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Complex complete synchronization of hyperchaotic complex nonlinear systems with fully uncertain parameters and its applications in secure communications

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Abstract In this manuscript, we discuss the definition of complex complete synchronization (CCS) of hyperchaotic nonlinear systems inclusive complex variables and uncertain parameters. This type of synchronization can analyze only for complex nonlinear systems. The CCS contains or incorporates two kinds of synchronizations (complete synchronization and anti-synchronization). In CCS the attractors of the master and slave systems are moving perpendicular or orthogonal to each other with the same shape, this phenomenon does not exist in the literature. On the evidence of Lyapunov function and adaptive control technique, a scheme is created to perform the CCS of two identical hyperchaotic attractors of these systems. The effectiveness of the acquired results is shown by a simulation example. Numerical issues are plotted to determine state variables, synchronization errors, modules errors, and phases errors of those hyperchaotic attractors after synchronization to ascertain that CCS is achieved. The above results will provide the theoretical foundation for the secure communication applications. CCS of hyperchaotic complex systems in which a state variable of the master system synchronizes with a different state variable of the slave system is an encouraging type of synchronization as it contributes excellent security in secure communication. During this secure communication, synchronization among transmitter and receiver is closed and message signals are recovered. The encryption and restoration of the signals are simulated numerically.

Keywords Complex complete synchronization; Hyperchaotic; Adaptive control; Lyapunov function; Complex

1. Introduction

Synchronization approaches have numerous possible applications in various fields such as secure communication, physics and engineering applications [1-4]. These approaches applicable in the chaotic and hyperchaotic nonlinear systems with real variables. There also exist, however, significant characters of nonlinear dynamical systems, where the principal variables are complex. For illustration, when amplitudes of electromagnetic fields are implicated. Another illustration, when chaos synchronization is utilized to secure communications, where the complex variable (doubling the number of variables) may be adopted to enhance the comfortable of transmitting information signals and heighten their security [5].

Newly, Mahmoud et al. proposed numerous chaotic and hyperchaotic nonlinear systems with complex variables [5-9] and interpreted their chaotic and hyperchaotic forms and investigated different types of synchronizations for those systems, such as complete synchronization (CS) [10], anti-synchronization (AS) [11], lag synchronization [12], anti-lag synchronization [13], phase synchronization [11], projective and modified projective synchronizations [14], modified projective phase synchronizations [15] and etc.

All those kinds of synchronization can be considered and scrutinized for real or complex dynamical systems. Newly, there are several new kinds of synchronization cannot be studied for real dynamical systems, for example, module-phase synchronization [16], complex projective synchronization [17], complex lag synchronization [18], complex complete synchronization CCS [19] and complex modified projective synchronization [20]. These new kinds of synchronizations are reviewed for chaotic and hyperchaotic complex nonlinear systems. In complex space, there are two significant amounts module and phase. Therefore, the actions of module and phase are studied in [16-20].

Remarkable issues on chaos synchronization are obtained by using the known (certain) parameters of master and slave systems, and the controller is constructed by those known parameters. To our best understanding, some system's parameters cannot be accurately known in advance. The synchronization will be stopped with the consequences of these uncertainties. Furthermore, in complex physical systems or laboratory situations, chaotic or hyperchaotic complex systems may have some uncertain parameters and may change from time to time [21-24]. Thus, it is a very important problem to realize complex chaos synchronization for these uncertain chaotic systems. The adaptive control is one of the common and valuable ways to control and synchronize the solutions of nonlinear systems with unknown or uncertain parameters [9-20].

Newly, chaos and hyperchaos synchronization have been comprehensively reviewed for applications in secure communication [25-27]. The intention is that the chaotic or hyperchaotic signal can be used as a vehicle and transmitted together with an information signal to a receiver. Furthermore, chaos or hyperchaos synchronization is utilized to recover the information signal in the receiver. Moreover, the hyperchaotic systems represented essentially a chaotic attractor with more than one positive Lyapunov exponents, are widely used for cryptography and secure communication because the existence of extra than one Lyapunov exponent apparently improves the security of communication schemes by creating further complex dynamics [28].

The urge of this theme is to complete our investigations about the novel type of complex synchronization which we called complex complete synchronization (CCS) [19]. This kind of complex synchronization can be analyzed just for chaotic and hyperchaotic nonlinear systems with complex variables. The concept of CCS can be recognized as consolidating between CS [10] and AS [11]. The state variable of the master system synchronizes with a different state variable of the slave system. In addition, the attractors of the master and slave systems after achieving the CCS are moving perpendicular or orthogonal to each other with the same shape. These events and phenomena do not appear or exist in the all types of synchronizations in the literature.

We wish to investigate CCS of two identical hyperchaotic complex nonlinear models with uncertain parameters based on the adaptive control method. The results of the CCS will provide the theoretical foundation for the secure communication applications.

The plan of this paper is as follows: Section 2 pronounces the definition and applications of hyperchaotic complex nonlinear systems and illustrates the description of hyperchaotic complex Lü system. The design of the suggested scheme to achieve CCS of two identical hyperchaotic complex nonlinear systems is stated in Section 3. In Section 4 we study CCS between hyperchaotic complex Lü systems. Numerical simulation is used to demonstrate the validity of this study. A simple application for the secure communication, based on the results of the CCS, is shown Section 5. Finally, the main conclusions of our investigations are summarized in Section 6.

2. A hyperchaotic complex nonlinear system

A complex dynamical system is named hyperchaotic if it is deterministic and displays the sentient dependence on the initial conditions. A hyperchaotic complex conformity is characterized as a complex chaotic model including at limited two positive Lyapunov exponents. The aggregate of Lyapunov exponents must be negative to guarantee that system is dissipative. It is indeed more complicated than chaotic complex systems and has extra unstable manifolds. Due to hyperchaotic complex systems with components of huge proportions, huge security, and huge productivity, it has a broadly applied potential in nonlinear circuits, secure communications, lasers, neural networks, biological systems and so on. Accordingly, research on hyperchaotic complex nonlinear systems is greatly necessary nowadays [7, 9, 13].

Suppose the complex nonlinear system with the hyperchaotic action in the form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})\mathbf{A} + \mathbf{f}(\mathbf{x}), \qquad (1)$$
$$\dot{\boldsymbol{z}} = g(\mathbf{x}, \bar{\mathbf{x}}, \boldsymbol{z}),$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ is a state complex vector, $\mathbf{x} = \mathbf{x}^r + j\mathbf{x}^i, \mathbf{x}^r = (u_1, u_3, ..., u_{2n-1})^T$,

$\mathbf{x}^i = (u_{2,} u_4, ..., u_{2n})^T$, $j = \sqrt{-1}$, T denotes transpose, $\mathbf{F}(\mathbf{x})$ is $n \times n$ complex matrix and the elements of this matrix are state complex variables, $\mathbf{A}$ is $n \times 1$ vector of system parameters, $\mathbf{f} = (f_1, f_2, ..., f_n)^T$ is a vector of linear or nonlinear complex functions, $\boldsymbol{z}$ is the vector of the real variables and superscripts r and i stand for the real and imaginary parts of the state complex vector $\mathbf{x}$.

In this paper we introduce and investigate the phenomenon of the adaptive CCS of two identical systems of the form (1) with fully unknown parameters by designing a control scheme. We tested its validity numerically.

Remark 1. Utmost of the hyperchaotic complex systems can be described by (1), such as hyperchaotic complex Lorenz, Chen and Lü systems [6, 7, 9].

In order to show the results of our scheme of two identical systems of the form (1) we choose, as an example, the hyperchaotic complex Lü system which have been introduced and studied recently in our work [6]. The hyperchaotic complex Lü system is:

$$\begin{aligned}
\dot{x} &= \rho(y - x) + w, \\
\dot{y} &= \nu y - xz + w, \qquad (2)\\
\dot{z} &= 1/2\left(\overline{x}y + x\overline{y}\right) - \mu z, \\
\dot{w} &= 1/2\left(\overline{x}y + x\overline{y}\right) - \sigma w,
\end{aligned}$$

where $\mathbf{x} = (x_1, x_2)^T = (x, y)^T, \boldsymbol{z} = (z, w)^T, \rho, \mu$ and ν are positive parameters, $x = u_1 + ju_2, y = u_3 + ju_4$ are complex functions, and $u_l (l = 1, ..., 4), z = u_5, w = u_7$ are real functions and σ is control parameter. Dots designate derivatives with honor to time and an overbar indicates complex conjugate variables. The complex nonlinear Lü model in hyperchaotic action is a six dimensional continuous real autonomous system. For the case $\rho = 42, \mu = 6, \nu = 25$ and $\sigma = 5$ system (2) has hyperchaotic attractor see Fig. 1. It is fascinating to remark that the two positive Lyapunov exponents for the hyperchaotic complex Lü system are $\lambda_1 = 3.43, \lambda_2 = 0.55$ [6].

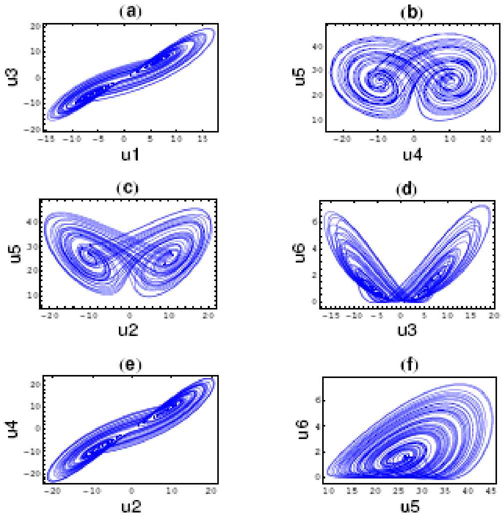

Figure 1: Hyperchaotic attractors of system (2) in some planes

3. A scheme to achieve CCS

We consider two hyperchaotic complex nonlinear systems of the form (1), one is the master system (with the subscript *m* as:

$$\begin{aligned}
\dot{\mathbf{x}}_m &= \dot{\mathbf{x}}_m^r + j\dot{\mathbf{x}}_m^i = \mathbf{F}(\mathbf{x}_m)\mathbf{A} + \mathbf{f}(\mathbf{x}_m), \qquad (3)\\
\dot{\boldsymbol{z}} &= g(\mathbf{x}, \overline{\mathbf{x}}, \boldsymbol{z}),
\end{aligned}$$

and the second is the controlled slave system (with subscript s) as:

$$\dot{\mathbf{y}}_s = \dot{\mathbf{y}}_s^r + j\dot{\mathbf{y}}_s^i = \mathbf{G}(\mathbf{y}_s)\mathbf{B} + \mathbf{g}(\mathbf{y}_s) + \mathbf{L}, \qquad (4)$$

where the additive complex controller

$\mathbf{L} = (L_1, L_2, ..., L_n)^T = \mathbf{L}^r + j\mathbf{L}^i$,

$\mathbf{L}^r = (v_1, v_3, ..., v_{2n-1})^T, \mathbf{L}^i = (v_2, v_4, ..., v_{2n})^T$

Definition [19]. *Two joined complex dynamical systems in a master-slave shape can display CCS if there exists a vector of the complex error function illustrate such as:*

$$\mathbf{e} = \mathbf{e}^r + j\mathbf{e}^i = \lim_{t\to\infty} \|\mathbf{y}_s - j\mathbf{x}_m\| = 0, \quad (5)$$

Where $\mathbf{e} = (e_1, e_2, ..., e_n)^T$, $\mathbf{x}_m$ *and* $\mathbf{y}_s$ *are the state complex vectors of the master and slave systems, respectively,*

$$\mathbf{e}^r = (e_{u_1}, e_{u_3}, ..., e_{u_{2n-1}})^T = \mathbf{y}_s^r + \mathbf{x}_m^i = 0, \quad (6)$$

and

$$\mathbf{e}^i = (e_{u_2}, e_{u_4}, ..., e_{u_{2n}})^T = \mathbf{y}_s^i - \mathbf{x}_m^r = 0. \quad (7)$$

Remark 2. From Eq. (6) the sum of the imaginary part of master system $\mathbf{x}_m^i$ and the real part of slave system $\mathbf{y}_s^r$ is vanishing when $t \to \infty$ (AS) [11].

Remark 3. In Eq. (7) the error between the imaginary part of slave system $\mathbf{y}_s^i$ and the real part of master system $\mathbf{x}_m^r$ goes to zero as $t \to \infty$ (CS) [10].

Remark 4. From remarks 2, 3 the CCS combines between AS and CS and the state variable of the master system synchronizes with a different state variable of the slave system. So, this type of synchronization is fit in the application of secure communication.

Remark 5. The CCS has a new phenomenon, not exist in all types of synchronization in the literature, the attractors of the master and slave systems are moving perpendicular or orthogonal to each other with the same shape.

Remark 6. In complex state:

$$j = \rho(\cos\theta + j\sin\theta), \quad (8)$$

where $\rho = |j| = 1$ (is the module of j) and $\theta = \frac{\pi}{2}$ (is the phase of j).

Theorem 1. *We may be able to achieve the adaptive complex complete synchronization between systems (3) and (4) by a choice of the controller* $\mathbf{L}$ *as*:

$$\begin{aligned}\mathbf{L} = \mathbf{L}^r + j\mathbf{L}^i &= [-\mathbf{G}(\mathbf{y}_s)(\tilde{\mathbf{B}}) + j\mathbf{F}(\mathbf{x}_m)(\tilde{\mathbf{A}})] \\ &+[-\mathbf{g}(\mathbf{y}_s) + j\mathbf{f}(\mathbf{x}_m)] - \boldsymbol{\Psi}\mathbf{e} \\ &= -\mathbf{G}^r(\mathbf{y}_s)(\tilde{\mathbf{B}}) - \mathbf{F}^i(\mathbf{x}_m)(\tilde{\mathbf{A}}) + [-\mathbf{g}^r(\mathbf{y}_s) - \mathbf{f}^i(\mathbf{x}_m)] \quad (9) \\ &-\boldsymbol{\Psi}\mathbf{e}^r + j(-\mathbf{G}^i(\mathbf{y}_s)(\tilde{\mathbf{B}}) + \mathbf{F}^r(\mathbf{x}_m)(\tilde{\mathbf{A}}) \\ &+[-\mathbf{g}^i(\mathbf{y}_s) + \mathbf{f}^r(\mathbf{x}_m)] - \boldsymbol{\Psi}\mathbf{e}^i),\end{aligned}$$

and the adaptive laws of parameters are selected as:

$$\dot{\tilde{\mathbf{A}}} = (-\mathbf{F}^r(\mathbf{x}_m))^T\mathbf{e}^r + (\mathbf{F}^i(\mathbf{x}_m))^T\mathbf{e}^i + \zeta\hat{\mathbf{A}}, \quad (10)$$

$$\dot{\tilde{\mathbf{B}}} = (\mathbf{G}^r(\mathbf{y}_s))^T\mathbf{e}^r + (\mathbf{G}^i(\mathbf{y}_s))^T\mathbf{e}^i + \zeta\hat{\mathbf{B}},$$

where $\mathbf{e}(t) = \mathbf{y}_s - j\mathbf{x}_m = \mathbf{e}^r + j\mathbf{e}^i = (e_1, e_2, ..., e_n)^T$ *is the vector of the complex error function* $\mathbf{e}^r = (e_{u_1}, e_{u_3}, ..., e_{u_{2n-1}})^T, \mathbf{e}^i = (e_{u_2}, e_{u_4}, ..., e_{u_{2n}})^T$. *The elements of the vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *are the parameters estimations of elements of the vectors* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively, the parameters errors are defined as* $\hat{\mathbf{A}} = \mathbf{A} - \tilde{\mathbf{A}}, \hat{\mathbf{B}} = \mathbf{B} - \tilde{\mathbf{B}}$ and $\boldsymbol{\Psi} = diag(\psi_1, \psi_2, ..., \psi_n)$, $\boldsymbol{\zeta} = diag(\zeta_1, \zeta_2, ..., \zeta_n)$, ψ_l, ζ_l *are positive constants,* $l = 1, 2, ..., n$.

Proof: Respect to the meaning of CCS and utilizing systems (3) and (4) we get:

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{y}}_s - j\dot{\mathbf{x}}_m = \dot{\mathbf{e}}^r + j\dot{\mathbf{e}}^i \\ &= [\mathbf{G}(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) - j\mathbf{F}(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}(\mathbf{y}_s) - j\mathbf{f}(\mathbf{x}_m)] + \mathbf{L}, \\ &= \mathbf{G}^r(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) - j\mathbf{F}^r(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^r(\mathbf{y}_s) - j\mathbf{f}^r(\mathbf{x}_m)] + \mathbf{L}^r \\ &\quad + j(\mathbf{G}^i(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) - j\mathbf{F}^i(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^i(\mathbf{y}_s) - j\mathbf{f}^i(\mathbf{x}_m)] + \mathbf{L}^i),\end{aligned} \tag{11}$$

So,

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{e}}^r + j\dot{\mathbf{e}}^i \\ &= \mathbf{G}^r(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) + \mathbf{F}^i(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^r(\mathbf{y}_s) + \mathbf{f}^i(\mathbf{x}_m)] + \mathbf{L}^r \\ &\quad + j(\mathbf{G}^i(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) - \mathbf{F}^r(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^i(\mathbf{y}_s) - \mathbf{f}^r(\mathbf{x}_m)] + \mathbf{L}^i),\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}\dot{\mathbf{e}}^r &= \mathbf{G}^r(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) + \mathbf{F}^i(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^r(\mathbf{y}_s) + \mathbf{f}^i(\mathbf{x}_m)] + \mathbf{L}^r \\ \dot{\mathbf{e}}^i &= \mathbf{G}^i(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) - \mathbf{F}^r(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^i(\mathbf{y}_s) - \mathbf{f}^r(\mathbf{x}_m)] + \mathbf{L}^i.\end{aligned} \tag{13}$$

Therefore, we will use Lyapunov function as:

$$\begin{aligned}\mathbf{V}(t) &= \tfrac{1}{2}[(\mathbf{e}^r)^T\mathbf{e}^r + (\mathbf{e}^i)^T\mathbf{e}^i + (\mathbf{A}-\tilde{\mathbf{A}})^T(\mathbf{A}-\tilde{\mathbf{A}}) \\ &\quad +(\mathbf{B}-\tilde{\mathbf{B}})^T(\mathbf{B}-\tilde{\mathbf{B}})] \\ &= 1/2\left(\sum_{l=1}^{n} e^2_{u_{2l-1}} + \sum_{l=1}^{n} e^2_{u_{2l}} + \hat{\mathbf{A}}^T\hat{\mathbf{A}} + \hat{\mathbf{B}}^T\hat{\mathbf{B}}\right).\end{aligned} \tag{14}$$

The total time derivative of $V(t)$ along the pathway of the error system (12) is as follows:

$$\begin{aligned}\dot{V}(t) &= (\dot{\mathbf{e}}^r)^T\mathbf{e}^r + (\dot{\mathbf{e}}^i)^T\mathbf{e}^i + \hat{\mathbf{A}}^T\dot{\hat{\mathbf{A}}} + \hat{\mathbf{B}}^T\dot{\hat{\mathbf{B}}} \\ &= (\mathbf{G}^r(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) + \mathbf{F}^i(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^r(\mathbf{y}_s) + \mathbf{f}^i(\mathbf{x}_m)] + \mathbf{L}^r)^T\mathbf{e}^r \\ &\quad +(\mathbf{G}^i(\mathbf{y}_s)(\tilde{\mathbf{B}}+\hat{\mathbf{B}}) - \mathbf{F}^r(\mathbf{x}_m)(\tilde{\mathbf{A}}+\hat{\mathbf{A}})] \\ &\quad +[\mathbf{g}^i(\mathbf{y}_s) - \mathbf{f}^r(\mathbf{x}_m)] + \mathbf{L}^i)^T\mathbf{e}^i \\ &\quad +\hat{\mathbf{A}}^T(-\dot{\tilde{\mathbf{A}}}) + \hat{\mathbf{B}}^T(-\dot{\tilde{\mathbf{B}}}),\end{aligned} \tag{15}$$

where $\dot{\hat{\mathbf{A}}} = -\dot{\tilde{\mathbf{A}}}$ and $\dot{\hat{\mathbf{B}}} = -\dot{\tilde{\mathbf{B}}}$.

By substituting from equations (9) and (10) in (15) we obtain:

$$\begin{aligned}\dot{V}(t) &= (\dot{\mathbf{e}}^r)^T\mathbf{e}^r + (\dot{\mathbf{e}}^i)^T\mathbf{e}^i + \hat{\mathbf{A}}^T\dot{\hat{\mathbf{A}}} + \hat{\mathbf{B}}^T\dot{\hat{\mathbf{B}}} \\ &= (\mathbf{G}^r(\mathbf{y}_s)(\hat{\mathbf{B}}) + \mathbf{F}^r(\mathbf{x}_m)(\hat{\mathbf{A}}) - \Psi\mathbf{e}^r)^T\mathbf{e}^r \\ &\quad +(\mathbf{G}^i(\mathbf{y}_s)(\hat{\mathbf{B}}) - \mathbf{F}^i(\mathbf{x}_m)(\hat{\mathbf{A}}) - \Psi\mathbf{e}^i)^T\mathbf{e}^i \\ &\quad +\hat{\mathbf{A}}^T(-\mathbf{F}^r(\mathbf{x}_m))^T\mathbf{e}^r + (\mathbf{F}^i(\mathbf{x}_m))^T\mathbf{e}^i - \zeta\hat{\mathbf{A}}) \\ &\quad +\hat{\mathbf{B}}^T(-\mathbf{G}^r(\mathbf{y}_s))^T\mathbf{e}^r + (-\mathbf{G}^i(\mathbf{y}_s))^T\mathbf{e}^i - \zeta\hat{\mathbf{B}}) \\ &= -[(\Psi\mathbf{e}^r)^T\mathbf{e}^r + (\Psi\mathbf{e}^i)^T\mathbf{e}^i] - \hat{\mathbf{B}}^T(\zeta\hat{\mathbf{B}}) - \hat{\mathbf{A}}^T(\zeta\hat{\mathbf{A}}) \\ &= -\left(\sum_{l=1}^{n}\psi_{2l-1}e^2_{u_{2l-1}} + \sum_{l=1}^{n}\psi_l e^2_{u_{2l}}\right) - \hat{\mathbf{B}}^T(\zeta\hat{\mathbf{B}}) - \hat{\mathbf{A}}^T(\zeta\hat{\mathbf{A}}).\end{aligned} \tag{16}$$

Considering, $V(t)>0$ and its derivative is negative definite. Therefore, Lyapunov's direct method implies that the equilibrium point $e_{u_{2l}}$ and $e_{u_{2l-1}}=0, l=1,...,n$. So, the events of the slave system and the master system will be globally synchronized asymptotically. This finishes the proof.

Remark 7: If systems (3) and (4) satisfy $\mathbf{f}(.)=\mathbf{g}(.)$ and $\mathbf{F}(.)=\mathbf{G}(.)$, then the structure of system (3) and system (4) are identical. So, our scheme is fitting to the adaptive CCS of two identical hyperchaotic systems with fully uncertain parameters.

Remark 8. When systems (3) and (4) are identical $\mathbf{A}=\mathbf{B},$ and the adaptive laws of parameters are selected as:

$$\dot{\hat{\mathbf{A}}}=\dot{\hat{\mathbf{B}}}=[\mathbf{G}^r(\mathbf{y}_s))^T+(-\mathbf{F}^r(\mathbf{x}_m))^T]\boldsymbol{e}^r+ \quad (17)$$
$$[(\mathbf{G}^i(\mathbf{y}_s))^T+(\mathbf{F}^i(\mathbf{x}_m))^T]\boldsymbol{e}^i-\zeta\hat{\mathbf{A}}$$

4. An example

4.1. Formula of the controller

Let us now review the CCS of two identical hyperchaotic complexes Lü systems with uncertain parameters as an example for Section 3. The master and the slave models are consequently represented, respectively, as follows:

$$\begin{aligned}\dot{x}_m&=\rho\left(y_m-x_m\right)+w_m,\\ \dot{y}_m&=\nu y_m-x_m z_m+w_m,\\ \dot{z}&=1/2\left(\bar{x}y+x\bar{y}\right)-\mu z,\\ \dot{w}&=1/2\left(\bar{x}y+x\bar{y}\right)-\sigma w,\end{aligned} \quad (18)$$

and

$$\begin{aligned}\dot{x}_s&=\rho\left(y_s-x_s\right)+w_s+L_1,\\ \dot{y}_s&=\nu y_s-x_s z_s+w_s+L_2,\end{aligned} \quad (19)$$

where $L_1=v_1+jv_2$, $L_2=v_3+jv_4$ are complex control functions, respectively, which are to be determined. The complex systems (18) and (19) can be formed respectively as**:**

$$\begin{pmatrix}\dot{x}_m\\ \dot{y}_m\end{pmatrix}=\begin{pmatrix}y_m-x_m & 0\\ 0 & y_m\end{pmatrix}\begin{pmatrix}\rho\\ \nu\end{pmatrix}+\begin{pmatrix}w_m\\ w_m-x_m z_m\end{pmatrix}, \quad (20)$$

and

$$\begin{pmatrix}\dot{x}_s\\ \dot{y}_s\end{pmatrix}=\begin{pmatrix}y_s-x_s & 0\\ 0 & y_s\end{pmatrix}\begin{pmatrix}\rho\\ \nu\end{pmatrix}+\begin{pmatrix}w_s\\ w_s-x_s z_s\end{pmatrix}+\begin{pmatrix}L_1\\ L_2\end{pmatrix}. \quad (21)$$

So, by comparing the complex systems (20) and (21) with the form of systems (3) and (4) respectively we find:

$$\mathbf{F}(\mathbf{x}_m)=\begin{pmatrix}y_m-x_m & 0\\ 0 & y_m\end{pmatrix},\mathbf{G}(\mathbf{y}_s)=\begin{pmatrix}y_s-x_s & 0\\ 0 & y_s\end{pmatrix},$$
$$\mathbf{A}=\mathbf{B}=\begin{pmatrix}\rho\\ \nu\end{pmatrix},\mathbf{f}(\mathbf{x}_m)=\begin{pmatrix}w_m\\ w_m-x_m z_m\end{pmatrix},\mathbf{g}(\mathbf{y}_s)=\begin{pmatrix}w_s\\ w_s-x_s z_s\end{pmatrix}.$$

According to Theorem 1, the controller is created as:

$$\mathbf{L}=[-\mathbf{G}(\mathbf{y}_s)(\tilde{\mathbf{B}})+j\mathbf{F}(\mathbf{x}_m)(\tilde{\mathbf{A}})]+[-\mathbf{g}(\mathbf{y}_s)+j\mathbf{f}(\mathbf{x}_m)]-\mathbf{\Psi e},$$

$$\begin{pmatrix}L_1\\ L_2\end{pmatrix}=\begin{pmatrix}-\tilde{\rho}\left(y_s-x_s\right)-w_s+j(\tilde{\rho}(y_m-x_m)+w_m)-\psi_1 e_1\\ -\tilde{\nu}y_s+x_s z_s-w_s+j(\tilde{\nu}y_m+w_m(t)-x_m(t)z_m(t))-\psi_2 e_2\end{pmatrix},$$

$$\mathbf{L}=\begin{pmatrix} -\tilde{\rho}(u_{3s}-u_{1s}+u_{4m}-u_{2m})-u_{7s}-\psi_1 e_{u_1} \\ -\tilde{\nu}(u_{3s}+u_{4m})+u_{1s}u_{5s}-u_{7s}+u_{2m}u_{5m}-\psi_2 e_{u_3} \end{pmatrix} \quad (22)$$
$$+j\begin{pmatrix} -\tilde{\rho}(u_{4s}-u_{2s}+u_{1m}-u_{3m})+u_{7m}-\psi_{1u_2} \\ -\tilde{\nu}(u_{4s}(t)-u_{3m}(t))+u_{2s}u_{5s}+u_{7m}-u_{2m}u_{5m}-\psi_2 e_{u_4} \end{pmatrix}.$$

Since $\mathbf{A}=\mathbf{B}=(\rho,\nu,\mu,\sigma)^T$ we can calculate the adaptive laws of parameters by using (17) as:

$$\dot{\tilde{\mathbf{A}}}=\dot{\tilde{\mathbf{B}}}=\begin{pmatrix}\dot{\tilde{\rho}}\\ \dot{\tilde{\nu}}\end{pmatrix}=\begin{pmatrix}(u_{3s}-u_{1s}-u_{3m}+u_{1m})e_{u_1}+\Omega e_{u_2}-\zeta_1\hat{\rho}\\ (u_{3s}-u_{3m})e_{u_3}+(u_{4s}+u_{4m})e_{u4}-\zeta_2\hat{\nu}\end{pmatrix}, \quad (23)$$

where $\Omega=(u_{4s}-u_{2s}+u_{4m}-u_{2m})$.

4.2. Numerical simulation

To prove and confirm the usefulness of the recommended scheme, we explain the simulation results of the CCS among two identical hyperchaotic complex Lü systems (18) and (19). Systems (18) and (19) with the controller (22) are solved numerically, and the parameters are chosen as $\rho=42,\mu=6,\nu=25$ and $\sigma=5.$ The initial requirement of the master model, the initial condition of the slave model, and the diagonal constant matrices are considered essentially $(x_m(0),y_m(0),z(0),w(0))^T=(1+2j,3+4j,5,6)^T$, $(x_s(0),y_s(0))^T=(6+8j,3+4j)^T$ and $\mathbf{\Psi}=diag(10,5)$, $\zeta=diag(6,7)$. The initial values of estimate for unknown parameters vector are considered as $(\tilde{\rho}(0),\tilde{\nu}(0))^T=(4,5)^T$. The results are outlined in Figures 2,3,4,5 and 6. In Figure 2 the solutions of (18) and (19) are plotted subject to different initial conditions and show that CCS is indeed achieved after a very little time t. We can see that each u_{1m},u_{3m} have the same sign of $u_{2s},.u_{4s}$ while u_{2m},u_{4m} have opposite sign of u_{1s},u_{3s}. This means CS achieves between the real part of system (18) and the imaginary part of system (19) while AS occurs between the real part of slave system (18) and the imaginary part of the master system (19). Figure 2 shows CCS is achieved after small time interval. The CCS errors are plotted in Figure 3. As expected from the above analytical considerations the CCS errors $e_{u_{2l-1}},e_{u_{2l}}$ converge to zero as $t\to\infty, l=1,2$. In Figure 3 it can be noticed that the errors will approach zero after small value of t. Figure 4 shows that the estimated values of the unknown parameter $\tilde{\rho}(t),\tilde{\nu}(t)$ converge to $42,6$ respectively.

In the numerical simulations, we compute the module errors and phases errors of master and slave models, respectively. For each complex number, the module and phase are determined as follow:

$$\rho_x=\sqrt{(x^r)^2+(x^i)^2},$$

and

$$\theta_x=\begin{cases}\arctan(x^i/x^r), & x^r>0,x^i ?0,\\ 2\pi+\arctan(x^i/x^r), & x^r>0,x^i<0,\\ \pi+\arctan(x^i/x^r), & x^r<0.\end{cases}$$

Figure 5 shows the modules errors and phases errors of the master system (18) and slave systems (19) with $\rho=1,\theta=\frac{\pi}{2}$. It is clear from Figure 5a, b the modules errors $\rho_{x_m}-\rho_{x_s},\rho_{y_m}-\rho_{y_s}$ converge to zero as $t\to\infty$ While the phases errors $\theta_{x_m}-\theta_{x_s},\theta_{y_m}-\theta_{y_s}$ go to $\frac{\pi}{2}$ as $t\to\infty$, see Figure 5c, d. A new phenomenon is illustrated in Figure 6 and does not appear in all kinds of synchronizations in the literature. The attractors of the master and slave systems in CCS are moving perpendicular or orthogonal to each other with the same shape as seen in Figure 6a, b.

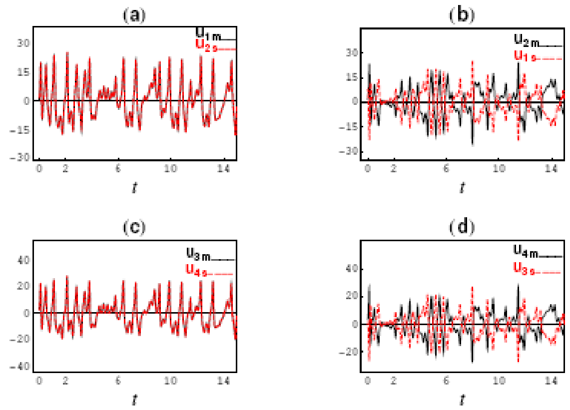

Figure 2: CCS between two non-identical systems (18) and (19) with the controller (22)

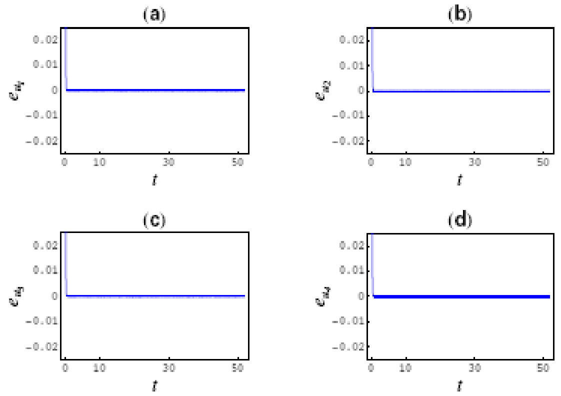

Figure 3: CCS errors between systems (18) and (19)

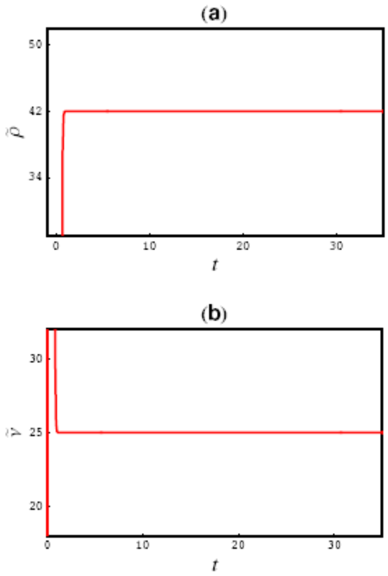

Figure 4: Adaptive parameters estimation laws versus t

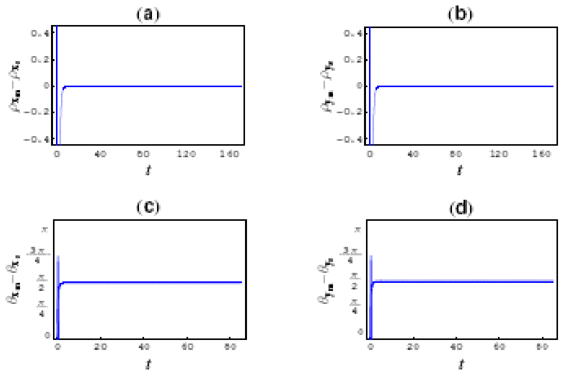

Figure 5: The modules errors and phases errors of systems (17) and (18)

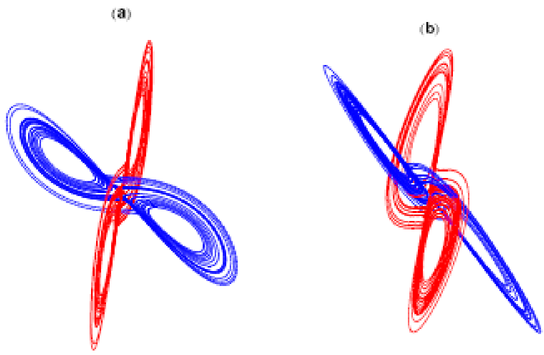

Figure 6: The orthogonal attractors of systems (17) and (18)

4.3 Application to secure communication

The chaos (or hyperchaos) synchronous control and its application have become the hot spot in nonlinear fields, in particular, the applications to secure communication [25-27]. The idea of secure communication is communicating a message from the transmitter to the receiver within chaotic (or hyperchaotic) systems. In other words, the message is inoculated or added into chaotic (or hyperchaotic) systems, transmitted, and then discovered and recovered by the receiver. Various sorts of secure communication schemes have been introduced like chaotic (or hyperchaotic) masking [26, 27] In chaotic (or hyperchaotic) masking, the message which we need to send it is added to a one of chaotic (or hyperchaotic) signal in order to hide it, then the signal is transmitted to the receiver. Under certain conditions, the message may be recovered at the receiver.

CCS of hyperchaotic complex systems in which a state variable of the master system synchronizes with a different state variable of the slave system is an encouraging type of synchronization as it contributes excellent security in secure communication. We consider system (18) as transmitter system and system (19) as receiver system. For one thing, we choose arbitrarily the information signal as $r(t) = 4\cos t - 21$. Take $\hat{r}(t) = r(t) + u_{1m}$ and suppose that $\hat{r}(t)$ is added to the variable $u_{2m} \Rightarrow \breve{r}(t) = \hat{r}(t) + u_{2m} = r(t) + u_{1m} + u_{2m}$. Numerical results of application to secure communication are shown in Figure 7. The information signal $r(t)$ and the transmitted signal $\breve{r}(t)$ are shown in Figures 7(a) and (b), respectively. The recovered data signal, whatever is expressed by $r^{*}(t) = r(t) - u_{2s} + u_{1s}$, is shown in Figure 7(c) (because after achieving CCS $u_{1m} - u_{2s} = 0, u_{2m} + u_{1s} = 0$). Figure 7(d) displays the error signal between the original information signal and the recovered one. From Figure 7(d), it is easy to find that the information signal $r(t)$ is recovered exactly after a very short transient.

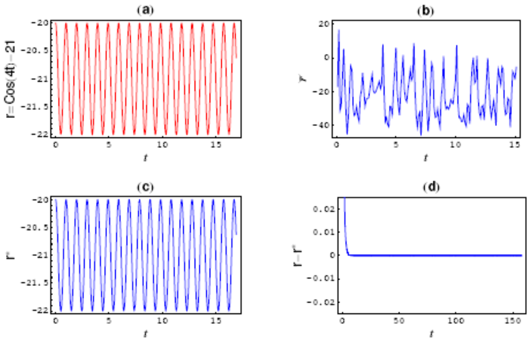

Figure 7: Simulation results of secure communication using CCS of two identical chaotic complex Lu systems

5. Conclusions

In this concern, we develop our investigation's of the new type of complex synchronization which called complex complete synchronization CCS [20]. We analyze and study the CCS concerning two identical hyperchaotic complex nonlinear systems with fully uncertain parameters. Of particular importance are some interesting and perhaps unusual properties for CCS, such as: (i) the CCS can be studied only in complex nonlinear systems. (ii) the CCS can be considered as syncretizing between CS and AS (see Figure 2). (iii) the state variable of the master system synchronizes with a different state variable of the slave system. (iv) the most notable characteristic of the CCS is the attractors of the master and slave systems are moving perpendicular or orthogonal to each other with the same shape (see Figure 6a, b). These phenomena in (i), (ii), (iii) and (iv) did not transpire and seem for any types of synchronization in the literature.

A scheme is outlined to realize CCS of two identical hyperchaotic complex nonlinear systems with uncertain parameters based on adaptive approaches and Lyapunov functions. During this scheme, we concluded analytically the control complex functions to achieve CCS. Moreover, via adaptive approaches, the parameter estimation rule is presented. It is easy and convenient to use this scheme for chaotic and hyperchaotic complex systems. We employ our scheme for two identical hyperchaotic complex Lü systems with different initial values, as an example, and gain valid outcomes. All the theoretical results are verified by numerical simulations of our example. An excellent agreement is found as shown in Figures 2, 3, 4. In Figures 5 we compute the modules errors and phases errors because, in the complex nonlinear dynamical system, the observable or measurable physical quantities usually are module and phase.

Finally, based on the state variable of the transmitter system (18) synchronizes with a different state variable of the receiver system (19), a simple secure communication project is designed via hyperchaotic-masking. The results indicate that using the hyperchaotic-masking method which adds the transmitted signal directly to a comparatively strong hyperchaotic signal to form the information carrier wave not only has strong security but can also recover the information signal effectively as shown in Figure 7.

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