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Numerical Analysis of a Mathematical Model of Hepatitis B Virus Transmission Dynamics in the Presence of Vaccination and Treatment

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Abstract In this paper, a mathematical model of hepatitis B virus transmission dynamics, incorporating vaccination and treatment as control parameters is presented. Using published values of the model parameters, the stability properties of the disease-free and the endemic equilibrium states were numerically investigated. The numerical results show that if the basic reproduction number is less than unity, the disease-free equilibrium state is locally and asymptotically stable. The numerical results also show that the endemic equilibrium state is locally and asymptotically stable if the basic reproduction number is above one. The numerical results further show that the disease-free (respectively the endemic) equilibrium state is globally and asymptotically stable if the basic reproduction number is less than (is greater than) unity. Finally, this study suggests that effective vaccination and treatment in combination guarantees eradication within the shortest possible time.

Keywords Hepatitis B, mathematical model, basic reproduction number, disease-free equilibrium state, endemic equilibrium state, stability

1. Introduction

Hepatitis B is a disease that is characterized by inflammation of the liver and is caused by infection by the hepatitis B virus [1-2]. Hepatitis B is a serious health challenge. More than a billion people across the globe have been infected with hepatitis B virus (HBV) and over 300 million people are carriers of the virus [3-7]. The modes of transmission of HBV are: mother to child (vertical), contact with an infected person, sexual contact with infected partners, exposure to blood or other infected fluids and contact with HBV contaminated instruments [1]. HBV control interventions include vaccination, education, screening of blood and blood products; and treatment [2].

As reported in Zou *et al* [8], mathematical models have been used to study the transmission dynamics of HBV in various communities, regions and countries. Anderson and May [9] used a simple deterministic, compartmental mathematical model to study the effects of carriers on the transmission of HBV. Anderson & May [10] and Williams *et al* [11] presented models of sexual transmission of HBV, which include heterogeneous mixing with respect to age and sexual activity. Edmunds *et al* [12] investigated the relation between the age at infection with HBV and the development of the carrier state. Medley *et al* [13] proposed a model to show that the prevalence of infection is largely determined by a feedback mechanism that relates the rate of transmission, average age at infection and age-related probability of developing carriage following infection. Thornley *et al* [14] applied the model of Medley *et al* [13] to predict chronic hepatitis B infection in New Zealand. The prevalence of HBV in developing countries is different from that in developed countries, since it appears that the rate of transmission in childhood is the major determinant of the level of HBV endemicity and little is known on the rates and patterns of sexual contact in developing countries [15]. Mclean and Blumberg [16] and Edmunds *et al* [17] studied models of HBV transmission in developing countries and Williams *et al* [11] described a model of HBV in UK. Zou *et al* [8] proposed a mathematical model to

investigate the transmission dynamics and prevalence of HBV in mainland China. The model is formulated from that of Medley *et al* [13] based on the characteristics of HBV in China.
The model by Zou *et al* [8] forms the motivation for this study. Zou and his collaborators assumed that the newborns to carrier mothers infected at birth do not stay in a latent period, so that they instantaneously become carriers. However, as pointed out by Anderson and May [9] and White and Fenner [3], person infected with HBV must have harboured the virus in the blood for at least six months to become a carrier. By this newborns to carrier mothers infected at birth are latently infected individuals. Mehmood [18] supported the same view in his study and assumed that the proportion of the infected newborns to carrier mothers is latent. The role of treatment of HBV carriers as a measure of control was not considered in their model.
In this paper, the above amendments and inclusion of the treatment parameter in their model have been effected. The analytical method of stability was earlier carried out by us in Kimbir et al [19]. We now dwell on numerical analysis.
The plan of this paper is as follows. The model equations are presented in section 2. Section 3 is devoted to the numerical stability of the equilibria. Numerical simulations of the model are treated in section 4. Section 5 gives the discussion of the results. Conclusive remarks are passed in section 6.

2. The Model Equations

2.1. The Existing Model

We begin by introducing the model by Zou *et al* [8]. We, first, present the parameters of the existing model.

2.2. Variables and Parameters of the Existing Model

The population is partitioned into six compartments described as follows: $S(t)$ = proportion of the susceptible individuals at time t,
$L(t)$ = proportion of the latent individuals at time t,
$I(t)$ = proportion of the acutely infected individuals at time t,
$C(t)$ = proportion of the chronic carriers at time t,
$R(t)$ = proportion of the recovered individuals at time t,
$V(t)$ = proportion of the vaccinated individuals at time t.
The following are the parameters of the existing model:
μ =birth rate,
μ_0=natural mortality rate,
μ_1 =HBV-related mortality rate,
ω =proportion of births without vaccination,
$(1-\omega)$ =proportion of births vaccinated,
ν =proportion of births vertically infected,
Ψ =rate of waning vaccine-induced immunity,
σ =rate of moving from latent state to acute state,
β =transmission coefficient,
γ_1 = rate of moving from acute to other compartments,
q = average probability that an individual fails to clear an acute infection and develops to carrier state,
$q\gamma_1$ = rate of moving from acute to carrier,
$(1-q)\gamma_1$ = rate of moving from acute to recovered class,
γ_2 = rate of moving from carrier to immune,
γ_3 = vaccination rate of the susceptible individuals,
ε = reduced transmission rate relative to acute infection by carriers.

2.3. The Equations of the Existing Model

Using the earlier assumptions and parameters, Zou *et al* [8] derived the following model equations.

$$\frac{dS}{dt} = \mu\omega(1-\nu C) + \Psi V - \left(\mu_0 + \beta I + \varepsilon\beta C + \gamma_3\right)S \quad (1.1)$$

$$\frac{dL}{dt} = (\beta I + \varepsilon\beta C)S - (\sigma + \mu_0)L \quad (1.2)$$

$$\frac{dI}{dt} = \sigma L - (\mu_0 + \gamma_1)I \quad (1.3)$$
$$\frac{dC}{dt} = \mu\nu\omega C + q\gamma_1 I - (\mu_0 + \mu_1 + \gamma_2)C \quad (1.4)$$
$$\frac{dR}{dt} = (1-q)\gamma_1 I + \gamma_2 C - \mu_0 R \quad (1.5)$$
$$\frac{dV}{dt} = \mu(1-\omega) + \gamma_3 S - (\mu_0 + \Psi)V \quad (1.6)$$

2.4. The Extended Model

2.5. Assumptions of the Extended Model

In addition to the assumptions by Zou *et al* [8], Kimbir, *et al* [20], make the following assumptions:

i. The chronic carriers are treated at the rate α. Acute infections are not subjected to antiviral treatment because of possibility of relapse and resistance [21],
ii. The newborns to carrier mothers infected at birth, first, enter the latent class [18],
iii. The treated individuals recover [22].

2.6. Equations of the Extended Model

The infected newborns are now moved to the second equation instead of the fourth equation in the existing model. Also, chronic individuals are now treated at a rate α and this is incorporated in the last term in the fourth equation.

Based on the above assumptions and the parameters the extended model is as follows.

$$\frac{dS}{dt} = \mu\omega(1-\nu C) + \Psi V - (\mu_0 + \beta I + \varepsilon\beta C + \gamma_3)S \quad (2.1)$$
$$\frac{dL}{dt} = \mu\nu\omega C + (\beta I + \varepsilon\beta C)S - (\sigma + \mu_0)L \quad (2.2)$$
$$\frac{dI}{dt} = \sigma L - (\mu_0 + \gamma_1)I \quad (2.3)$$
$$\frac{dC}{dt} = q\gamma_1 I - (\mu_0 + \mu_1 + \gamma_2 + \alpha)C \quad (2.4)$$
$$\frac{dV}{dt} = \mu(1-\omega) + \gamma_3 S - (\mu_0 + \Psi)V \quad (2.5)$$
$$\frac{dR}{dt} = (1-q)\gamma_1 I + (\gamma_2 + \alpha)C - \mu_0 R \quad (2.6)$$

$S(0) \geq 0, L(0) \geq 0, I(0) \geq 0, C(0) \geq 0, V(0) \geq, R(0) \geq 0.$

Because the model variables are in terms of proportions,

$$S(t) + L(t) + I(t) + C(t) + R(t) + V(t) = 1 \quad (2.7)$$

for all time t.

The model is defined in the subset $D \times [0, \infty)$ of R_+^7, where

$D = \{(S, L, I, C, V, R) \in R_+^6 : 0 \leq S, L, I, C, V, R \leq 1, S + L + I + C + V + R \leq 1\}$

Table1: Parameter values used in numerical simulations

Parameter/Variable	Value	Reference
ν	0.11	[8]
ψ	0.1	[8]
σ	6 per year	[8]
β	0.95	[8]
γ_1	4 per year	[8]
q	0.885	[8]
γ_2	0.025	[8]
ε	0.16	[8]
μ	0.0367	[23]
μ_0	0.0166	[23]

$S(0)$	0.7	Assumed
$L(0)$	0.05	Assumed
$I(0)$	0.05	Assumed
$C(0)$	0.08	[24]
$R(0)$	0.12	Assumed

3. Stability of Equilibria

We now calculate the disease-free equilibrium state of the extended model. As done in Zou *et al* (2009), we begin this by setting the left hand sides of equations $(2.1)-(2.5)$ to zero and get the disease-free equilibrium state as follows.

The disease-free equilibrium state, $E_0=(S_0,0,0,0,V_0)$, where $S_0=\frac{\mu(\Psi+\mu_0\omega)}{\mu_0(\mu_0+\gamma_3+\Psi)}$ and $V_0=\frac{\mu(\mu_0+\gamma_3-\mu_0\omega)}{\mu_0(\mu_0+\gamma_3+\Psi)}$.

Next generation method [25-28] gives the basic reproduction number as follows.

$$R_0=\rho(F_xV^{-1})=\frac{\sigma\beta S_0}{(\sigma+\mu_0)(\mu_0+\gamma_1)}+\frac{q\gamma_1\sigma(\mu\nu\omega+\varepsilon\beta S_0)}{(\sigma+\mu_0)(\mu_0+\gamma_1)(\mu_0+\mu_1+\gamma_2+\alpha)} \qquad (2.8)$$

3.1. Existence and Local Stability Analysis of the Disease-free Equilibrium State (DFEs)

We will now examine the existence and local stability of DFEs. We shall first compute the Jacobian matrix for the disease-free equilibrium state using equations $(2.1)-(2.5)$ as done in Zou *et al* [8].

The Jacobian matrix for the disease-free state J_{E_0} is given as

$$J_{E_0}=\begin{pmatrix}-(\mu_0+\gamma_3+\Psi) & 0 & -\beta S_0 & -(\mu\nu\omega+\varepsilon\beta S_0) & \Psi\\ 0 & -(\sigma+\mu_0) & \beta S_0 & \mu\nu\omega+\varepsilon\beta S_0 & 0\\ 0 & \sigma & -(\mu_0+\gamma_1) & 0 & 0\\ 0 & 0 & q\gamma_1 & -(\mu_0+\mu_1+\gamma_2+\alpha) & 0\\ 0 & 0 & -\beta S_0 & -(\mu\nu\omega+\varepsilon\beta S_0) & -\mu_0\end{pmatrix}$$

We investigate the local stability of the disease-free equilibrium state for the following values of the control parameters. For

Table 2	γ_3=0	α=0	$\omega=1$

$$J_{E_0}=\begin{pmatrix}-0.0166 & 0.0000 & -2.2108 & -0.3578 & 0.1000\\ 0.0000 & -6.0166 & 2.2108 & 0.3578 & 0.0000\\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166\end{pmatrix}.$$

Thus, the characteristic equation in λ becomes

$$\lambda^5+10.2280\lambda^4+12.8658\lambda^3-5.3737\lambda^2-0.9005\lambda-0.0134=0 \qquad (2.9)$$

Solving (2.9) gives $\lambda_1=-0.0166,\lambda_2=-8.6749,\lambda_3=-1.8513,\lambda_4=0.4314,\lambda_5=-0.1166$ and $R_0=5.6536$.

For

Table 3	γ_3=0	α=0	$\omega=0$

$$J_{E_0}=\begin{pmatrix}-0.0166 & 0.0000 & -1.8961 & -0.3034 & 0.1000\\ 0.0000 & -6.0166 & 1.8961 & 0.3034 & 0.0000\\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166\end{pmatrix}.$$

Thus, the characteristic equation in λ becomes

$$\lambda^5+10.2280\lambda^4+14.7543\lambda^3-3.8504\lambda^2-0.7274\lambda-0.0109=0 \qquad (2.10)$$

Solving (2.10) gives $\lambda_1=-0.0166,\lambda_2=-8.4233,\lambda_3=-2.0061,\lambda_4=0.3347,\lambda_5=-0.1166$ and $R_0=4.7993$.

For

Table 4	γ_3=0.17	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.1866 & 0.0000 & -.7714 & -0.1234 & 0.1000 \\ 0.0000 & -6.0166 & 0.7714 & 0.1234 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.1700 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

The characteristic equation is

$\lambda^5 + 10.3980\lambda^4 + 23.2214\lambda^3 + 4.7413\lambda^2 - 0.3340\lambda - 0.0067 = 0$ (2.11).

Thus, $\lambda_1 = -0.2866, \lambda_2 = -0.0166, \lambda_3 = -7.3116, \lambda_4 = -2.8513, \lambda_5 = 0.0680$ and $R_0 = 1.9526$.

For

Table 5	γ_3=0.2	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.2166 & 0.0000 & -0.6983 & -0.1117 & 0.1000 \\ 0.0000 & -6.0166 & 0.6983 & 0.1117 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.2000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$\lambda^5 + 10.4280\lambda^4 + 23.9633\lambda^3 + 5.7725\lambda^2 - 0.2725\lambda - 0.0060 = 0$ (2.12) is the characteristic equation,

$\lambda_1 = -0.3166, \lambda_2 = -0.0166, \lambda_3 = -7.2208, \lambda_4 = -2.9281, \lambda_5 = 0.0540$ are the characteristic roots and $R_0 = 1.7675$.

For

Table 6	γ_3=0.3	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.3166 & 0.0000 & -0.5307 & -0.0849 & 0.1000 \\ 0.0000 & -6.0166 & 0.5307 & 0.0849 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.3000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$\lambda^5 + 10.5280\lambda^4 + 25.9802\lambda^3 + 8.9160\lambda^2 - 0.0720\lambda - 0.0035 = 0$ (2.13)

$\lambda_1 = -0.4166, \lambda_2 = -0.0166, \lambda_3 = -6.9975, \lambda_4 = -3.1207, \lambda_5 = 0.0234$ are the characteristic roots and $R_0 = 1.3433$.

For

Table 7	γ_3=0.4	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.4166 & 0.0000 & -0.4280 & -0.0685 & 0.1000 \\ 0.0000 & -6.0166 & 0.4280 & 0.0685 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.4000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$\lambda^5 + 10.6280\lambda^4 + 27.6077\lambda^3 + 11.8085\lambda^2 + 0.1244\lambda - 0.0011 = 0$ (2.14)

$\lambda_1 = -0.5166, \lambda_2 = -0.0166, \lambda_3 = 0.0056, \lambda_4 = -3.2525, \lambda_5 = -6.8478$ are the characteristic roots and $R_0 = 1.0832$.

For

Table 8	γ_3=0.5	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.5166 & 0.0000 & -0.3586 & -0.0574 & 0.1000 \\ 0.0000 & -6.0166 & 0.3586 & 0.0574 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.5000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$\lambda^5 + 10.7280\lambda^4 + 29.0353\lambda^3 + 14.5722\lambda^2 + 0.3188\lambda + 0.0014 = 0$ (2.15)

$\lambda_1 = -0.6166, \lambda_2 = -0.0166, \lambda_3 = -6.7396, \lambda_4 = -3.3491, \lambda_5 = -0.0061$ are the characteristic roots and

$R_0 = 0.9076$.

For

Table 9	γ_3=0.6	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.6166 & 0.0000 & -0.3085 & -0.0494 & 0.1000 \\ 0.0000 & -6.0166 & 0.3085 & 0.0494 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.6000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.8280\lambda^4 + 30.3466\lambda^3 + 17.2609\lambda^2 + 0.5119\lambda + 0.0039 = 0 \quad (2.16)$$

$\lambda_1 = -0.7166, \lambda_2 = -0.0166, \lambda_3 = -6.6574,\ \lambda_4 = -3.4231, \lambda_5 = -0.0143$ are the characteristic roots and

$R_0 = 0.7809.$

For

Table 10	γ_3=0.7	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.7166 & 0.0000 & -0.2707 & -0.0433 & 0.1000 \\ 0.0000 & -6.0166 & 0.2707 & 0.0433 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.7000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.9280\lambda^4 + 31.5844\lambda^3 + 19.9022\lambda^2 + 0.7043\lambda + 0.0064 = 0 \quad (2.17)$$

$\lambda_1 = -0.8166, \lambda_2 = -0.0166, \lambda_3 = -6.5925,\ \lambda_4 = -3.4819, \lambda_5 = -0.0204$ are the characteristic roots and

$R_0 = 0.6835.$

For

Table 11	γ_3=0.8	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.8166 & 0.0000 & -0.2412 & -0.0386 & 0.1000 \\ 0.0000 & -6.0166 & 0.2412 & 0.0386 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.8000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 11.0280\lambda^4 + 32.7728\lambda^3 + 22.5116\lambda^2 + 0.8962\lambda + 0.0088 = 0 \quad (2.18)$$

$\lambda_1 = -0.9166, \lambda_2 = -0.0166, \lambda_3 = -0.0251,\ \lambda_4 = -3.5297, \lambda_5 = -6.5400$ are the characteristic roots and

$R_0 = 0.6105.$

For

Table 12	γ_3=0.9	α=0	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.9166 & 0.0000 & -0.2175 & -0.0348 & 0.1000 \\ 0.0000 & -6.0166 & 0.2175 & 0.0348 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.9000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}$$

$$\lambda^5 + 11.1280\lambda^4 + 33.9263\lambda^3 + 25.0985\lambda^2 + 1.0876\lambda + 0.0113 = 0 \quad (2.19)$$

$\lambda_1 = -1.0166, \lambda_2 = -0.0166, \lambda_3 = -0.0289,\ \lambda_4 = -3.5695, \lambda_5 = -6.4965$ and

$R_0 = 0.5505.$

For

Table 13	γ_3=0	α=0.2	$\omega = 1$

$$J_{E_0} = \begin{pmatrix} -0.0166 & 0.0000 & -0.2108 & -0.3578 & 0.1000 \\ 0.0000 & -6.0166 & 0.2108 & 0..3578 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.5616 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.7280\lambda^4 + 17.9490\lambda^3 + 0.7461\lambda^2 - 0.1647\lambda - 0.0029 = 0 \quad (2.20)$$

$\lambda_1 = -0.0166, \lambda_2 = -8.6673, \lambda_3 = -2.0122,\ \lambda_4 = 0.0847, \lambda_5 = -0.1166$ and

$R_0 = 1.1088.$

For

Table 14	γ_3=0	α=0.4	$\omega = 1$

$$J_{E_0} = \begin{pmatrix} -0.0166 & 0.0000 & -0.2108 & -0.3578 & 0.1000 \\ 0.0000 & -6.0166 & 0.2108 & 0..3578 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.4616 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.6280\lambda^4 + 16.9324\lambda^3 - 0.4778\lambda^2 - 0.3119\lambda - 0.0050 = 0 \quad (2.21)$$

$\lambda_1 = -0.0166, \lambda_2 = -8.6689, \lambda_3 = -1.9758,\ \lambda_4 = 0.1499, \lambda_5 = -0.1166$ and $R_0 = 1.2301$.

For

Table 15	γ_3=0	α=0.5	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.0166 & 0.0000 & -1.8961 & -0.3034 & 0.1000 \\ 0.0000 & -6.0166 & 1.8961 & 0.3034 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.5616 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}$$

$$\lambda^5 + 10.6280\lambda^4 + 16.9324\lambda^3 - 0.4778\lambda^2 - 0.3119\lambda - 0.0050 = 0 \quad (2.22)$$

$\lambda_1 = -0.0166, \lambda_2 = -8.4160, \lambda_3 = -2.1377,\ \lambda_4 = -0.1411, \lambda_5 = -0.1166$ and $R_0 = 0.9455$.

For

Table 16	γ_3=0.1	α=0.2	$\omega = 0$

$$J_{E_0} = \begin{pmatrix} -0.1166 & 0.0000 & -1.0207 & -0.1633 & 0.1000 \\ 0.0000 & -6.0166 & 1.0207 & 0.1633 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.2616 & 0.0000 \\ 0.1000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.5280\lambda^4 + 23.0711\lambda^3 + 6.1075\lambda^2 + 0.3661\lambda + 0.0045 = 0 \quad (2.23)$$

$\lambda_1 = -0.0166, \lambda_2 = -0.2166, \lambda_3 = -7.5956,\ \lambda_4 = -2.6367, \lambda_5 = -0.0625$ and $R_0 = 0.8021$.

3.2. Existence and Local Stability Analysis of the Endemic Equilibrium State (EEs)

We investigate the stability of EEs for the following values of the control parameters.

For

Table 17	γ_3=0	α=0	$\omega = 1$

$$J_{E_E} = \begin{pmatrix} -0.0938 & 0.0000 & -0.3723 & -0.0636 & 0.1000 \\ 0.0772 & -6.0166 & 0.3723 & 0.0636 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}$$

$$\lambda^5 + 10.3052\lambda^4 + 24.6858\lambda^3 + 5.0288\lambda^2 + 0.3819\lambda + 0.0134 = 0 \quad (2.24)$$

$\lambda_1 = -6.7647, \lambda_2 = -3.3245, \lambda_3 = -0.0497 + 0.0514i,\ \lambda_4 = -0.0497 - 0.0514i, \lambda_5 = -0.1166$ and $R_0 = 5.6536$.

For

Table 18	γ_3=0	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.0797 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ 0.0631 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.2911\lambda^4 + 24.4045\lambda^3 + 4.6424\lambda^2 + 0.3195\lambda + 0.0109 = 0 \quad (2.25)$$

$\lambda_1 = -6.8031, \lambda_2 = -3.2867, \lambda_3 = -0.0423 + 0.0491i,\ \lambda_4 = -0.0423 - 0.0491i, \lambda_5 = -0.1166$ and $R_0 = 4.7993$

For

Table 19	γ_3=0.2	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.2512 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ 0.0346 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.2000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}$$

$$\lambda^5 + 10.4626\lambda^4 + 26.1360\lambda^3 + 8.4195\lambda^2 + 0.2693\lambda + 0.0060 = 0 \qquad (2.26)$$

$\lambda_1 = -6.8006, \lambda_2 = -3.2912, \lambda_3 = -0.3379,\ \lambda_4 = -0.0164 + 0.0229i, \lambda_5 = -0.0164 - 0.0229i$ and $R_0 = 1.7675$.

Table 20	γ_3=0.3	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.3370 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ 0.0204 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.3000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.5484\lambda^4 + 27.0017\lambda^3 + 10.3080\lambda^2 + 0.2441\lambda + 0.0035 = 0 \qquad (2.27)$$

$\lambda_1 = -6.7993, \lambda_2 = -3.2937, \lambda_3 = -0.4311,\ \lambda_4 = -0.0121 + 0.0148i, \lambda_5 = -0.0121 - 0.0148i$ and $R_0 = 1.3433$

Table 21	γ_3=0.4	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.4227 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ 0.0061 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.4000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.6341\lambda^4 + 27.8675\lambda^3 + 12.1966\lambda^2 + 0.2190\lambda + 0.0011 = 0 \qquad (2.29)$$

$\lambda_1 = -6.7980, \lambda_2 = -3.2963, \lambda_3 = -0.5214,\ \lambda_4 = -0.0093 + 0.0023i, \lambda_5 = -0.0093 - 0.0023i$ and $R_0 = 1.0832$.

For

Table 22	γ_3=0.5	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.5085 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ -0.0081 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.5000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}$$

$$\lambda^5 + 10.7199\lambda^4 + 27.7333\lambda^3 + 14.0851\lambda^2 + 0.1939\lambda - 0.0014 = 0 \qquad (2.30)$$

$\lambda_1 = -6.7966, \lambda_2 = -3.2990, \lambda_3 = -0.6099,\ \lambda_4 = -0.0196, \lambda_5 = 0.0052$ and $R_0 = 0.9076$.

For

Table 23	γ_3=0.45	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.4656 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ -0.0010 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.5000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.6770\lambda^4 + 28.3004\lambda^3 + 13.1408\lambda^2 + 0.2065\lambda - 0.0002 = 0 \qquad (2.31)$$

$\lambda_1 = -6.7973, \lambda_2 = -3.2976, \lambda_3 = -0.5658,\ \lambda_4 = -0.0171, \lambda_5 = 0.0008$ and $R_0 = 0.9876$.

For

Table 24	γ_3=0.42	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.4399 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ -0.0033 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.4200 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.6513\lambda^4 + 28.0407\lambda^3 + 12.5743\lambda^2 + 0.2140\lambda + 0.0006 = 0 \qquad (2.32)$$

$\lambda_1 = -6.7977, \lambda_2 = -3.2968, \lambda_3 = -0.5392,\ \lambda_4 = -0.0143, \lambda_5 = -0.0033$ and $R_0 = 1.0429$.

For

Table 25	γ_3=0.43	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.4485 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ 0.0019 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.4300 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.6599\lambda^4 + 28.1272\lambda^3 + 12.7631\lambda^2 + 0.2115\lambda + 0.0003 = 0 \qquad (2.33)$$

$\lambda_1 = -6.7975, \lambda_2 = -3.2968, \lambda_3 = -0.5392,\ \lambda_4 = -0.0143, \lambda_5 = -0.0033$ and $R_0 = 1.0238$.

For

Table 26	γ_3=0.44	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.4570 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ 0.0004 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.4400 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.6684\lambda^4 + 28.2138\lambda^3 + 12.95201\lambda^2 + 0.2090\lambda + 0.0001 = 0 \qquad (2.34)$$

$\lambda_1 = -6.7974, \lambda_2 = -3.2974, \lambda_3 = -0.5569,\ \lambda_4 = -0.0164, \lambda_5 = -0.0004$ and $\lambda_0 = 1.0054$.

For

Table 27	λ_3=0.6	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.5942 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ -0.0224 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.6000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.8056\lambda^4 + 29.5990\lambda^3 + 15.9737\lambda^2 + 0.1688\lambda - 0.0039 = 0 \qquad (2.35)$$

$\lambda_1 = -6.7951, \lambda_2 = -3.3020, \lambda_3 = -0.6972,\ \lambda_4 = -0.0224, \lambda_5 = 0.0111$ and $R_0 = 0.7809$.

For

Table 28	γ_3=0.7	α=0	$\omega = 0$

$$J_{E_E} = \begin{pmatrix} -0.6800 & 0.0000 & -0.3951 & -0.0632 & 0.1000 \\ -0.0366 & -6.0166 & 0.3951 & 0.0632 & 0.0000 \\ 0.0000 & 6.0000 & -4.0166 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.5400 & -0.0616 & 0.0000 \\ 0.7000 & 0.0000 & 0.0000 & 0.0000 & -0.1166 \end{pmatrix}.$$

$$\lambda^5 + 10.8914\lambda^4 + 30.4648\lambda^3 + 17.8622\lambda^2 + 0.1436\lambda - 0.0064 = 0 \qquad (2.36)$$

$\lambda_1 = -6.7937, \lambda_2 = -3.3052, \lambda_3 = -0.7838,\ \lambda_4 = -0.0239, \lambda_5 = 0.0151$ and $R_0 = 0.6853$.

4. Global Stability of the Equilibria

We carry out the following numerical simulations to investigate global stability of both the disease-free and the endemic equilibrium states.

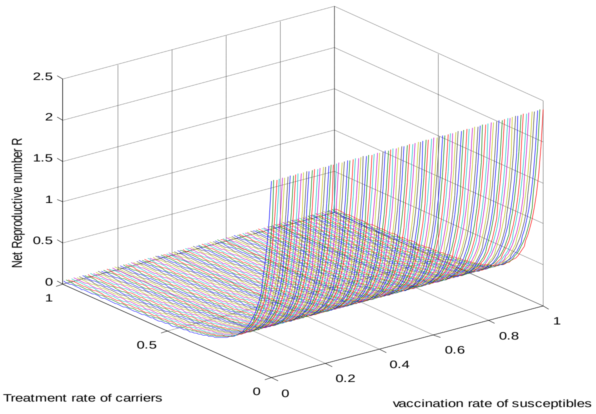

Figure 1: Impact of vaccination and treatment on the net reproduction number

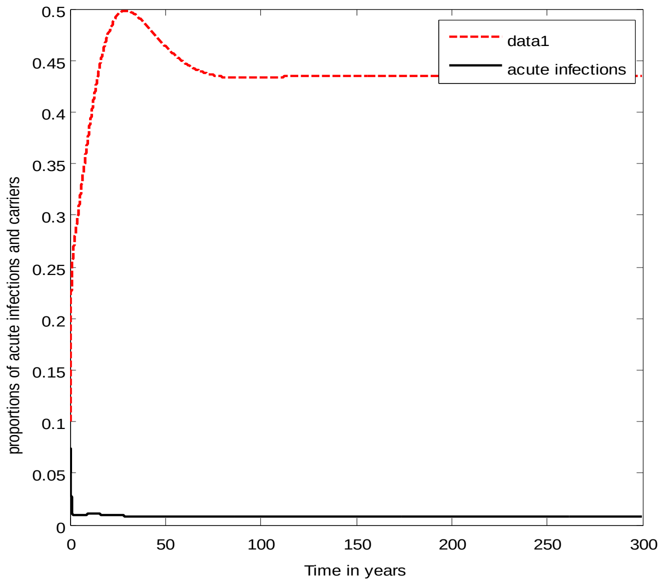

Figure 2: γ_3=0; ω=1; α=0

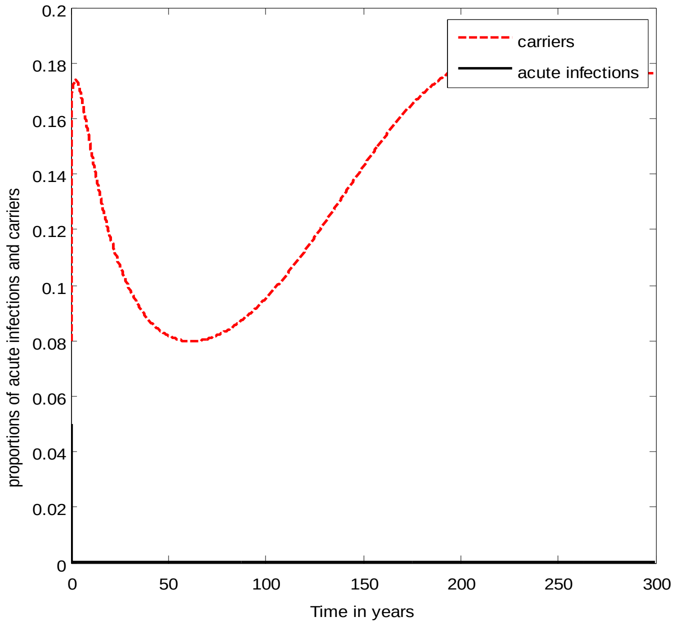

Figure 3: γ_3=0.2; ω=0; α=0

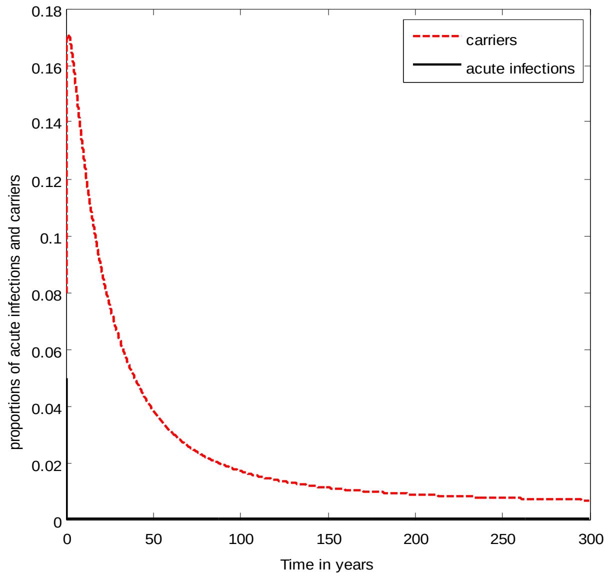

Figure 4: γ_3=0.4; ω=0; α=0

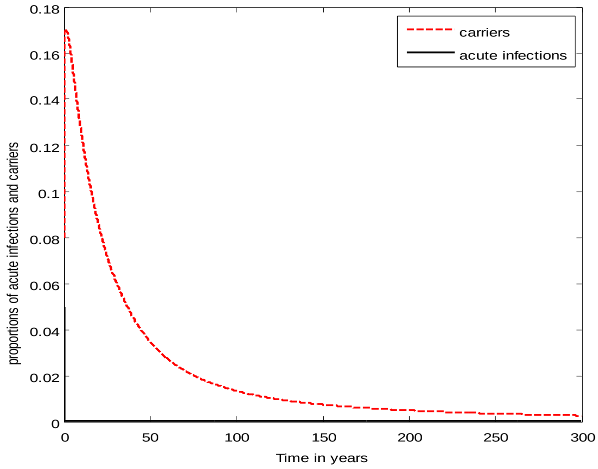

Figure 5: γ_3=0.44; ω=0; α=0

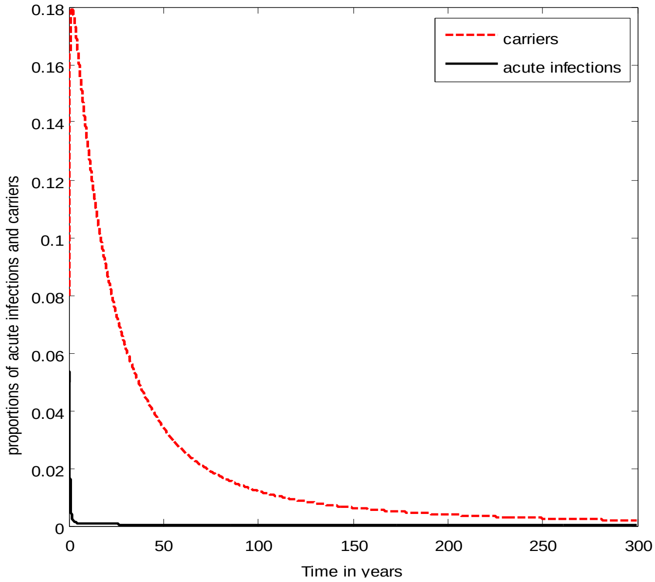

Figure 6: γ_3=0.5; ω=0; α=0

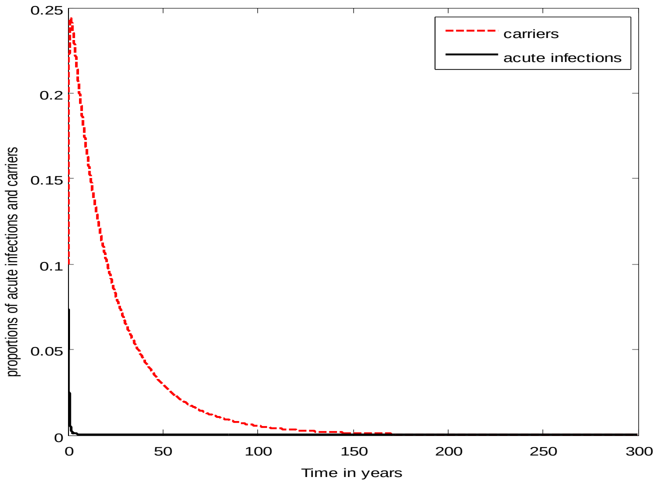

Figure 7: γ_3=0.8; ω=0; α=0

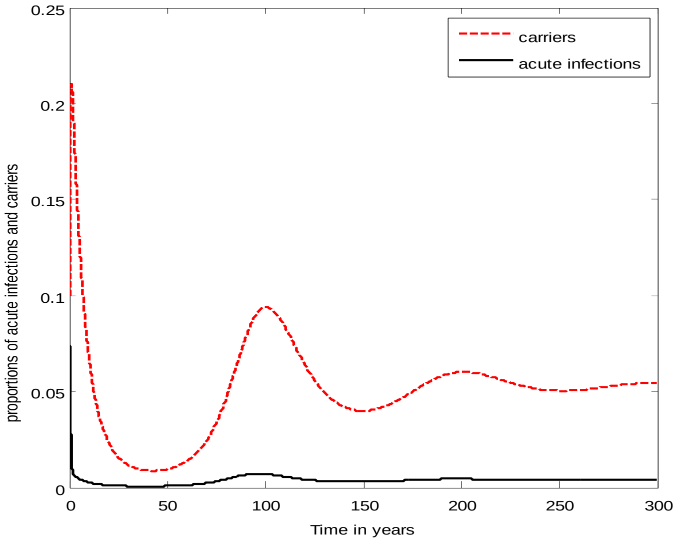

Figure 8: γ_3=0.0; ω=1; α=0.2

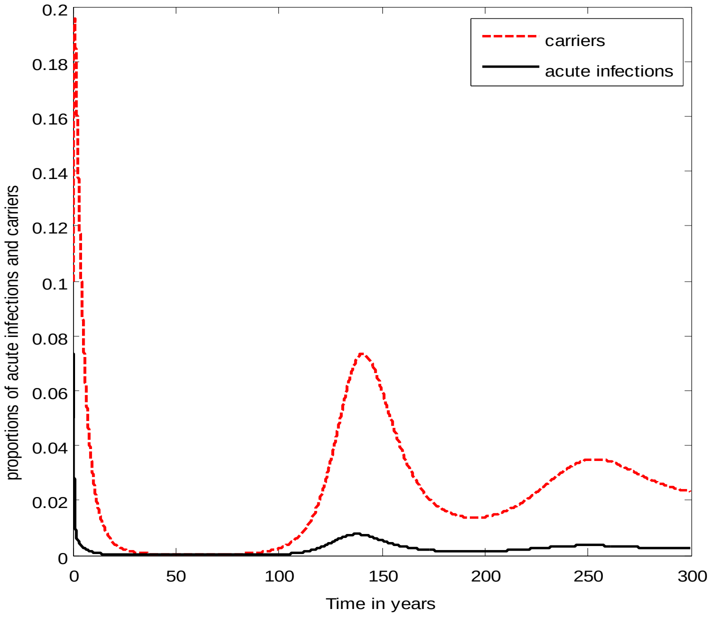

Figure 9: γ_3=0.0; ω=1; α=0.3

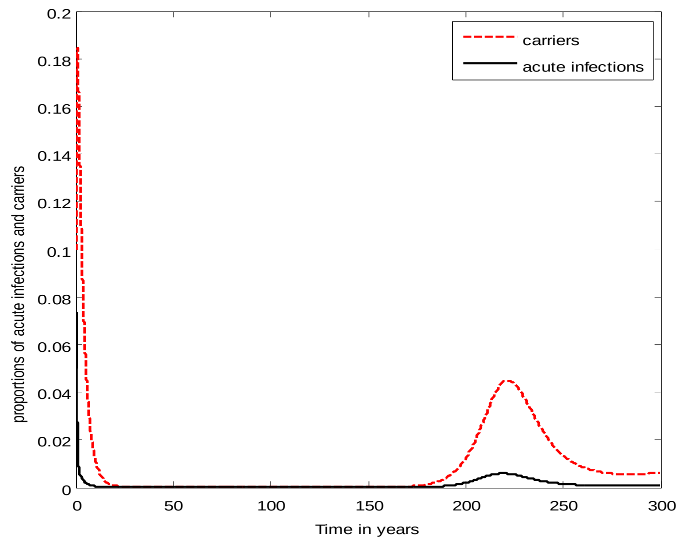

Figure 10: γ_3=0.0; ω=1; α=0.4

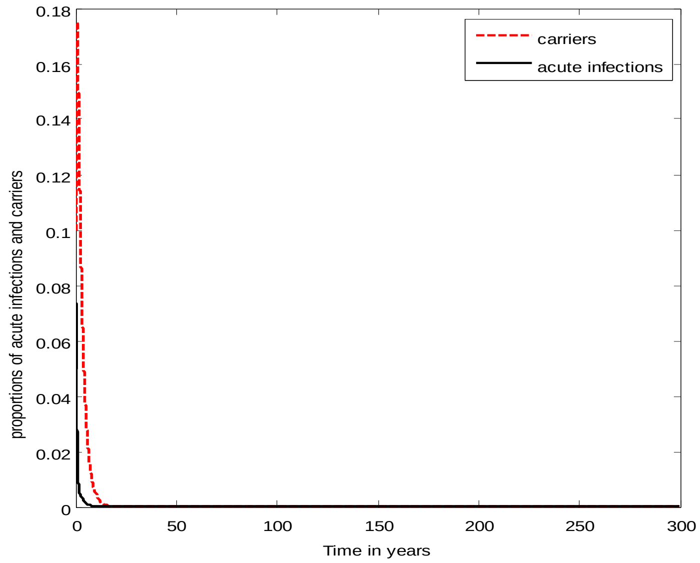

Figure 11: γ_3=0.0; ω=1; α=0.5

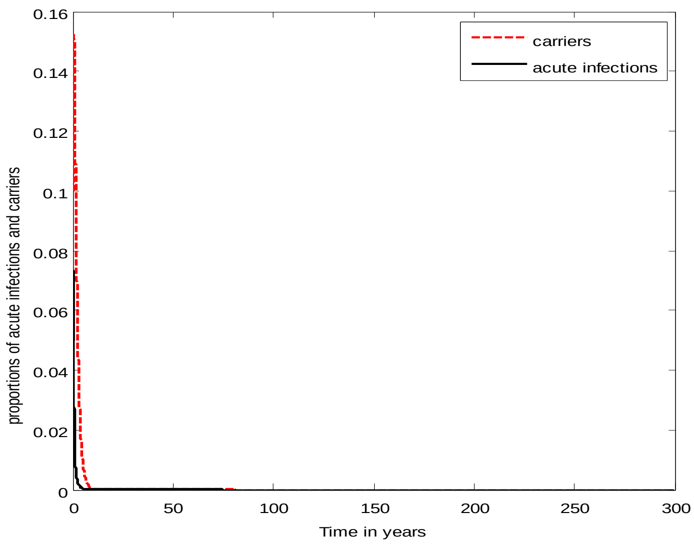

Figure 12: γ_3=0.0; ω=1; α=0.8

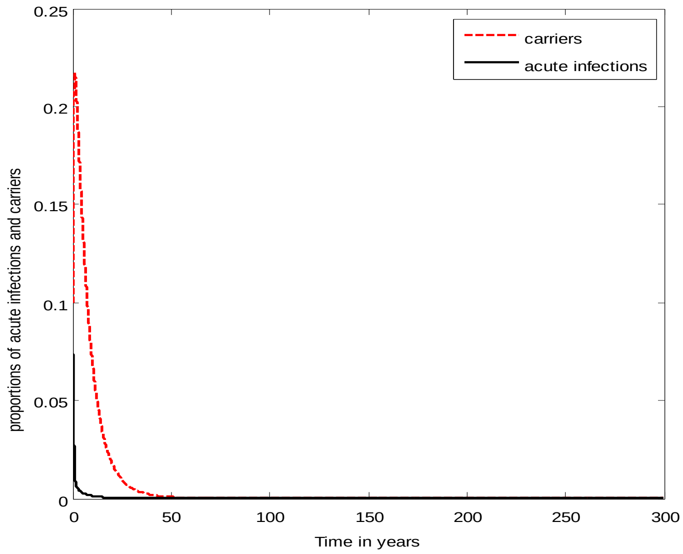

Figure 13: γ_3=0.1; ω=0.5; α=0.15

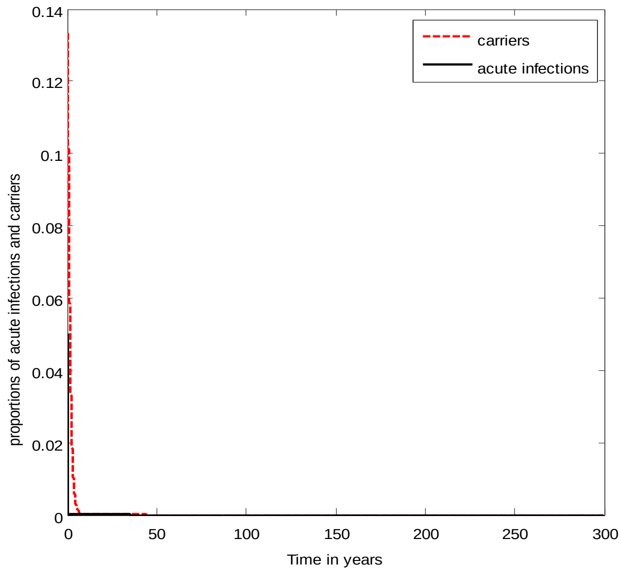

Figure 14: γ_3=0.8; ω=0; α=0.8

5. Discussion

In this study a mathematical model of hepatitis B transmission dynamics incorporating vaccination and treatment as control parameters is numerically explored. The control parameters in this model are γ_3, ω and α representing the rates of vaccination of the susceptible individuals, vaccination of the neonates and treatment of carriers respectively.

First, we investigated the local stability of the disease-free equilibrium state for different values of the control parameters. The numerical results show that reducing R_0 below 1 through γ_3, α or combinations of γ_3, ω and α guarantees local asymptotic stability of the disease-free equilibrium state. The numerical results also reveal that the disease-free equilibrium state becomes unstable for R_0 greater than 1. These can be seen for various values of the control parameters in Tables 2 through 16.

Secondly, we examined the stability properties of the endemic equilibrium state for varying values of the control parameters. The numerical results show that if R_0 is greater than 1, the endemic equilibrium state becomes locally asymptotically stable. The numerical results further show that reducing R_0 below 1 makes the endemic equilibrium state unstable. These can be observed for the values of the control parameters shown in Tables 17 through 26.

Thirdly, some numerical simulations of the model were performed. These were to investigate the global stability properties of the equilibria. The numerical results show that values of $R_0 < 1$ are sufficient for global stability of the disease-free equilibrium state. Furthermore, the values of $R_0 > 1$ are sufficient for the global stability of the endemic equilibrium state. These are depicted by figures 2 through 14.

6. Conclusion

In this paper, we carried out numerical analysis of a mathematical model of HBV considering vaccination and treatment parameters. The model was earlier formulated in our article [19-20] and the analytical method as well as the sensitivity analysis of the control parameters to the basic reproduction number was done. Here we carried out numerical analysis of the local and global properties of the disease-free and the endemic equilibria. The numerical results show that the values of $R_0 < 1$ guarantee both local and global stability of the disease-free equilibrium state. The numerical results also reveal that values of $R_0 > 1$ are adequate for both the local and global stability of the endemic equilibrium state.

References

[1]. WHO (2002).Hepatitis B Factsheet.

[2]. CDC (2005). Centre for prevention and control of diseases.

[3]. White, O. D. and Fenner, J.F. (1994). Viruses of humans, medical virology (4th ed.) Academic Press Ltd.

[4]. Platkov, E., Shlyakov, E., Glick,V., Khalemsky, S., and Fisehbein, A. (2001) Humoral immune response of hospital employees induced by a recombinant hepatitis vaccine: 5 years after the primary standard immunization, the Journal of preventive medicine 9(3): 59-66.

[5]. Carriappa, M.M., Jayaram, B.J., Bhalwar, C.R., Praharaj, A., Mehta, V and Kpur, L. (2004). Epidemiological differentials of hepatitis B carrier state in the army: a community-based sero-epidemiological study MJAFI, vol. 60, no.3.

[6]. Fernadez, E., Rodrigo, L., Garcia S., Riestra S. and Blanco C. (2006). Hepatitis B surface antigen detection using pooled sera: A cost benefit analysis. Rev. esp enferm dig vol.98: no.2 pp.112-121.

[7]. Onuzulike, N. and Ogueri, E.O., (2007). Seroprevalence of hepatitis B surface antigen (HBsAg) in pregnant women in Owerri, Imo state of Nigeria, Research. J. of boil. Sc. 2(2):178-182.

[8]. Zou, L. and Zhang, W. and Ruan, S. (2009). Modeling the transmission dynamics and control of hepatitis B virus in China. J. Theor. Biol., Vol. 10, pp. 1-9.

[9]. Anderson, R.M. and May, R.M. (1991). Infectious diseases of humans: Dynamics and Control. Oxford University Press.

[10]. Anderson, R.M. and May, R.M.(1992). Directly transmitted infectious diseases: Control by vaccination, Science, Vol. 215, pp 1053-1060.

[11]. Williams, J.R., Nokes, D. J., Medley, G. F., and Anderson,R.M. (1996).The transmission dynamics of hepatitis B in the UK: A mathematical model for evaluating costs and effectiveness of immunization programmes. J. of Epidemiol. Vol. 116, 71–89.

[12]. Edmunds, W.J., Medley, G.F., Nokes, D.J., Hall, A.J., and Whittle, H.C. (1993). The influence of age on the development of the hepatitis B carrier state. Proc. R. Soc. London. B 253, 197-201.

[13]. Medley, G.F., Lindop, N.A, Edmunds, W.J. and Nokes, D.J. (2001). Hepatitis B virus endemicity: heterogeneity, catastrophic dynamics and control. J of nature medicine, Vol.7: No.5, Nature Publishing Group.

[14]. Thornley, S., Bullen, C., and Roberts, M. (2008).Hepatitis B in a high prevalence New Zealand population: A mathematical model applied to infection control policy. J. Theor. Biol. 254, 599–603.

[15]. Edmunds, W.J., Medley, G.F., and Nokes, D.J. (1996). Epidemiological patterns of hepatitis B virus (HBV) in highly endemic areas. *Epidemiology of Infection.* 117, 313-325.

[16]. McLean, A.R., and Blumberg, B.S., (1994). Modelling the impact of mass vaccination against hepatitis B: Model formulation and parameter estimation. Proc.R. Soc. Lond.B256, 7–15.

[17]. Edmunds, W.J., Medley, G.F., and Nokes, D.J. (1996). The transmission dynamics and control of hepatitis B virus in Gambia. *Journal of Statistics in Medical Research* 15: 2215-2233.

[18]. Mehmood, N. (2011). Modelling the transmission dynamics of hepatitis B and optimal control, J. Theor. Biol., Vol. 13, pp. 1-17.

[19]. Kimbir, A.R., Aboiyar, T, Abu, O. and Onah, E.S. (2014). Modeling the HBV transmission dynamics in the presence of vaccination and treatment. Mathematical Theory and Modeling. ISSN 2224-5804 (paper), ISSN 2225-0522 (online) vol.4, No. 12 :29-43

[20]. Kimbir, A.R., Aboiyar, T, Abu, O. and Onah, E.S. (2014). A Simulation of a mathematical model of HBV transmission dynamics in the presence of vaccination and treatment. Mathematical Theory and Modeling. ISSN 2224-5804 (paper), ISSN 2225-0522 (online) vol.4, No. 12 :44-59

[21]. WHO (2001).Hepatitis B Factsheet.

[22]. O'Leary, C., Hong, Z., Zhang, F., Dawood, M., Smart, G., Kaita, K. and Wu, J.(2008). A mathematical model to study the effect of hepatitis B virus vaccine and anti-virus treatment among the Canadian inuit population, CDC.

[23]. USAID (2009).

[24]. Olumuyiwa, O.O., Evan, A., Francis, P.M. and Vincent, I.A. (2011) Five-years post-vaccination efficacy of hepatitis B vaccine in rural Nigeria. *Landes Bioscience.* 7(6):625-629.

[25]. Vanden Driessche, P., Watmough, J.,(2002).Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Bio Sci.180, 29–48.

[26]. Heffernan, J.M. (2005). Perspectives on the basic reproductive ratio. J. R. Soc. Interface,2, 281-293.

[27]. Mukandavire, Z., Das, P., Chiyaka, C. and Nyabadza, F. (2010). Global analysis of an HIV/AIDS epidemic model, WJMS, 6(3):231-240, UK.

[28]. Ameh, E.J. (2009) .The basic reproduction number: Bifurcation and Stability, PGD project at AIMS.