boldsymbol{T}(t)$ and its derivative $\boldsymbol{T}^{(1)}(t)$ is

$$\boldsymbol{T}^{(1)}(t)=\boldsymbol{T}(t)\boldsymbol{C}^T,\qquad \boldsymbol{T}^{(0)}(t)=\boldsymbol{T}(t) \tag{7}$$

where

$$\boldsymbol{C}^T=\begin{bmatrix}0&1&0&\cdots&0&0&0\\0&0&2&\cdots&0&0&0\\0&0&0&\cdots&0&0&0\\\vdots&\vdots&\vdots&&\vdots&&\vdots\\0&0&0&\cdots&0&0&N\\0&0&0&\cdots&0&0&0\end{bmatrix}$$

It follows from (6) and (7) that

$$\begin{aligned}&\boldsymbol{T}^{(0)}(t)=\boldsymbol{T}(t)\\&\boldsymbol{T}^{(1)}(t)=\boldsymbol{T}(t)\boldsymbol{C}^T\\&\boldsymbol{T}^{(2)}(t)=\boldsymbol{T}^{(1)}(t)\boldsymbol{C}^T=\boldsymbol{T}(t)(\boldsymbol{C}^T)^2\\&\vdots\ \vdots\\&\boldsymbol{T}^{(k)}(t)=\boldsymbol{T}^{(k-1)}(\mathrm{t})(\boldsymbol{C}^T)^{k-1}=\boldsymbol{T}(t)(\boldsymbol{C}^T)^k\end{aligned} \tag{8}$$

and therefore

$$\boldsymbol{B}^{(k)}(t)=\boldsymbol{T}^{(k)}(t)\boldsymbol{R}^T=\boldsymbol{T}(t)(\boldsymbol{C}^T)^k\boldsymbol{R}^T \tag{9}$$

Using the relations (8) and (9), we have recurrence relations

$$\begin{aligned}\boldsymbol{y}^{(k)}(t)&=\boldsymbol{B}^{(k)}(t)\boldsymbol{A}\\&=\boldsymbol{T}^{(k)}(t)\boldsymbol{R}^T\boldsymbol{A}\\&=\boldsymbol{T}(t)(\boldsymbol{C}^T)^k\boldsymbol{R}^T\boldsymbol{A},\quad k=0,1,2,\dots m\end{aligned} \tag{10}$$

Similarly, the matrix relations are obtained as follows

$$\begin{aligned}&\boldsymbol{T}(\lambda_{jk}t+\mu_{jk})=\boldsymbol{T}(t)\boldsymbol{B}(\lambda_{jk},\mu_{jk})\\&y(\lambda_{jk}t+\mu_{jk})=\boldsymbol{T}(\lambda_{jk}t+\mu_{jk})\boldsymbol{R}^T\boldsymbol{A}\\&y^{(k)}(\lambda_{jk}t+\mu_{jk})=\boldsymbol{T}(t)\boldsymbol{B}(\lambda_{jk},\mu_{jk})(\boldsymbol{C}^T)^k\boldsymbol{R}^T\boldsymbol{A}\end{aligned} \tag{11}$$

where for $\lambda_{jk}\neq 0$ and $\mu_{jk}\neq 0$,

$$\boldsymbol{B}(\lambda_{jk},\mu_{jk})=\begin{bmatrix}\binom{0}{0}(\lambda_{jk})^0(\mu_{jk})^0&\binom{1}{0}(\lambda_{jk})^0(\mu_{jk})^1&\cdots&\binom{N}{0}(\lambda_{jk})^0(\mu_{jk})^N\\0&\binom{1}{1}(\lambda_{jk})^1(\mu_{jk})^0&\cdots&\binom{N}{1}(\lambda_{jk})^1(\mu_{jk})^{N-1}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\binom{N}{N}(\lambda_{jk})^N(\mu_{jk})^0\end{bmatrix}$$

and for $\lambda_{jk} \neq 0$ and $\mu_{jk} = 0$,

$$B(\lambda_{jk},0)=\begin{bmatrix} (\lambda_{jk})^0 & 0 & \dots & 0 \\ 0 & (\lambda_{jk})^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\lambda_{jk})^N \end{bmatrix}$$

3. Method of Solution

Now, we can construct the fundamental matrix equation for Eq. (1). For this purpose, we substitute the matrix relations (10) and (11) into Eq. (1) and obtain the matrix equation

$$\boldsymbol{T}(t)(\boldsymbol{C}^T)^m \boldsymbol{R}^T \boldsymbol{A} - \sum_{j=0}^{J}\sum_{k=0}^{m-1} \boldsymbol{P_{jk}}(t)\boldsymbol{T}(t)\boldsymbol{B}(\lambda_{jk},\mu_{jk})(\boldsymbol{C}^T)^k \boldsymbol{R}^T \boldsymbol{A} = g(t) \tag{12}$$

The collocation points t_i are defined as

$$t_i = \frac{b}{N} i, \quad i = 0,1,\dots,N. \tag{13}$$

Substituting Eq. (13) in Eq. (12), we obtain the system of matrix equations

$$\boldsymbol{T}(t_i)(\boldsymbol{C}^T)^m \boldsymbol{R}^T \boldsymbol{A} - \sum_{j=0}^{J}\sum_{k=0}^{m-1} \boldsymbol{P_{jk}}(t_i)\boldsymbol{T}(t_i)\boldsymbol{B}(\lambda_{jk},\mu_{jk})(\boldsymbol{C}^T)^k \boldsymbol{R}^T \boldsymbol{A} = g(t_i), \quad i = 0,1,\dots N$$

or briefly the fundamental matrix equation

$$\left\{ \boldsymbol{T}(\boldsymbol{C}^T)^m \boldsymbol{R}^T - \sum_{j=0}^{J}\sum_{k=0}^{m-1} \boldsymbol{P_{jk}}\boldsymbol{T}\boldsymbol{B}(\lambda_{jk},\mu_{jk})(\boldsymbol{C}^T)^k \boldsymbol{R}^T \right\} \boldsymbol{A} = \boldsymbol{G} \tag{14}$$

where

$$\boldsymbol{P}_{jk} = \begin{bmatrix} P_{jk}(t_0) & 0 & \cdots & 0 \\ 0 & P_{jk}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & P_{jk}(t_N) \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix}, \boldsymbol{T} = \begin{bmatrix} T(t_0) \\ T(t_1) \\ \vdots \\ T(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_0 & \cdots & t_0^N \\ 1 & t_1 & \cdots & t_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & \cdots & t_N^N \end{bmatrix}$$

Thus, the fundamental matrix equation (14) for Eq. (1) can be written in the form

$$\boldsymbol{WA} = \boldsymbol{G} \text{ or } [\boldsymbol{W};\boldsymbol{G}] \tag{15}$$

where

$$\boldsymbol{W} = \boldsymbol{T}(\boldsymbol{C}^T)^m \boldsymbol{R}^T - \sum_{j=0}^{J}\sum_{k=0}^{m-1} \boldsymbol{P_{jk}}\boldsymbol{T}\boldsymbol{B}(\lambda_{jk},\mu_{jk})(\boldsymbol{C}^T)^k \boldsymbol{R}^T, \ \boldsymbol{W} = [\boldsymbol{W_{ij}}], \ i, j = 0,1,\dots N$$

Here, Eq. (15) corresponds to a system of $N+1$ linear algebraic equations with unknown Morgan-Voyce coefficients $a_0, a_1, \dots, a_N$.

By means of the relation (10), for the conditions (2), we can obtain the matrix forms as follows,

$$\sum_{k=0}^{m-1} c_{ik}\boldsymbol{T}(0)(\boldsymbol{C}^T)^k \boldsymbol{R}^T \boldsymbol{A} = [\lambda_i], \quad i = 0,1,2,\dots m-1.$$

On the other hand, we can write the matrix form for conditions as

$$U_i A = [\lambda_i] \text{ or } [\boldsymbol{U_i};\lambda_i], \quad i = 0,1,2,\dots,m-1 \tag{16}$$

where

$$\boldsymbol{U}_i = \sum_{k=0}^{m-1} c_{ik}\boldsymbol{T}(0)(\boldsymbol{C}^T)^k \boldsymbol{R}^T = [u_{i0} \ \ u_{i1} \ \ u_{i2} \ \ \dots \ \ u_{iN}], \quad i = 0,1,2,\dots,m-1$$

Under conditions (2), to obtain the solution of Eq. (1), we replace the row matrices (16) by the last m rows of the matrix (15) and have the new augmented matrix [15,30],

$$[\tilde{W};\tilde{G}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(t_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(t_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1-m)0} & w_{(N-1-m)1} & \cdots & w_{(N-1-m)N} & ; & g(t_{N-1-m}) \\ w_{(N-m)0} & w_{(N-m)1} & \cdots & w_{(N-m)N} & ; & g(t_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(m-1)1} & u_{(m-1)2} & \cdots & u_{(m-1)N} & ; & \lambda_{m-1} \end{bmatrix} \tag{17}$$

If rank $\tilde{W} = rank[\tilde{W};\tilde{G}] = N+1$, then we can write $A = (\tilde{W})^{-1}\tilde{G}$. Thus, we uniquely determine the matrix A (thereby the coefficients $a_0, a_1, ..., a_N$). So Eq. (1) with conditions (2) has a unique solution and this solution is given by Morgan-Voyce series solution (4). On the other hand, when $|\tilde{W}| = 0$, that is if rank $\tilde{W} = rank[\tilde{W};\tilde{G}] < N+1$,then one can be found a particular solution. Otherwise if rank $\tilde{W} \neq rank[\tilde{W};\tilde{G}] < N+1$,then there is no solution.

4. Residual Correction and Error Estimation

In this section, we will give an error estimation for the Morgan-Voyce polynomial solution (4) with the residual error function [25-28] and will improve the Morgan-Voyce polynomial solution (4) with the help of the residual error function. For this purpose, we get the residual function of the Morgan-Voyce collocation method as

$$R_N(t) = L[y_N(t)] - g(t). \tag{18}$$

Here $y_N(t)$ is the Morgan-Voyce polynomial solution given by (4) of problem (1) and (2). Thus, $y_N(t)$ satisfies the problem

$$\begin{cases} L[y(t)] = y^{(m)}(t) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t) y^{(k)}(\lambda_{jk}t + \mu_{jk}) = g(t) + R_N(t) \\ \sum_{k=0}^{m-1} c_{ik} y_N^{(k)}(0) = \lambda_i, \quad i = 0,1,...,m-1. \end{cases}$$

Also, the error function $e_N(t)$

$$e_N(t) = y(t) - y_N(t) \tag{19}$$

such that $y(t)$ is the exact solution of problem (1) and (2).

By using Eqs. (1), (2), (18) and (19) we can get the error differential equation

$$L[e_N(t)] = L[y(t)] - L[y_N(t)] = -R_N(t)$$

with the condition

$$\sum_{k=0}^{m-1} c_{ik} e_N^{(k)}(0) = 0, \quad i = 0,1,...,m-1.$$

or clearly, the error problem is

$$\begin{cases} e_N^{(m)}(t)-\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}(t)e_N^{(k)}(\lambda_{jk}t+\mu_{jk})=-R_N(t) \\ \sum_{k=0}^{m-1}c_{ik}e_N^{(k)}(0)=0, \quad i=0,1,\ldots,m-1. \end{cases} \tag{20}$$

Here, we note that the nonhomogeneous condition

$$\sum_{k=0}^{m-1}c_{ik}y^{(k)}(0)=\lambda_i, \quad i=0,1,\ldots,m-1.$$

and

$$\sum_{k=0}^{m-1}c_{ik}y_N^{(k)}(0)=\lambda_i, \quad i=0,1,\ldots,m-1.$$

has been reduced to the homogeneous condition

$$\sum_{k=0}^{m-1}c_{ik}e_N^{(k)}(0)=0, \quad i=0,1,\ldots,m-1.$$

By solving problem (20) with the method introduced section (2) and (3), we get the approximation

$$e_{N,M}(t)=\sum_{n=0}^{M}a_n^{*}B_n(t) \; M\geq N$$

to $e_N(t)$.

Consequently, by means of the polynomials $y_N(t)$ and $e_{N,M}(t),\ (M\geq N)$, we get the correct Morgan-Voyce polynomial solution $y_{N,M}(t)=y_N(t)+e_{N,M}(t)$. Also, we construct the error function $e_N(t)=\left|y(t)-y_N(t)\right|$, the correct error function $E_{N,M}(t)=e_N(t)-e_{N,M}(t)=y(t)-y_{N,M}(t)$ and the estimated error function $e_{N,M}(t)$.

If the exact solution of Eq. (1) unknown, then the absolute errors $\left|e_N(t_i)\right|=\left|y(t_i)-y_N(t_i)\right|,(0\leq x_i\leq b)$ is not found. However the absolute errors can be approximately computed with the aid of the estimated absolute error function $\left|e_{N,M}(t)\right|$

5. Numerical Examples

In this section, we want to show the accuracy and efficiency properties of the present method. For this reason, we give several numerical examples. We have performed all calculations on MATLAB. The values of the exact solution $y(t)$, the polynomial approximate solution $y_N(t)$, the corrected Morgan-Voyce polynomial solution $y_{N,M}(t)=y_N(t)+e_{N,M}(t)$, the absolute error function $e_N(t)=\left|y(t)-y_N(t)\right|$, the corrected absolute error function $\left|E_{N,M}(t)\right|=\left|y(t)-y_{N,M}(t)\right|$ and the estimated absolute error function $e_{N,M}(t)$ have been illustrated in the Tables and Figures at the selected points of the given interval.

Example 1: (Sezer and Akyüz-Daşcıoğlu [30]). **:** With the exact solution $y(t)=\cos(t)$ [24], we consider the pantograph equation of third order

$$y'''(t)=ty''(2t)-y'(t)-y(\tfrac{t}{2})+t\cos(2t)+\cos(\tfrac{t}{2}), \quad 0\leq t\leq 1. \tag{21}$$

The initial conditions are $y(0)=1,\ y'(0)=0,\ y''(0)=-1.$

The approximate solution $y_7(t)$ by the truncated Morgan-Voyce series for $N=7$ is given by

$$y_7(t)=\sum_{n=0}^{7}a_nB_n$$

Now, let us compute the coefficients a_n, $(n=0,1,2,...,N)$ of the approximate solution in the above form. First, the set of collocation points (4) for $a=0.1$, $b=1$ and $N=7$ is calculated as

$$\left\{t_0=\frac{1}{10},t_1=\frac{8}{35},t_2=\frac{5}{14},t_3=\frac{17}{35},t_4=\frac{43}{70},t_5=\frac{26}{35},t_6=\frac{61}{70},t_7=1\right\}$$

and from Eq. 14, the fundamental matrix equation of the problem is

$$\{\boldsymbol{T}(\boldsymbol{C}^T)^3\boldsymbol{R}^T-\boldsymbol{P_{00}}\boldsymbol{TB}(\lambda_{00},\mu_{00})(\boldsymbol{C}^T)^0\boldsymbol{R}^T-\boldsymbol{P_{01}}\boldsymbol{TB}(\lambda_{01},\mu_{01})(\boldsymbol{C}^T)^1\boldsymbol{R}^T$$
$$-\boldsymbol{P_{02}}\boldsymbol{TB}(\lambda_{02},\mu_{02})(\boldsymbol{C}^T)^2\boldsymbol{R}^T\}\boldsymbol{A}=\boldsymbol{G}$$

So, by applying the procedure given in Section 2 and 3, we get the Morgan-Voyce polynomial solution

$$y_7(t)=1-(0.433355608343e-16)t-0.499999999999t^2$$
$$+(0.292958337116e-6)t^3+(0.415238563586e-2)t^4+(0.359406712252e-4)t^5$$
$$-(0.182525669465e-3)t^6+(0.183940111353e-4)t^7$$

Since, we compute the corrected Morgan-Voyce polynomial solution, let us first consider the error problem

$$\begin{cases} e_7'''(t)-te_7''(2t)+e_7'(t)+e(\frac{t}{2})-t\cos(2t)-\cos(\frac{t}{2})=-R_7(t) \\ e_7(0)=0,\quad e_7{}'(0)=0,\quad e_7{}''(0)=0 \end{cases}$$

Such that the residual function is

$$R_7(t)=y_7'''(t)-ty_7''(2t)+y_7'(t)+y_7(\frac{t}{2})-t\cos(2t)-\cos(\frac{t}{2})$$

By solving the error problem (20) for $M=10$ with the method in Sections 2 and 3, the estimated Morgan-Voyce error function approximation $e_{7,10}(t)$ to $e_7(t)$ is obtained as

$$e_{7,10}(t)=(0.693607751936e-15)-(0.343664303350e-14)t-(0.236539244477e-17)t^2$$
$$-(0.292961608873e-04)t^3+(0.142893886870e-03)t^4-(0.360125981672e-03)t^5$$
$$+(0.439066507396e-03)t^6-(0.188815184812e-03)t^7+(0.285169208951e-04)t^8$$
$$-(0.134151800991e-05)t^9-(0.732360508292e-07)t^{10}$$

Hence, we calculate the corrected Morgan-Voyce polynomial solution

$$y_{7,10}(t)=1.0000000000000008-(0.386999864184e-14)t-0.49999999999t^2$$
$$-(0.327175726047e-09)t^3+0.041666750245t^4-(0.71926942003186352e-06)t^5$$
$$-(0.138619018725e-02)t^6-(0.487507345842e-05)t^7+(0.285169208951e-04)t^8$$
$$-(0.134151800991e-05)t^9-(0.732360508292e-08)t^{10}$$

In Table 1, we compare the numerical values of the exact solution, the Morgan-Voyce polynomial solutions and corrected Morgan-Voyce polynomial solutions. In Table 2, the actual absolute errors are compared with absolute errors estimated by the presented technique. These errors are almost identical. Table 3 shows the corrected absolute errors by our method for $N=7,10$ and $M=10,13,15$. Also the actual error function and the estimated error function for $N=7,10$ and $M=10,13,15$ are compared in Figures 1a and 1b. Figures 1c and 1d display the corrected absolute error functions for $N=7,10$ and $M=10,13,15$

Table 1: Numerical results of the exact and the approximate solutions for $N=7,10$ and $M=10,13,15$ Eq. (21)

t_i	Exact solution $y(t_i)=\cos(t_i)$	Morgan-Voyce polynomial solution $y_7(t_i)$	Corrected Morgan-Voyce polynomial solution $y_{7,10}(t_i)$	$y_{7,15}(t_i)$
0	1	1	0.999999999999	0.999999999999
0.2	0.98006657784	0.98006667309	0.980066577861	0.980066577841
0.4	0.92106099400	0.92106139110	0.921060993927	0.921060994003
0.6	0.82533561491	0.82533575712	0.825335609283	0.825335614910
0.8	0.69670670935	0.69670103637	0.696706657652	0.696706709347
1	0.54030230587	0.54027124232	0.540302067554	0.540302305869
	$y(t_i)=\cos(t_i)$	$y_{10}(t_i)$	$y_{10,13}(t_i)$	$y_{10,15}(t_i)$
0	1	1	1	1
0.2	0.98006657784	0.980066577802	0.980066577841	0.980066577841
0.4	0.92106099400	0.921060993804	0.921060994003	0.921060994003
0.6	0.82533561491	0.825335613768	0.825335614911	0.825335614910
0.8	0.69670670935	0.696706702114	0.696706709359	0.696706709347
1	0.54030230587	0.540302275198	0.540302305922	0.540302305868

Table 2: Comparison of the absolute error functions for $N=7,10$ and $M=10,13,15$ of Eq. (21)

t_i	Absolute errors for Morgan-Voyce polynomial solution $\left\|e_7(t_i)\right\|=\left\|y(t_i)-y_7(t_i)\right\|$	Estimated absolute errors for Morgan-Voyce polynomial solution $\left\|e_{7,10}(t_i)\right\|$	$\left\|e_{7,15}(t_i)\right\|$
0	6.4835e-17	2.2413e-016	3.5477e-016
0.2	9.5244e-08	9.5224e-008	9.5244e-008
0.4	3.9709e-07	3.9717e-007	3.9709e-007
0.6	1.4221e-07	1.4784e-007	1.4221e-007
0.8	5.6730e-06	5.6213e-006	5.6730e-006
1	3.1064e-05	3.0825e-005	3.1064e-005
	$\left\|e_{10}(t_i)\right\|=\left\|y(t_i)-y_{10}(t_i)\right\|$	$\left\|e_{10,13}(t_i)\right\|$	$\left\|e_{10,15}(t_i)\right\|$
0	5.1088e-16	1.2804e-016	3.4645e-016
0.2	3.9585e-11	3.9587e-011	3.9585e-011
0.4	1.9885e-10	1.9889e-010	1.9885e-010
0.6	1.1415e-09	1.1428e-009	1.1415e-009
0.8	7.2329e-09	7.2448e-009	7.2329e-009
1	3.0671e-08	3.0725e-008	3.0671e-008

Table 3: Numerical results of the corrected error functions for $N=7,10$ and $M=10,13,15$ Eq. (21)

t_i Improved absolute errors $\left\|E_{N,M}(t_i)\right\|=\left\|y(t_i)-y_{N,M}(t_i)\right\|$

	$\lvert E_{7,10}(t_i)\rvert$	$\lvert E_{7,15}(t_i)\rvert$	$\lvert E_{10,13}(t_i)\rvert$	$\lvert E_{10,15}(t_i)\rvert$
0	7.7716e-016	8.8818e-016	4.4409e-016	0
0.2	2.0102e-011	2.2204e-016	1.3323e-015	0
0.4	7.5690e-011	7.7716e-016	4.3077e-014	2.2204e-016
0.6	5.6266e-009	2.8311e-014	1.3413e-012	4.4409e-015
0.8	5.1696e-008	2.5269e-013	1.1908e-011	3.9968e-014
1	2.3831e-007	1.1473e-012	5.4110e-011	1.8063e-013

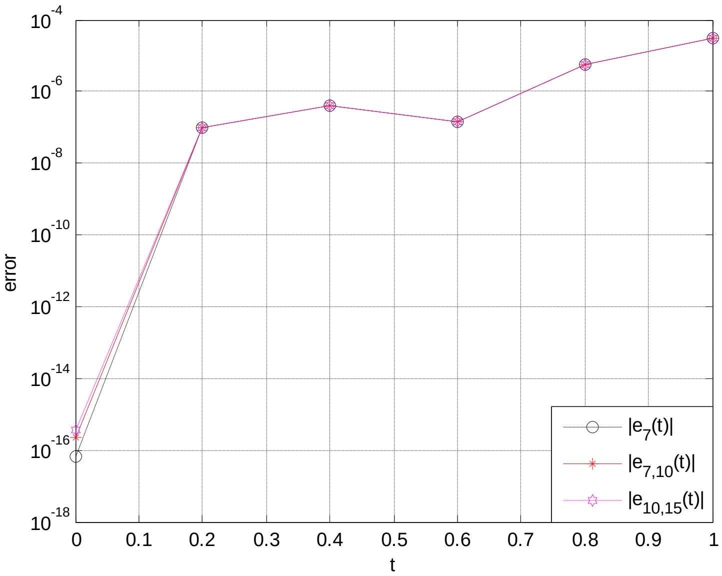

Figure.1a. Comparison of the absolute error functions $|e_N(t)| = |y(t) - y_N(t)|$ *and the estimated error functio* $|e_{N,M}(t)|$ *for* $N = 7$ *and* $M = 10,15$ *of Eq. (21).*

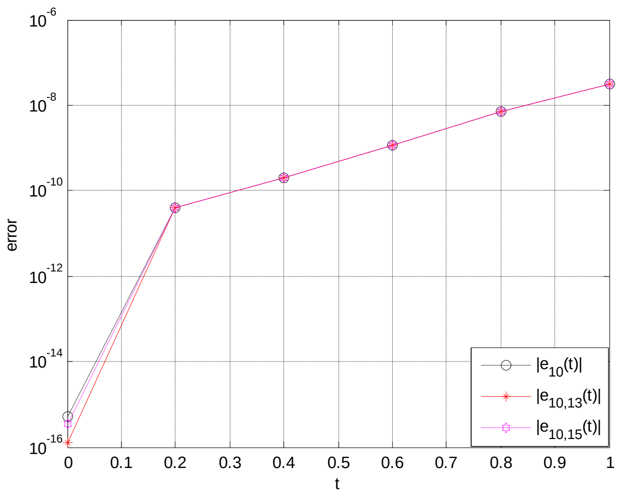

Figure 1b: Comparison of the absolute error functions $|e_N(t)| = |y(t) - y_N(t)|$ *and the estimated error functio* $|e_{N,M}(t)|$ *for* $N = 10$ *and* $M = 13,15$ *of Eq. (21).*

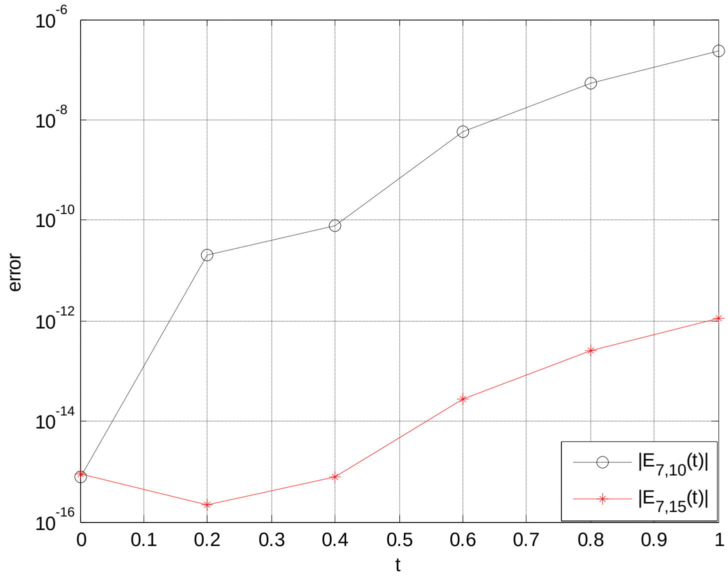

Figure 1c: Comparison of the improved absolute error functions $\left|E_{N,M}(t)\right| = \left|y(t) - y_{N,M}(t)\right|$ *for* $N = 7$ *and* $M = 10,15$ *of Eq. (21).*

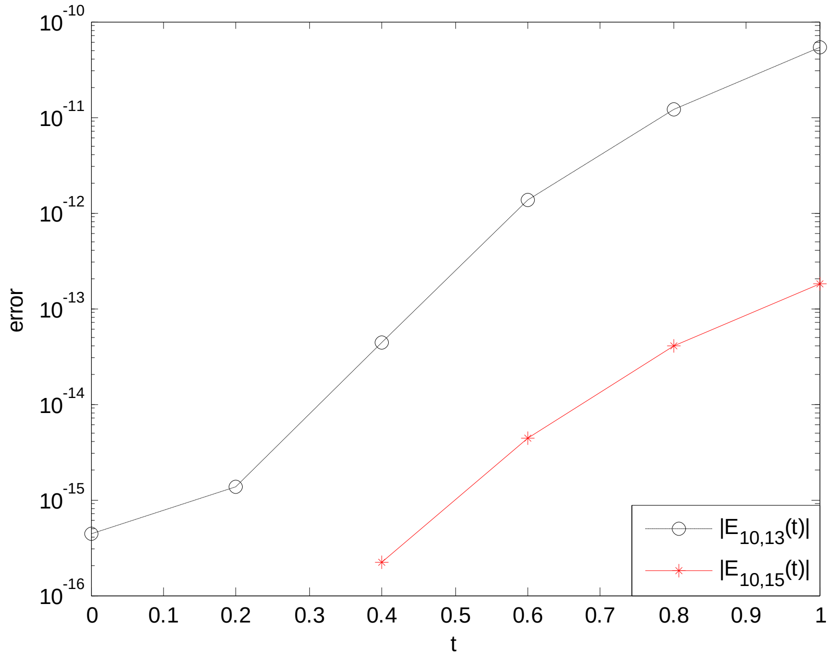

Figure 1d: Comparison of the improved absolute error functions $\left|E_{N,M}(t)\right| = \left|y(t) - y_{N,M}(t)\right|$ *for* $N = 10$ *and* $M = 13,15$ *of Eq. (21).*

Example 2. Consider the pantograph equation of first order

$$y'(t) = \frac{1}{2} e^{\frac{t}{2}} y(\frac{t}{2}) + \frac{1}{2} y(t), \quad 0 \le t \le 1. \qquad (22)$$

with the boundary condition $y(0)=1$. The exact solution of the problem is $y(t)=e^{t}$.

By applying the presented method for $N=10$, $M=12$ and $M=15$, we obtain the approximate solutions. We compute the approximate solution for $N=10$ and $M=12$ as follows,

$$\begin{aligned} y_{10,12}(t) &= 0.999999999999+0.999999999999t+0.499999999999t^{2}+0.166666666669t^{3} \\ &+(0.416666666469e-2)t^{4}+(0.833333343867e-3)t^{5}+(0.138888855692e-3)t^{6} \\ &+(0.198413280567e-4)t^{7}+(0.248012627121e-5)t^{8}+(0.275491205118e-6)t^{9} \\ &+(0.277608273630e-7)t^{10}+(0.230382578085e-8)t^{11}+(0.304608381562e-9)t^{12} \end{aligned}$$

In the table 4, we compare the absolute errors obtained by the present method, Spline method [32], Adomian Method [23], Taylor Method [30], Variational method [31]. In addition, the absolute error functions are compared in the Figure.2.

Table 4: Comparison of the solutions and the absolute errors of Eq. (22)

t_i	Spline method $h=0.001$ $m=3$[32]	Adomian method with 13 terms[23]	Taylor method[30] $N=12$	Variational Method [31]	Present method $\lvert E_{10,12}(t_i)\rvert$	$\lvert E_{10,15}(t_i)\rvert$
0.2	1.37e-11	0.00	2.220e-16	2.44e-05	2.220e-16	0
0.4	3.27e-11	2.22e-16	1.332e-15	2.28e-04	2.220e-16	2.220e-16
0.6	5.86e-11	2.22e-16	2.189e-13	9.00e-04	2.220e-16	0
0.8	9.54e-11	1.33e-15	9.361e-12	2.50e-03	8.882e-16	4.441e-16
1	1.43e-10	4.88e-15	1.729e-10	5.71e-03	4.441e-16	8.882e-16

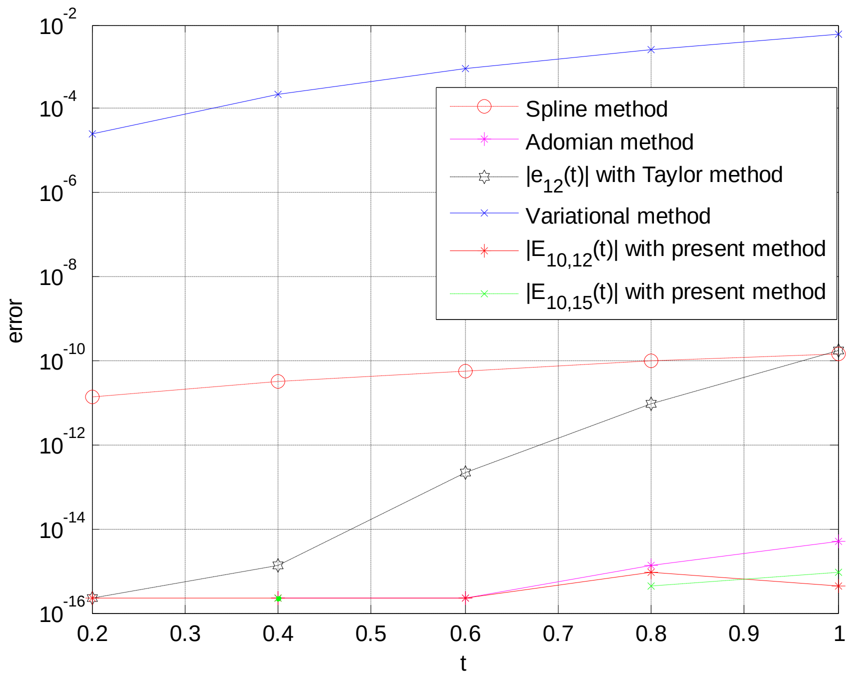

Figure 2: Comparison of the absolute error functions $\lvert e_N(t)\rvert=\lvert y(t)-y_N(t)\rvert$ *and* $\lvert E_{N,M}(t)\rvert=\lvert y(t)-y_{N,M}(t)\rvert$ *for* $N=10$ *and* $M=12,15$ *of Eq. (22)*

6. Conclusions

In this article, we have improved the Morgan-Voyce collocation method, based on Morgan-Voyce polynomials, for generalized pantograph equations. This improvement is based on the residual error function. In addition, an error estimation is given with the residual error function. Morever, if the exact solution of the problem is unknown, then the absolute errors $\left|e_N(t_i)\right| = \left|y(t_i) - y_N(t_i)\right|$, $(0 \le t \le b)$ can be estimated by the approximation $\left|e_{N,M}(t)\right|$. It is seen from Tables 1-3 that the estimated absolute errors $\left|e_{N,M}(t_i)\right|$ are quite close to the actual absolute errors $\left|e_N(t_i)\right| = \left|y(t_i) - y_N(t_i)\right|$. We see from the tables and the figures that the errors decrease when N and M are increased. The comparisons of the present method by the other methods show that our method is very effective. A consirable advantage of the method is that the approximate solutions are computed very easily by using a well-known symbolic software such as Matlab, Maple and Mathematica.

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