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First Derivative Method for the Integration of Oscillatory Problems

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Abstract A first derivative method for the integration of first-order oscillatory problems is derived in this research. The integration is carried out within a two-step interval. The method of collocation and interpolation of power series basis function was adopted in deriving the method. The method derived was tested on some sampled oscillatory differential problems to verify its reliability. The results obtained in terms of the point wise absolute errors show that the first derivative method developed approximates the exact solutions closely. The method derived was also tested for zero-stability, consistency and convergence.

Keywords Differential equations, first derivative, integration, oscillatory, two-step
2010 AMS Subject Classification: 65L05, 65L06, 65D30

1. Introduction

Most of the phenomena, processes and situations that describe the world in which we live are contained in what are known as differential equations. Such equations appear not only in the physical sciences, but also in biology, engineering, management sciences, and all scientific disciplines that attempt to understand the world in which we live. It is also important to state that many of these equations that govern the physical world have no solution in closed form. Therefore, to find the answer to questions about the world in which we live, we must resort to solving these equations numerically. Also, the subject is by no means closed, so researchers should be on the lookout for new techniques that provide greater efficiency and higher accuracy [1].

A very challenging set of these differential equations being encountered in our day-to-day activities is the oscillatory differential equations. They are differential equations whose solutions are composed of smooth varying and of a 'nearly periodic' functions, i.e. they are oscillations whose wave form and period varies slowly with time (relative to the period), and where the solution is sought over a very large number of cycles [2].

In this research, oscillatory problems of the form,

$$y'(x) = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

shall be considered; where $f : \Re \times \Re^m \to \Re^m$, $y, y_0 \in \Re^m$, $y(x)$ is assumed to possess oscillatory solution and f satisfy the existence and uniqueness theorem stated in the theorem below.

Theorem 1.1 [3]

Let,

$$U^{(n)} = f(x, u, u', ..., u^{(n-1)}), \; U^k(x_0) = C_k \tag{2}$$

$k = 0, 1, ..., (n-1)$; u and f are scalars. Let be the region defined by the inequalities

$x_0 \le x \le x_0 + a, \left| s_j - c_j \right| \le b, j = 0,1,...,(n-1),(a > 0,\ b > 0).$ Suppose the function $f(x, s_0, s_1, ..., s_{n-1})$ is defined in □ and in addition:

(a) f is non-negative and non-decreasing in each $x, s_0, s_1, ..., s_{n-1}$ in □ ;

(b) $f(x, c_0, c_1, ..., c_{n-1}) > 0$ for $x_0 \le x \le x_0 + a;$ and

(c) $C_k \ge 0,\ k = 0,1,2,...,(n-1).$

Then, the differential equation (1) has a unique solution in □ .

A lot of methods have been proposed for the solution of problems of the form (1). Linear Multistep Methods (LMMs) have been developed varying from discrete LMMs to continuous ones. Continuous LMMs have greater advantages over the discrete methods such that they give better error estimation, provide a simplified form of coefficients for further evaluation at different grid points and provide approximate solution at all interior points within the interval of integration, [4]. These methods are first derivative methods that are implemented in predictor-corrector mode and Taylor series expansions are adopted to supply starting values, [5]. The setbacks of the predictor-corrector methods are that they are very costly to implement, longer computer time, greater human effort and reduced order of accuracy which affect the accuracy of the method.

Scholars latter developed block methods to cater for some setbacks of the predictor-corrector methods mentioned above. Block methods generate independent solutions at all selected grid point without overlapping. It is less expensive in terms of the number of function evaluation compared to predictor-corrector methods and moreover it possesses the properties of Runge-Kutta methods of being self-starting, see [6, 7, 8]. The block method was modified by incorporating function evaluation at off-step points to afford the opportunity of circumventing the Dahlquist 'zero stability barrier' (as stated in Theorem 1.2 below) and this made it possible to obtain convergent k-step methods with order $2k+1$ up to $k = 7$, [9]. Even higher orders are available if two or more offstep points are used. This method was called 'hybrid method'. The method is useful in reducing the step number of a method and still remains zero-stable, see the works of [10, 11].

Theorem 1.2 [12]

There is no consistent, zero-stable linear k-step method whose order exceeds $k+1$ if k is odd or $k+2$ if k is even.

Definition 1.1 [13, 14]

A differential equation is said to be oscillatory if,

(i) all the nontrivial solution of (1.1) have an infinite number of zeros (roots) on $x_0 \le x < \infty$, and

(ii) it has at least one oscillating solution.

Definition 1.2: [15]

A numerical integration scheme is said to be $A(\alpha)$-stable for some $\alpha \in [0, \pi/2]$ if the wedge

$$S_\alpha = \left\{ z : \left| Arg(-z) \right| < \alpha,\ z \ne 0 \right\}$$

is contained in its region of absolute stability. The largest $\alpha \left(i.e.\ \alpha_{\max} \right)$ is called the angle of absolute stability.

Definition 1.3: [15]

A numerical integration scheme is said to be $A(0)$-stable if it is $A(\alpha)$-stable for some $\alpha \in \left(0, \pi/2\right)$. Note that $A\left(\frac{\pi}{2}\right)$-stability $\equiv A$-stability.

2. Derivation of the First Derivative Method

A first derivative method of the form,

$$A^{(0)}\mathbf{Y}_m = E\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \tag{3}$$

shall be derived for the integration of oscillatory problems of the form (1), where $A^{(0)}, E, d$ *and* b are $r \times r$ matrices (r is the number of collocation points). On the other hand, $Y_m, y_n, F(Y_m)$ *and* $f(y_n)$ are vector matrices with r entries.

This derivation shall be achieved by employing power series as basis function as shown in equation (4) below,

$$y(x) = \sum_{n=0}^{r+s-1} a_n x^n \tag{4}$$

where r *and* s are the numbers of collocation and interpolation points respectively.

Let the approximate solution to (1) be given by power series of degree 7, by allowing $r+s-1=7$ in equation (4), that is,

$$y(x) = \sum_{n=0}^{7} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \tag{5}$$

with the first derivative given by,

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 \tag{6}$$

Substituting (6) into (1) gives,

$$f(x,y) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 \tag{7}$$

Now, interpolating (5) at point $x_{n+s}, s = \frac{5}{3}$ and collocating (7) at points $x_{n+r}, r = 0\left(\frac{1}{3}\right)2$, leads to a system of nonlinear equation of the form,

$$XA = U \tag{8}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n & f_n & f_{n+\frac{1}{3}} & f_{n+\frac{2}{3}} & f_{n+1} & f_{n+\frac{4}{3}} & f_{n+\frac{5}{3}} & f_{n+2} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_{n+\frac{5}{3}} & x^2_{n+\frac{5}{3}} & x^3_{n+\frac{5}{3}} & x^4_{n+\frac{5}{3}} & x^5_{n+\frac{5}{3}} & x^6_{n+\frac{5}{3}} & x^7_{n+\frac{5}{3}} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x^2_{n+\frac{1}{3}} & 4x^3_{n+\frac{1}{3}} & 5x^4_{n+\frac{1}{3}} & 6x^5_{n+\frac{1}{3}} & 7x^6_{n+\frac{1}{3}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^4_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} \\ 0 & 1 & 2x_{n+\frac{4}{3}} & 3x^2_{n+\frac{4}{3}} & 4x^3_{n+\frac{4}{3}} & 5x^4_{n+\frac{4}{3}} & 6x^5_{n+\frac{4}{3}} & 7x^6_{n+\frac{4}{3}} \\ 0 & 1 & 2x_{n+\frac{5}{3}} & 3x^2_{n+\frac{5}{3}} & 4x^3_{n+\frac{5}{3}} & 5x^4_{n+\frac{5}{3}} & 6x^5_{n+\frac{5}{3}} & 7x^6_{n+\frac{5}{3}} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} & 4x^3_{n+2} & 5x^4_{n+2} & 6x^5_{n+2} & 7x^6_{n+2} \end{bmatrix}$$

Solving (8) by Gauss elimination method for the $a_j 's, j = 0(1)5$ and substituting back into the power series basis function gives a linear multistep method of the form,

$$y(x) = \alpha_{\frac{5}{3}}(x) y_{n+\frac{5}{3}} + h \sum_{j=0}^{2} \beta_j(x) f_{n+j}, \ j = 0\left(\frac{1}{3}\right)2 \tag{9}$$

where the coefficients of y_n *and* f_{n+j} are given as,

$$\left.\begin{aligned}
&\alpha_{\frac{5}{3}} = 1 \\
&\beta_0 = \frac{1}{139104}(209952t^7 - 1711206t^6 + 5688144t^5 - 9913995t^4 + 9675792t^3 - 5202792t^2 + 1391040t - 137575) \\
&\beta_{\frac{1}{3}} = -\frac{1}{7245}(6561t^7 - 51030t^6 + 158193t^5 - 246645t^4 + 197316t^3 - 68040t^2 + 3625) \\
&\beta_{\frac{2}{3}} = \frac{1}{1391040}(3149280t^7 - 23320710t^6 + 67482072t^5 - 95111415t^4 + 64978200t^3 - 17010000t^2 - 171875) \\
&\beta_1 = -\frac{1}{86940}(262440t^7 - 1845585t^6 + 4997538t^5 - 6475140t^4 + 4009320t^3 - 982800t^2 + 44375) \\
&\beta_{\frac{4}{3}} = \frac{1}{463680}(1049760t^7 - 6991110t^6 + 17799264t^5 - 21662235t^4 + 12791520t^3 - 3061800t^2 - 124375) \\
&\beta_{\frac{5}{3}} = -\frac{1}{86940}(78732t^7 - 494991t^6 + 1194102t^5 - 1394820t^4 + 802872t^3 - 190512t^2 + 15325) \\
&\beta_2 = \frac{1}{463680}(69984t^7 - 413910t^6 + 957096t^5 - 1087695t^4 + 617064t^3 - 146160t^2 + 5125)
\end{aligned}\right\} \tag{10}$$

and t is given by

$$t = \frac{x - x_n}{h} \tag{11}$$

Evaluating (9) at $t = \frac{1}{3}\left(\frac{1}{3}\right)2$, gives a discrete two-step algorithm of the form (3) given by,

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{3}}\\y_{n+\frac{2}{3}}\\y_{n+1}\\y_{n+\frac{4}{3}}\\y_{n+\frac{5}{3}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{3}}\\y_{n-\frac{2}{3}}\\y_{n-1}\\y_{n-\frac{4}{3}}\\y_{n-\frac{5}{3}}\\y_{n}\end{bmatrix}+h\begin{bmatrix}0&0&0&0&0&\frac{428149}{417312}\\0&0&0&0&0&\frac{8419}{86940}\\0&0&0&0&0&\frac{3043}{30912}\\0&0&0&0&0&\frac{6346}{65205}\\0&0&0&0&0&\frac{27515}{278208}\\0&0&0&0&0&\frac{907}{9660}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{3}}\\f_{n-\frac{2}{3}}\\f_{n-1}\\f_{n-\frac{4}{3}}\\f_{n-\frac{5}{3}}\\f_{n}\end{bmatrix}$$

$$+h\begin{bmatrix}\frac{2713}{7245}&\frac{-410461}{139104}&\frac{67477}{260820}&\frac{-69737}{463680}&\frac{4381}{86940}&\frac{-30701}{4173120}\\\frac{752}{1449}&\frac{-4447}{86940}&\frac{4112}{21735}&\frac{-3629}{28980}&\frac{976}{21735}&\frac{-199}{28980}\\\frac{81}{161}&\frac{6201}{51520}&\frac{3803}{9660}&\frac{-8289}{51520}&\frac{171}{3220}&\frac{-1207}{154560}\\\frac{3712}{7245}&\frac{1784}{21735}&\frac{39232}{65205}&\frac{58}{7245}&\frac{832}{21735}&\frac{-416}{65205}\\\frac{725}{1449}&\frac{34375}{278208}&\frac{8875}{17388}&\frac{24875}{92736}&\frac{3065}{17388}&\frac{-1025}{92736}\\\frac{432}{805}&\frac{27}{3220}&\frac{1744}{2415}&\frac{27}{3220}&\frac{432}{805}&\frac{907}{9660}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{3}}\\f_{n+\frac{2}{3}}\\f_{n+1}\\f_{n+\frac{4}{3}}\\f_{n+\frac{5}{3}}\\f_{n+2}\end{bmatrix} \quad (12)$$

This is a first derivative method for the solution of oscillatory problems of the form (1).

3. Analysis of the First Derivative Method

In this section, some basic properties of the first derivative method shall be discussed.

3.1 Order of Accuracy of the First Derivative Method

The order of accuracy of a method quantifies the rate of convergence of computed solution of a differential equation to the exact solution. According to [16], the first derivative method (12) is said to be of uniform accurate order p, if p is the largest positive integer for which $\bar{c}_0=\bar{c}_1=\bar{c}_2=...=\bar{c}_p=0,\ \bar{c}_{p+1}\neq 0$. $\bar{c}_{p+1}$ is called the error constant and the local truncation error of the method is given by;

$$\bar{t}_{n+k}=\bar{c}_{p+1}h^{(p+1)}y^{(p+1)}(t)+O\left(h^{(p+2)}\right) \quad (13)$$

The larger the order of accuracy, the faster the error is reduced as h decreases. Therefore, for our first derivative method $\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{c}_6=\bar{0}$, implying that the order $p=\begin{bmatrix}6&6&6&6&6\end{bmatrix}^T$ and the error constant is given by $\begin{bmatrix}3.5672\times10^{-6}&4.9438\times10^{-6}&4.7926\times10^{-6}&4.8807\times10^{-6}&4.7663\times10^{-6}&5.1121\times10^{-6}\end{bmatrix}^T$.

3.2 Consistency of the First Derivative Method

Consistency refers to a quantitative measure of the extent to which the exact solution satisfies the discrete problem. The first derivative method (12) is consistent since it has order $p\geq 1$. Consistency controls the magnitude of the local truncation error committed at each stage of the computation, [17].

3.3 Zero Stability of the First Derivative Method

Definition 3.1 [17]: A block method is said to be zero-stable, if the roots $z_{s,} s = 1,2,...,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - E)$ satisfies $|z_s| \le 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation.

For the first derivative method (12), the first characteristic polynomial is given by,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} z & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & z-1 \end{vmatrix} = z^5(z-1)$$

Thus, solving for z in

$$z^5(z-1) = 0 \tag{14}$$

gives $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ *and* $z_6 = 1$. Hence, the first derivative method (12) is zero-stable.

3.4 Convergence of the First Derivative Method

The first derivative method is convergent since it is consistent and zero-stable.

Theorem 3.1 [18]

A linear multistep method is convergent if and only if it is zero stable and consistent

3.5 Region of Absolute Stability of the First Derivative Method

Definition 3.2 [19]

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

Applying the boundary locus method, we obtain the stability polynomial for the first derivative method derived as;

$$\begin{aligned} \bar{h}(w) = {} & h^6\left(\frac{8}{39123}w^6 - \frac{1348847}{314940150}w^5\right) - h^5\left(\frac{79}{27945}w^6 + \frac{17751889}{787350375}w^5\right) + h^4\left(\frac{29}{1242}w^6 - \frac{49712251}{299943000}w^5\right) \\ & - h^3\left(\frac{475}{3726}w^6 + \frac{24256997}{52490025}w^5\right) + h^2\left(\frac{569}{1242}w^6 - \frac{67419419}{69986700}w^5\right) - h\left(\frac{206}{207}w^6 + \frac{208}{207}w^5\right) + w^6 - w^5 \end{aligned} \tag{15}$$

The region of absolute stability of the first derivative method is shown in Figure 3.1.

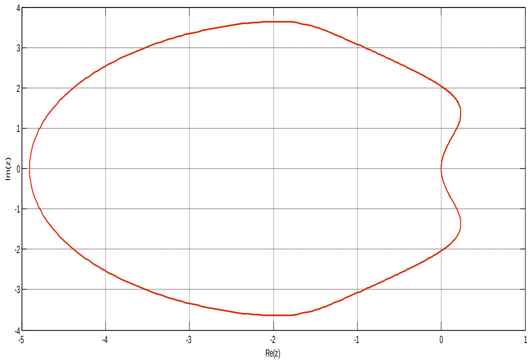

Figure 3.1: Stability region of the first derivative method

The stability region obtained in Figure 3.1 is A(0)-stable, see [15]. The unstable region consists of the complex plane outside the enclosed figure while the stable region is the interior of the curve.

4. Results

4.1 Numerical Experiments

The first derivative method developed shall be applied on some sampled oscillatory differential equations that find application in science and engineering. The following notations shall be used in the Tables below:

ERR= Absolute error in the computational method

Eval t =Evaluation time per seconds

ESJ-Absolute error in [20]

EJS-Absolute error in [21]

Problem 4.1

Consider the Prothero-Robinson oscillatory problem

$$y' = Ly + \cos x - L\sin x,\ \ L = -1,\ y(0) = 0$$

with the exact solution

$$y(x) = \sin x$$

Source: [20]

Problem 4.2

Consider the oscillatory problem

$$y' = -\sin x - 200(y - \cos x),\ y(0) = 0$$

with the exact solution

$$y(x) = \cos x - e^{-200x}$$

Source: [21]

Table 4.1: The result for the Prothero-Robinson oscillatory problem 4.1

t	Exact Solution	Computed Solution	ERR	ESJ	Eval t
0.1000	0.0998334166468282	0.0998334166468281	5.551115e-017	1.452952e-011	0.0313
0.2000	0.1986693307950612	0.1986693307950610	1.942890e-016	1.621117e-011	0.0345
0.3000	0.2955202066613396	0.2955202066613391	4.440892e-016	2.131013e-011	0.0376
0.4000	0.3894183423086506	0.3894183423086496	9.436896e-016	1.379910e-011	0.0409
0.5000	0.4794255386042031	0.4794255386042017	1.443290e-015	2.744084e-011	0.0440
0.6000	0.5646424733950355	0.5646424733950336	1.887379e-015	1.111424e-011	0.0470
0.7000	0.6442176872376912	0.6442176872376887	2.553513e-015	2.865663e-011	0.0502
0.8000	0.7173560908995230	0.7173560908995198	3.219647e-015	1.921784e-010	0.0533
0.9000	0.7833269096274836	0.7833269096274799	3.774758e-015	1.239202e-010	0.0564
1.0000	0.8414709848078967	0.8414709848078923	4.440892e-015	1.471102e-010	0.0596

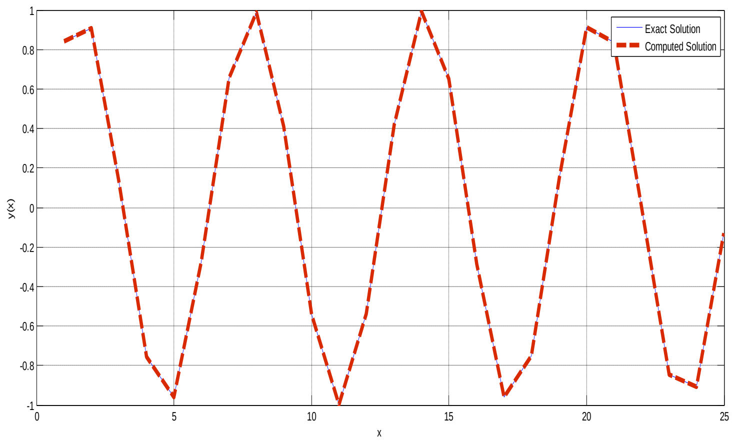

Figure 4.1: Graphical result showing the oscillatory nature of Prothero-Robinson Problem 4.1

Table 4.2: The result for the oscillatory problem 4.2

t	Exact Solution	Computed Solution	ERR	EJS	Eval t
0.0010	0.1812687469220599	0.1812687444444861	2.477574e-009	6.581226e-006	0.0360
0.0020	0.3296779539650273	0.3296776444451111	3.095199e-007	2.937887e-006	0.0525
0.0030	0.4511838639093485	0.4511836088351063	2.550742e-007	9.396094e-006	0.0731
0.0040	0.5506630358934451	0.5506626209385402	4.149549e-007	1.130466e-005	0.0952
0.0050	0.6321080588545993	0.6321077180050110	3.408496e-007	7.910709e-006	0.1200
0.0060	0.6987877881417979	0.6987873709128151	4.172290e-007	1.031328e-005	0.1358
0.0070	0.7533785361584351	0.7533781938140038	3.423444e-007	1.042596e-005	0.1362
0.0080	0.7980714821760110	0.7980711092733236	3.729027e-007	7.798045e-006	0.1365
0.0090	0.8346606120517877	0.8346603062446752	3.058071e-007	8.490002e-006	0.1369
0.0100	0.8646147171800526	0.8646144047247972	3.124553e-007	8.038839e-006	0.1372

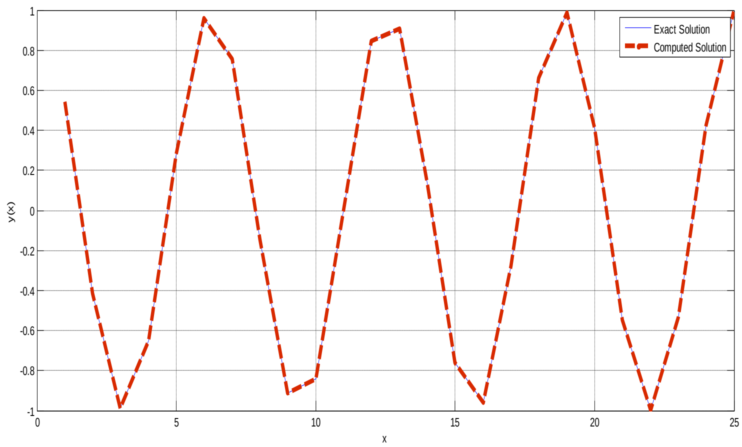

Figure 4.2: Graphical result showing the oscillatory nature of Problem 4.2

4.2. Discussion of Results

The numerical result (in Tables 4.1 and 4.2) and graphical results (in Figures 4.2 and 4.3) clearly show that the first derivative method derived is computationally reliable; this is because the computed solutions converge to the exact solutions. The stability region obtained also shows that the method can effectively handle even stiff equations since it is A(0)-stable.

Conclusion

The first derivative method developed in this research has been shown to be efficient in handling oscillatory problems of the form (1). It is also important to note that the method derived is consistent, convergent and zero-stable. Thus, the method derived is recommended for the solution of oscillatory problems of the form (1). Suffice to say that the method derived can handle stiff and other first order differential equations.

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