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***Journal of Scientific and Engineering Research*, 2017, 4(9):311-318**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Two-step Algorithm for the Solution of Nonlinear Differential Equations

J. Sunday *, P. Tumba

Department of Mathematics, Adamawa State University, Mubi, Nigeria

Abstract In this research, an efficient two-step algorithm is derived for the solution of nonlinear first order differential equations. The derivation is carried out with the aid of collocation and interpolation of power series basis function. The reliability and applicability of the two-step algorithm derived was established by solving some nonlinear differential equations. The results obtained in terms of the point wise absolute errors show that the two-step algorithm developed approximates the exact solutions closely. The research further investigated the basic properties of the two-step algorithm and found it to be zero-stable, consistent and convergent.

Keywords Algorithm, differential equations, nonlinear, two-step
2010 AMS Subject Classification: 65L05, 65L06, 65D30

1. Introduction

The problem of deriving efficient algorithms for the solution of differential equations has received a great deal of attention in recent years. This is the reason why a wide variety of methods have been proposed. Three important factors contribute to the efficiency of any algorithm for solving ordinary differential equations, [1]:

- the relative ease with which the step-size may be changed,
- the possibility of using high order, highly stable schemes and
- the relatively small amount of computational effort required per step given that an efficient differential equation integrator must be implicit.

In this paper, we shall consider first order nonlinear differential equations of the form

$$y'(t) = f(t, y), \;\; y(t_0) = y_0 \tag{1}$$

where $f : \Re \times \Re^m \to \Re^m, \; y, y_0 \in \Re^m$ *and* f is assumed to satisfy the Lipchitz condition stated in the theorem below.

Theorem 1.1 [2]

Let $f(t, y)$ *be defined and continuous for all points* (t, y) *in the region* D *defined by* $a \le t \le b, -\infty < y < \infty$, a *and* b *finite, and let there exist a constant* L *such that, for every* t, y, y^* *such that* (t, y) *and* (t, y^*) *are both in* D*;*

$$\left| f(t, y) - f(t, y^*) \right| \le L \left| y - y* \right| \tag{2}$$

Then, if η *is any given number, there exists a unique solution* $y(t)$ *of the initial value problem (1), where* $y(t)$ *is continuous and differentiable for all* $(t, y) \, in \, D$*. The requirement (2) is known as a Lipschitz condition and the constant* L *as a Lipschitz constant.*

Equations of the form (1) find application in areas of engineering, science and social sciences. It is a well known fact that some of these problems have proved to be either difficult to solve or cannot be solved analytically, hence the necessity of numerical techniques for such problems remains vital [3].

A lot of methods have been proposed for the solution of problems of the form (1). Linear Multistep Methods (LMMs) have been developed varying from discrete LMMs to continuous ones. Continuous LMMs have greater advantages over the discrete methods such that they give better error estimation, provide a simplified form of coefficients for further evaluation at different grid points and provide approximate solution at all interior points within the interval of integration [4]. These methods are first derivative methods that are implemented in predictor-corrector mode and Taylor series expansions are adopted to supply starting values [3]. The setbacks of the predictor-corrector methods are that they are very costly to implement, longer computer time, greater human effort and reduced order of accuracy which affect the accuracy of the method.

Scholars latter developed block methods to cater for some setbacks of the predictor-corrector methods mentioned above. Block methods generate independent solutions at all selected grid point without overlapping. It is less expensive in terms of the number of function evaluation compared to predictor-corrector methods and moreover it possesses the properties of Runge-Kutta methods of being self-starting, see [5-7]. The block method was modified by incorporating function evaluation at off-step points to afford the opportunity of circumventing the 'zero stability barrier' and this made it possible to obtain convergent k-step methods with order $2k+1$ up to $k=7$, [8]. Even higher orders are available if two or more offstep points are used. This method was called 'hybrid method'. The method is useful in reducing the step number of a method and still remains zero-stable, see the works of [9, 10].

Definition 1.1: [11]

A numerical integrator is said to be A-stable if its region of absolute stability R incorporates the entire left-half of the complex plane denoted by C, that is, $R=\{z\in C: re(z)<0\}$

Definition 1.2: [12]

A numerical integration scheme is said to be $A(\alpha)$-stable for some $\alpha\in[0,\pi/2]$ if the wedge

$$S_\alpha=\{z:|Arg(-z)|<\alpha,\ z\neq 0\}$$

is contained in its region of absolute stability. The largest $\alpha\left(i.e.\ \alpha_{\max}\right)$ is called the angle of absolute stability.

In view of the foregoing, a two-step algorithm shall be derived for the solution of nonlinear differential equations of the form (1).

2. Derivation of the Two-Step Algorithm

A two-step algorithm of the form,

$$A^{(0)}\mathbf{Y}_m=E\mathbf{y}_n+hd\mathbf{f}(\mathbf{y}_n)+hb\mathbf{F}(\mathbf{Y}_m) \tag{3}$$

shall be derived for the solution of nonlinear equations of the form (1). In doing this, power series will be employed as basis function. The power series is given by,

$$y(t)=\sum_{n=0}^{r+s-1}a_n t^n \tag{4}$$

where $r\ and\ s$ are the numbers of collocation and interpolation points respectively.

Let the approximate solution to (1) be given by power series of degree 5, by allowing $r+s-1=5$ in equation (4), that is,

$$y(t)=\sum_{n=0}^{5}a_n t^n=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5 \tag{5}$$

with the first derivative given by,

$$y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 \tag{6}$$

Substituting (6) into (1) gives,

$$f(t, y) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 \tag{7}$$

Now, interpolating (5) at point $t_{n+s}, s = \frac{3}{2}$ and collocating (7) at points $t_{n+r}, r = 0\left(\frac{1}{2}\right)2$, leads to a system of nonlinear equation of the form,

$$TA = U \tag{8}$$

where

$$A = \left[a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5\right]^T \qquad U = \left[y_n \;\; f_n \;\; f_{n+\frac{1}{2}} \;\; f_{n+1} \;\; f_{n+\frac{3}{2}} \;\; f_{n+2}\right]^T$$

$$T = \begin{bmatrix} 1 & t_{n+\frac{3}{2}} & t^2_{n+\frac{3}{2}} & t^3_{n+\frac{3}{2}} & t^4_{n+\frac{3}{2}} & t^5_{n+\frac{3}{2}} \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 1 & 2t_{n+\frac{1}{2}} & 3t^2_{n+\frac{1}{2}} & 4t^3_{n+\frac{1}{2}} & 5t^4_{n+\frac{1}{2}} \\ 0 & 1 & 2t_{n+1} & 3t^2_{n+1} & 4t^3_{n+1} & 5t^4_{n+1} \\ 0 & 1 & 2t_{n+\frac{3}{2}} & 3t^2_{n+\frac{3}{2}} & 4t^3_{n+\frac{3}{2}} & 5t^4_{n+\frac{3}{2}} \\ 0 & 1 & 2t_{n+2} & 3t^2_{n+2} & 4t^3_{n+2} & 5t^4_{n+2} \end{bmatrix}$$

Solving (8) by Gauss elimination method for the $a_j\text{'s}, j = 0(1)5$ and substituting back into the power series basis function gives a linear multistep method of the form,

$$y(t) = \alpha_{\frac{3}{2}}(t) y_{n+\frac{3}{2}} + h\sum_{j=0}^{2} \beta_j(t) f_{n+j}, \; j = 0\left(\frac{1}{2}\right)2 \tag{9}$$

where the coefficients of y_n *and* f_{n+j} are given as,

$$\left.\begin{aligned} \alpha_{\frac{3}{2}} &= 1 \\ \beta_0 &= \frac{1}{1440}(192x^5 - 1200x^4 + 2800x^3 - 3000x^2 + 1440x - 243) \\ \beta_{\frac{1}{2}} &= -\frac{1}{720}(384x^5 - 2160x^4 + 4160x^3 - 2880x^2 + 459) \\ \beta_1 &= \frac{1}{60}(48x^5 - 240x^4 + 380x^3 - 180x^2 - 27) \\ \beta_{\frac{3}{2}} &= -\frac{1}{720}(384x^5 - 1680x^4 + 2240x^3 - 960x^2 + 189) \\ \beta_2 &= \frac{1}{1440}(192x^5 - 720x^4 + 880x^3 - 360x^2 + 27) \end{aligned}\right\} \tag{10}$$

and x is given by

$$x=\frac{t-t_n}{h} \tag{11}$$

Evaluating (9) at $t=\frac{1}{2}\left(\frac{1}{2}\right)2$, gives a discrete two-step algorithm of the form (3) given by,

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{2}}\\y_{n-1}\\y_{n-\frac{3}{2}}\\y_{n}\end{bmatrix}+h\begin{bmatrix}0&0&0&\frac{251}{1440}\\0&0&0&\frac{29}{180}\\0&0&0&\frac{27}{160}\\0&0&0&\frac{7}{45}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{2}}\\f_{n-1}\\f_{n-\frac{3}{2}}\\f_{n}\end{bmatrix}$$
$$+h\begin{bmatrix}\frac{323}{720}&\frac{-11}{60}&\frac{53}{720}&\frac{-19}{1440}\\\frac{31}{45}&\frac{2}{45}&\frac{1}{45}&\frac{-1}{180}\\\frac{51}{80}&\frac{9}{20}&\frac{21}{80}&\frac{-3}{160}\\\frac{32}{45}&\frac{4}{15}&\frac{32}{45}&\frac{7}{45}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix} \tag{12}$$

The two-step algorithm derived in equation (12) is capable of solving nonlinear differential equation of the form (1).

3. Analysis of the Two-Step Algorithm

Some basic properties of the two-step algorithm derived shall be discussed in this section.

3.1. Order of the Two-Step Algorithm

According to [13], the two-step algorithm (12) is said to be of uniform accurate order p, if p is the largest positive integer for which $\bar{c}_0=\bar{c}_1=\bar{c}_2=...=\bar{c}_p=0,\ \bar{c}_{p+1}\neq \bar{0}$. $\bar{c}_{p+1}$ is called the error constant and the local truncation error of the method is given by;

$$\bar{t}_{n+k}=\bar{c}_{p+1}h^{(p+1)}y^{(p+1)}(t)+O\left(h^{(p+2)}\right) \tag{13}$$

Therefore, for our two-step algorithm $\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{0}$, implying that the order $p=[5\quad 5\quad 5\quad 6]^T$ and the error constant is given by $\left[2.9297\times10^{-4}\quad 1.7361\times10^{-4}\quad 2.9297\times10^{-5}\quad -6.6138\times10^{-5}\right]^T$.

3.2. Consistency of the Two-Step Algorithm

The two-step algorithm (12) is consistent since it has order $p\geq 1$. Consistency controls the magnitude of the local truncation error committed at each stage of the computation, [14].

3.3. Zero Stability of the Two-Step Algorithm

Definition 3.1 [14]: A block method is said to be zero-stable, if the roots $z_{s,}s=1,2,...,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z)=\det(zA^{(0)}-E)$ satisfies $|z_s|\leq 1$ and every root satisfying $|z_s|=1$ have multiplicity not exceeding the order of the differential equation.

For the two-step algorithm (12), the first characteristic polynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\right|$$

$$=\begin{vmatrix}z&0&0&-1\\0&z&0&-1\\0&0&z&-1\\0&0&0&z-1\end{vmatrix}=z^3(z-1)$$

Thus, solving for z in

$$z^3(z-1)=0 \tag{14}$$

gives $z_1=z_2=z_3=0$ *and* $z_4=1$. Hence, the two-step algorithm (12) is zero-stable.

3.4. Convergence of the Two-Step Algorithm

The two-step algorithm is convergent since it is consistent and zero-stable.

Theorem 3.1 [15]

A linear multistep method is convergent if and only if it is zero stable and consistent

3.5. Region of Absolute Stability of the Two-Step Algorithm

Definition 3.2 [16]

Region of absolute stability is a region in the complex z plane, where $z=\lambda h$. It is defined as those values of z such that the numerical solutions of $y'=-\lambda y$ satisfy $y_j\to 0$ as $j\to\infty$ for any initial condition.

Applying the boundary locus method, we obtain the stability polynomial for the two-step algorithm derived as;

$$\begin{aligned}\bar{h}(w)=&-h^4\left(\frac{1}{80}w^3-\frac{17}{600}w^4\right)-h^3\left(\frac{223}{3600}w^4+\frac{5}{48}w^3\right)-h^2\left(\frac{7}{16}w^3-\frac{1878}{3600}w^4\right)\\&-h\left(w^4+w^3\right)+w^4-w^3\end{aligned} \tag{15}$$

The region of absolute stability of the two-step algorithm is shown in Figure 3.1.

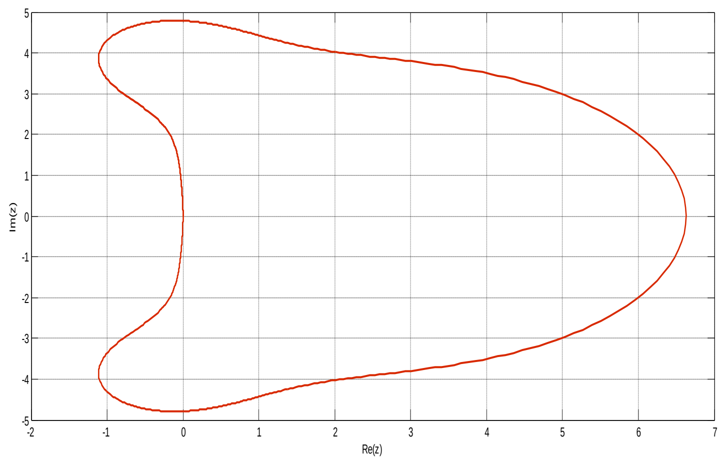

Figure 3.1: Stability region of the two-step method

The stability region obtained in Figure 3.1 is $A(\alpha)$-stable (see [12]), since the stability region consists of the complex plane outside the enclosed figure. Note that the unstable region is the interior of the curve (when the curve is on the positive plane) while the stability region contains the exterior part of the curve.

4. Results

4.1 Numerical Experiments

The two-step algorithm developed shall be applied on two important nonlinear differential equations that find application in science and engineering. This is with the view to testing how computationally reliable the two-step algorithm derived is.

The following notations shall be used in the Tables below:

ERR= Absolute error in the computational method

Eval t =Evaluation time per seconds

EFA-Absolute error in [17]

ENB-Absolute error in [18]

Problem 4.1:

Consider the nonlinear problem,

$$y'(t) = -(1+t)^{-1} + y(t) - y^2(t),\ y(0) = 1 \tag{16}$$

The exact solution is given by,

$$y(t) = (1+t)^{-1} \tag{17}$$

Source: [17]

Table 4.1: Showing the result for the nonlinear problem 4.1

t	Exact Solution	Computed Solution	ERR	EFA	Eval t
0.1000	0.9090909090909091	0.9090909090915035	5.944134e-013	3.8296e-07	0.0813
0.2000	0.8333333333333334	0.8333333333341414	8.080203e-013	3.8296e-07	0.0837
0.3000	0.7692307692307692	0.7692307692316456	8.764101e-013	5.7951e-07	0.0860
0.4000	0.7142857142857142	0.7142857142866030	8.888446e-013	6.8133e-07	0.0885
0.5000	0.6666666666666666	0.6666666666675477	8.810730e-013	7.3394e-07	0.0909
0.6000	0.6250000000000000	0.6250000000008679	8.678613e-013	7.6091e-07	0.0934
0.7000	0.5882352941176470	0.5882352941185024	8.554268e-013	7.7483e-07	0.0958
0.8000	0.5555555555555555	0.5555555555564012	8.457679e-013	7.8257e-07	0.0982
0.9000	0.5263157894736841	0.5263157894745238	8.397727e-013	7.8799e-07	0.1005
1.0000	0.4999999999999999	0.5000000000008377	8.377743e-013	7.9326e-07	0.1031

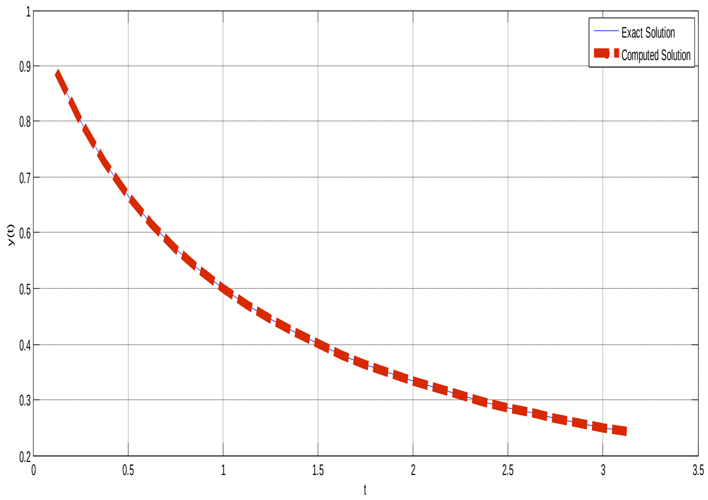

Figure 4.1: Graphical Results for the Nonlinear Problem 4.1.

Problem 4.2:

Consider the nonlinear problem,

$$y'(t) = -1 + y^2(t),\ y(0) = 0 \tag{18}$$

The exact solution is given by,

$$y(t) = -\tanh(t) \tag{19}$$

Source: [18]

Table 4.2: Showing the result for the nonlinear problem 4.2

t	Exact Solution	Computed Solution	ERR	ENB	Eval t
0.1000	-0.0996679946249558	-0.0996679946249528	3.053113e-015	1.8×10^{-7}	0.0171
0.2000	-0.1973753202249040	-0.1973753202248864	1.759703e-014	1.2×10^{-6}	0.0192
0.3000	-0.2913126124515909	-0.2913126124515429	4.796163e-014	2.7×10^{-6}	0.0215
0.4000	-0.3799489622552249	-0.3799489622551366	8.826273e-014	3.5×10^{-6}	0.0236
0.5000	-0.4621171572600099	-0.4621171572598834	1.264544e-013	2.9×10^{-6}	0.0258
0.6000	-0.5370495669980354	-0.5370495669978849	1.505462e-013	1.6×10^{-6}	0.0279
0.7000	-0.6043677771171636	-0.6043677771170095	1.540990e-013	8.7×10^{-7}	0.0301
0.8000	-0.6640367702678492	-0.6640367702677114	1.378897e-013	9.2×10^{-7}	0.0323
0.9000	-0.7162978701990246	-0.7162978701989173	1.072475e-013	1.1×10^{-6}	0.0345
1.0000	-0.7615941559557651	-0.7615941559556947	1.038814e-013	1.8×10^{-7}	0.0367

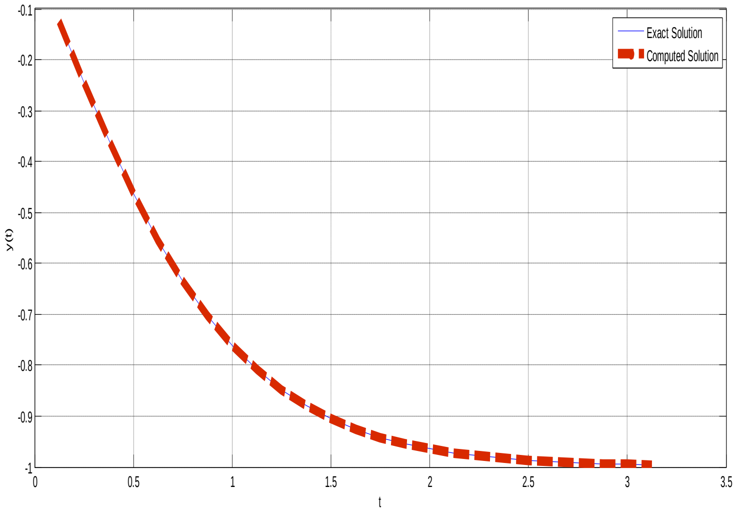

Figure 4.2: Graphical Results for the Nonlinear Problem 4.2

4.2. Discussion of Results

From the results above, it is clear that the two-step algorithm derived is efficient in handling nonlinear differential equations of the form (1). The stability region obtained also shows that the method can effectively handle even stiff equations since it is $A(\alpha)$-stable. The evaluation time per seconds obtained were also very small, showing that the method derived generates results faster. The analysis presented also show that the method is convergent, consistent and zero-stable.

5. Conclusion

The two-step algorithm derived in this work has been shown to be efficient in solving nonlinear problems of the form (1). Thus, the algorithm developed is recommended for the solution of problems of the form (1) and first order differential equations in general.

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