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Fekete-Szegö problem for certain subclass of analytic and bi-univalent functions

Nizami Mustafa

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Abstract In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions in the unit disk in the complex plane. Here, we find upper bound estimate for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ of the functions belonging to this class. Moreover, upper bound estimates for the initial coefficients $|a_2|$ and $|a_3|$ of the functions belonging to this class are obtained. Some interesting corollaries of the results obtained here are also discussed.

Keywords Bi-univalent functions, Analytic functions, Coefficient bound, Fekete-Szegö problem
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1. Introduction

Let's A be the class of the functions in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

It is well-known that a function $f: \mathbb{C} \to \mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1 = z_2$ if $f(z_1) = f(z_2)$ or $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$.

We denote by S the subclass of A consisting of univalent functions in U. Some of the important and well-investigated subclasses of S include the class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(\alpha \in [0,1))$.

By definition

$$S^*(\alpha) = \left\{ f \in S : \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, z \in U \right\}, \ \alpha \in [0,1)$$

and

$$C(\alpha) = \left\{ f \in S : \mathrm{Re}\left(1 + \frac{zf'(z)}{f(z)} \right) > \alpha, z \in U \right\}, \ \alpha \in [0,1).$$

The above mentioned function classes have been recently investigated rather extensively in [14, 22, 28, 27] and the references therein.

It is well-known that (see, for example, [7]) every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z))=z, z\in U,\ f(f^{-1}(w))=w, w\in D=\{w:|w|<r_0(f)\}, r_0(f)\geq\frac{1}{4},$$

where $f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots, w\in D$.

A function $f\in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent. Let us Σ denote the class of bi-univalent functions in U given (1.1).

Examples of functions in the class Σ are

$$\frac{z}{1-z},\quad \ln\frac{1}{1-z},\quad \ln\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in A such as

$$\frac{2z-z^2}{2}\ \text{and}\ \frac{z}{1-z^2}$$

are also not members of Σ.

Earlier, Brannan and Taha [2] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike function of order α denoted $S_\Sigma^*(\alpha)$ and bi-convex function of order α denoted $C_\Sigma(\alpha)$ corresponding to the function classes $S^*(\alpha)$ and $C(\alpha)$, respectively. Thus, following Brannan and Taha [2], a function $f\in\Sigma$ is in the classes $S_\Sigma^*(\alpha)$ and $C_\Sigma(\alpha)$, respectively, if each of the following conditions are satisfied:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha, z\in U,\ \operatorname{Re}\left(\frac{zg'(w)}{g(w)}\right)>\alpha, w\in D$$

and

$$\operatorname{Re}\left(1+\frac{zf'(z)}{f(z)}\right)>\alpha, z\in U,\ \operatorname{Re}\left(1+\frac{zg'(w)}{g(w)}\right)>\alpha, w\in D.$$

For a brief history and interesting examples of functions which are in the class Σ, together with various other properties of this bi-univalent function class, one can refer the work of Srivastava et al. [24] and references therein. In [24], Srivastava et al. reviewed the study of coefficient problems for bi-univalent functions. Also, various subclasses of bi-univalent function class were introduced and non-sharp estimates on the first two coefficients in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1, 3-5, 9, 10, 12, 16, 18, 21, 23, 25-27, 29, 30].

An analytic function f is subordinate to an analytic function ϕ, written $f(z)\prec\phi(z)$, provided there is an analytic function $u:U\to U$ with $u(0)=0$ and $|u(z)|<1$ satisfying $f(z)=\phi(u(z))$ (see, for example, [14]).

Ma and Minda [15] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in U, with $\phi(0)=1$, $\phi'(0)>0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike and Ma-Minda convex functions consists of functions $f\in A$ satisfying the subordination

$\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$, respectively. These classes denoted, respectively, by $S^*(\phi)$ and $C(\phi)$.

An analytic function $f \in S$ is said to be bi- starlike of Ma-Minda type or bi- convex of Ma-Minda type if both f and f^{-1} are, respectively, Ma-Minda starlike or Ma-Minda convex functions. These classes are denoted, respectively, by $S^*_\Sigma(\phi)$ and $C_\Sigma(\phi)$. In the sequel, it is assumed that ϕ is an analytic function with positive real part in U, satisfying $\phi(0)=1,\ \phi'(0)>0$ and $\phi(U)$ is starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the following form:

$$\phi(z)=1+b_1z+b_2z^2+b_3z^3+\cdots,\ b_1>0. \tag{1.2}$$

One of the important tools in the theory of analytic functions is the functional $H_2(1)=a_3-a_2^2$ which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3-\mu a_2^2$, where μ is some real number (see [8]). Estimating for the upper bound of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem. In 1969, Keogh and Merkes [13] solved the Fekete-Szegö problem for the class starlike and convex functions. Someone can see the Fekete-Szegö problem for the classes of starlike functions of order α and convex functions of order α at special cases in the paper of Orhan et al. [19]. On the other hand, recently, Çağlar and Aslan (see [5]) have obtained Fekete-Szegö inequality for a subclass of bi-univalent functions. Also, Zaprawa (see [32, 33]) have studied on Fekete-Szegö problem for some subclasses of bi-univalent functions. In special cases, he studied the Fekete-Szegö problem for the subclasses bi-starlike functions of order α and bi-convex functions of order α.

Motivated by the aforementioned works, we define a new subclass of bi-univalent functions Σ as follows.

Definition 1.1. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $M_\Sigma(\phi,\beta)$, $\beta \ge 0$, where ϕ is an analytic function given by (1.2), if the following conditions are satisfied:*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\beta} \prec \phi(z),\ z\in U,\quad \left(\frac{zg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{zg''(w)}{g'(w)}\right)^{1-\beta} \prec \phi(w), w\in D,$$

where $g=f^{-1}$.

Remark 1.1. *Choose $\beta=1$ in Definition 1.1, we have $M_\Sigma(\phi,1)=S^*_\Sigma(\phi)$; that is,*

$$\frac{zf'(z)}{f(z)} \prec \phi(z), z\in U \quad \text{and} \quad \frac{zg'(w)}{g(w)} \prec \phi(w), w\in D$$

if and only if $f \in S^*_\Sigma(\phi)$, where $g=f^{-1}$.

Remark 1.2. *Choose $\beta=0$ in Definition 1.1, we have $M_\Sigma(\phi,0)=C_\Sigma(\phi)$; that is,*

$$1+\frac{zf'(z)}{f(z)} \prec \phi(z), z\in U \quad \text{and} \quad 1+\frac{zg'(w)}{g(w)} \prec \phi(w), w\in D$$

if and only if $f \in K_\Sigma(\phi)$, where $g=f^{-1}$.

Remark 1.3. *These classes $S^*_\Sigma(\phi)$ and $C_\Sigma(\phi)$ was investigated by Ma and Minda* [15].

To prove our main results, we need require the following lemmas.

Lemma 1.1. *(See, for example,* [20]*) If $p \in \mathrm{P}$, then the estimates $\left|p_n\right| \le 2, n=1,2,3,\dots$ are sharp, where P is the family of all functions p, analytic in U for which $p(0)=1$ and $\mathrm{Re}\left(p(z)\right)>0\ (z\in U)$, and*

$$p(z)=1+p_1z+p_2z^2+\cdots,\ z\in U. \tag{1.3}$$

Lemma 1.2. *(See, for example,* [11]*) If the function* $p\in \mathrm{P}$ *is given by the series (1.2), then*

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$
$$4p_3=p_1^3+2\left(4-p_1^2\right)p_1x-\left(4-p_1^2\right)p_1x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

for some x *and* z *with* $|x|\le 1$ *and* $|z|\le 1$.

The object of the present paper is to find the upper bound estimate for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for the class $M_\Sigma(\phi,\beta)$, $\beta\in[0,1]$ and upper bound estimates for the initial coefficients $|a_2|$ and $|a_3|$ of the functions belonging to this class .

In the introduction and preliminaries section of the paper, we provide the necessary information to prove our main results. In the second section we give the main result. Ere, the special cases of the results obtained in presented paper have been examined and compared with known results. In the third section, we give concluding remarks.

2. The Fekete-Szegö problem for the class $M_\Sigma(\phi,\beta)$

In this section, we prove the following theorem on upper bound of the Fekete-Szegö functional for the functions belonging to the class $M_\Sigma(\phi,\beta)$.

Theorem 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\Sigma(\phi,\beta)$, $\beta\in[0,1]$, *where* ϕ *is an analytic function given by (1.2) and* $\mu\in\mathbb{C}$. *Then,*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{b_1}{2(3-2\beta)}, & \text{if } |1-\mu|\in[0,\mu_0),\\[2mm] \dfrac{b_1^2}{(2-\beta)^2}|1-\mu|, & \text{if } |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\mu_0=\dfrac{(2-\beta)^2}{4b_1(3-2\beta)}$.

Proof. Let us $f\in M_\Sigma(\phi,\beta)$, $\beta\in[0,1]$ and $g=f^{-1},\phi$ is an analytic function given by (1.2). Then, in view of Definition 1.1, there are analytic functions $u:U\to U$, $v:D\to D$ with $u(0)=0=v(0)$, $|u(z)|<1$, $|v(w)|<1$ and satisfying

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\beta}=\phi(u(z)) \text{ and } \left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}=\phi(v(w)). \tag{2.1}$$

Let us define the functions $p(z)$ and $q(w)$ by

$$p(z):=\frac{1+u(z)}{1-u(z)}=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U \text{ and } q(w):=\frac{1+v(w)}{1-v(w)}=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in D.$$

It follows that

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left\{p_1z+\left[p_2-\frac{p_1^2}{2}\right]z^2+\left[p_3-p_1p_2+\frac{p_1^3}{4}\right]z^3+\cdots\right\} \tag{2.2}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left\{q_1w+\left[q_2-\frac{q_1^2}{2}\right]w^2+\left[q_3-q_1q_2+\frac{q_1^3}{4}\right]w^3+\cdots\right\}. \tag{2.3}$$

Using (2.2) and (2.3) in (1.2), we can easily write

$$\begin{aligned}\phi(u(z))=1+\frac{b_1p_1}{2}z+\left[\frac{b_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}b_2p_1^2\right]z^2\\+\left[\frac{b_1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{b_2p_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{b_3p_1^3}{8}\right]z^3+\cdots\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\phi(v(w))=1+\frac{b_1q_1}{2}w+\left[\frac{b_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}b_2q_1^2\right]w^2\\+\left[\frac{b_1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)+\frac{b_2q_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{b_3q_1^3}{8}\right]w^3+\cdots.\end{aligned} \tag{2.5}$$

Also, using (2.4) and (2.5) in (2.1) and equating the coefficients, we get

$$(2-\beta)a_2=\frac{b_1p_1}{2}, \tag{2.6}$$

$$2(3-2\beta)a_3+\frac{1}{2}\left[(\beta-2)^2-3(4-3\beta)\right]a_2^2=\frac{b_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}b_2p_1^2, \tag{2.7}$$

and

$$-(2-\beta)a_2=\frac{b_1q_1}{2}, \tag{2.8}$$

$$-2(3-2\beta)a_3+\frac{1}{2}\left(\beta^2-11\beta+16\right)a_2^2=\frac{b_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}b_2q_1^2, \tag{2.9}$$

From (2.6) and (2.8), we have

$$a_2=\frac{b_1p_1}{2(2-\beta)}=\frac{-b_1q_1}{2(2-\beta)}, \tag{2.10}$$

it follows that

$$p_1=-q_1. \tag{2.11}$$

By subtracting from (2.7) to (2.9) and next considering (2.10) and (2.11), we can easily obtain

$$a_3=a_2^2+\frac{b_1(p_2-q_2)}{8(3-2\beta)}=\frac{b_1^2p_1^2}{4(2-\beta)^2}+\frac{b_1(p_2-q_2)}{8(3-2\beta)}. \tag{2.12}$$

From (2.12) and (2.10), we find that

$$a_3-\mu a_2^2=(1-\mu)a_2^2+\frac{b_1(p_2-q_2)}{8(3-2\beta)}. \tag{2.13}$$

Since $p_1=-q_1$, according to Lemma 1.1,

$$p_2-q_2=\frac{4-p_1^2}{2}(x-y) \tag{2.14}$$

for some x, y with $|x| \le 1,\ |y| \le 1$ and $p_1 \in [0,2]$.

In this case, since $p_1 \in [0,2]$, we may assume without any restriction that $t \in [0,2]$, where $t = |p_1|$.

Substituting the expression (2.14) in (2.13) and using triangle inequality, taking $|x| = \xi,\ |y| = \eta$, we can easily obtain that

$$\left|a_3 - \mu a_2^2\right| \le d_1(t) + d_2(t)(\xi + \eta) = F(\xi, \eta), \tag{2.15}$$

where

$$d_1(t) = |1-\mu| \frac{b_1^2 t^2}{4(2-\beta)^2} \ge 0 \quad \text{and} \quad d_2(t) = \frac{b_1\left(4-t^2\right)}{16(3-2\beta)} \ge 0.$$

From (2.15), we can write

$$\left|a_3 - \mu a_2^2\right| \le \max\left\{\max\left\{F(\xi,\eta) : (\xi,\eta) \in \Omega\right\} : t \in [0,2]\right\}, \tag{2.16}$$

where $\Omega = \{(\xi,\eta) \in \mathbb{R}^2 : \xi, \eta \in [0,1]\}$.

It is clear that

$$\max\left\{F(\xi,\eta) : (\xi,\eta) \in \Omega\right\} = F(1,1) = d_1(t) + 2d_2(t) = c_1(\phi,\beta,\mu)t^2 + c_2(\phi,\beta),$$

where

$$c_1(\phi,\beta,\mu) = \frac{b_1^2}{4(2-\beta)^2}\left[|1-\mu| - \frac{(2-\beta)^2}{4b_1(3-2\beta)}\right],\ c_2(\phi,\beta) = \frac{b_1}{4(3-2\beta)}.$$

Let us define the function $G: \mathbb{R} \to \mathbb{R}$ as follows

$$G(t) = c_1(\phi,\beta,\mu)t^2 + c_2(\phi,\beta),\ t \in [0,2]. \tag{2.17}$$

Differentiating both sides of (2.17), we have

$$G'(t) = 2c_1(\phi,\beta,\mu)t.$$

It is clear that $G'(t) < 0$ if $|1-\mu| \in \left[0, \frac{(2-\beta)^2}{4b_1(3-2\beta)}\right)$. Thus, the function $G(t)$ is a strictly decreasing function if $|1-\mu| \in [0,\mu_0)$, where $\mu_0 = \frac{(2-\beta)^2}{4b_1(3-2\beta)}$.

Therefore,

$$\max\left\{G(t) : t \in [0,2]\right\} = G(0) = 2d_2(0) = \frac{b_1}{2(3-2\beta)}. \tag{2.18}$$

Also, $G'(t) \ge 0$; that is, the function $G(t)$ is an increasing function for $|1-\mu| \ge \mu_0$.

Therefore,

$$\max\left\{G(t) : t \in [0,2]\right\} = G(2) = d_1(2) = \frac{|1-\mu| b_1^2}{(2-\beta)^2}. \tag{2.19}$$

Thus, from (2.15)-(2.19), we obtain that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{b_1}{2(3-2\beta)}, & \text{if } |1-\mu| \in [0,\mu_0), \\ \frac{b_1^2}{(2-\beta)^2}|1-\mu|, & \text{if } |1-\mu| \in [\mu_0, +\infty), \end{cases}$$

where $\mu_0 = \dfrac{(2-\beta)^2}{4b_1(3-2\beta)}$.

Thus, the proof of Theorem 2.1 is completed.

In the special cases from Theorem 2.1, we arrive at the following results.

Corollary 2.1. *Let the function $f(z)$ given by (1.1) be in the class $S_\Sigma^*(\phi)$, where ϕ is an analytic function given by (1.2) and $\mu \in \mathbb{C}$. Then,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{b_1}{2}, & \text{if } |1-\mu| \in \left[0, \dfrac{1}{4b_1}\right), \\ |1-\mu| b_1^2, & \text{if } |1-\mu| \in \left[\dfrac{1}{4b_1}, +\infty\right). \end{cases}$$

Corollary 2.2. *Let the function $f(z)$ given by (1.1) be in the class $C_\Sigma(\phi)$, where ϕ is an analytic function given by (1.2) and $\mu \in \mathbb{C}$. Then,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{b_1}{6}, & \text{if } |1-\mu| \in \left[0, \dfrac{1}{3b_1}\right), \\ \dfrac{|1-\mu| b_1^2}{4}, & \text{if } |1-\mu| \in \left[\dfrac{1}{3b_1}, +\infty\right). \end{cases}$$

The following theorem is direct result of Theorem 2.1.

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the class $M_\Sigma(\phi,\beta)$, $\beta \in [0,1]$. Then,*

$$|a_2| \le \frac{b_1}{2-\beta} \quad \textit{and} \quad |a_3| \le \begin{cases} \dfrac{b_1}{2(3-2\beta)}, & \text{if } b_1 < \dfrac{(2-\beta)^2}{4(3-2\beta)}, \\ \dfrac{b_1^2}{(2-\beta)^2}, & \text{if } b_1 \ge \dfrac{(2-\beta)^2}{4(3-2\beta)}. \end{cases}$$

Moreover

$$|a_3 - a_2^2| \le \frac{b_1}{2(3-2\beta)}.$$

Proof. Let us $f \in M_\Sigma(\phi,\beta)$, $\beta \in [0,1]$ and $g = f^{-1}$, ϕ is an analytic function given by (1.2). Then, from (2.10) the first estimate of theorem is clear. The second and third estimates is direct result of Theorem 2.1 for $\mu = 0$ and $\mu = 1$, respectively .

In the special cases from Theorem 2.2, we arrive at the following results.

Corollary 2.3. *Let the function $f(z)$ given by (1.1) be in the class $S_\Sigma^*(\phi)$, where ϕ is an analytic function given by (1.2). Then,*

$$|a_2| \le b_1 \quad \textit{and} \quad |a_3| \le \begin{cases} \dfrac{b_1}{2}, & \text{if } b_1 < \dfrac{1}{4}, \\ b_1^2, & \text{if } b_1 \ge \dfrac{1}{4}. \end{cases}$$

Moreover

$$\left|a_3 - a_2^2\right| \le \frac{b_1}{2}.$$

Corollary 2.4. *Let the function* $f(z)$ *given by (1.1) be in the class* $C_\Sigma(\phi)$, *where* ϕ *is an analytic function given by (1.2). Then,*

$$|a_2| \le \frac{b_1}{2} \text{ and } |a_3| \le \begin{cases} \frac{b_1}{6}, & \text{if } b_1 < \frac{1}{3}, \\ \frac{b_1^2}{4}, & \text{if } b_1 \ge \frac{1}{3}. \end{cases}$$

3. Concluding remarks

If the function $\phi(z)$, aforementioned in study, is given by

$$\phi(z) := \frac{1+az}{1+bz} = 1 + (a-b)z - b(a-b)z^2 + b^2(a-b)z^3 + \cdots \quad (-1 \le b < a \le 1), \quad (3.1)$$

then $b_1 = (a-b),\ b_2 = -b(a-b)$ and $b_3 = b^2(a-b)$.

Taking $a = 1-2\alpha,\ b = -1$ in (3.1), we have

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + 2(1-\alpha)z^3 + \cdots \quad (0 \le \alpha < 1). \quad (3.2)$$

Hence, $b_1 = b_2 = b_3 = 2(1-\alpha)$.

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.1, we can readily deduce the following results, respectively.

Corollary 3.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\Sigma\left(\frac{1+az}{1+bz}, \beta\right)$, $-1 \le b < a \le 1, 0 \le \beta \le 1$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{a-b}{2(3-2\beta)}, & \text{if } |1-\mu| \in [0, \mu_0), \\ \frac{|1-\mu|(a-b)^2}{(2-\beta)^2}, & \text{if } |1-\mu| \in [\mu_0, +\infty), \end{cases}$$

where $\mu_0 = \frac{(2-\beta)^2}{4(a-b)(3-2\beta)}$.

Corollary 3.2. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\Sigma\left(\frac{1+(1-2\alpha)z}{1-z}, \beta\right) = M_\Sigma(\alpha, \beta)$, $0 \le \alpha < 1, 0 \le \beta \le 1$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{1-\alpha}{3-2\beta}, & \text{if } \left|1-\mu\right| \in \left[0, \mu_0\right), \\ \dfrac{4\left|1-\mu\right|(1-\alpha)^2}{(2-\beta)^2}, & \text{if } \left|1-\mu\right| \in \left[\mu_0, +\infty\right), \end{cases}$$

where $\mu_0 = \dfrac{(2-\beta)^2}{8(1-\alpha)(3-2\beta)}$

Also, taking $\alpha = 0$ in (3.2), we get

$$\phi(z) = \frac{1+z}{1-z} = 1+2z+2z^2+2z^3+\cdots \quad . \tag{3.3}$$

Hence, $b_1 = b_2 = b_3 = 2$.

Choosing $\phi(z)$ of the form (3.3) in Theorem 2.1, we arrive at the following corollary.

Corollary 3.3. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\Sigma\left(\dfrac{1+z}{1-z}, \beta\right)$, $0 \le \beta \le 1$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{1}{3-2\beta}, & \text{if } \left|1-\mu\right| \in \left[0, \mu_0\right), \\ \dfrac{4\left|1-\mu\right|}{(2-\beta)^2}, & \text{if } \left|1-\mu\right| \in \left[\mu_0, +\infty\right), \end{cases}$$

where $\mu_0 = \dfrac{(2-\beta)^2}{8(3-2\beta)}$.

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.2, we can readily deduce the following results, respectively.

Corollary 3.4. *Let the function* $f(z)$ *given by (1.1) be in the class* $S_\Sigma^*\left(\dfrac{1+az}{1+bz}\right)$, $-1 \le b < a \le 1$. *Then,*

$$\left|a_2\right| \le \frac{a-b}{2-\beta} \quad \text{and} \quad \left|a_3\right| \le \begin{cases} \dfrac{a-b}{2(3-2\beta)}, & \text{if } a < b+b_0, \\ \dfrac{(a-b)^2}{(2-\beta)^2}, & \text{if } a \ge b+b_0, \end{cases}$$

where $b_0 = \dfrac{(2-\beta)^2}{4(3-2\beta)}$.

Moreover

$$\left|a_3-a_2^2\right| \le \frac{a-b}{2(3-2\beta)}.$$

Corollary 3.5. *Let the function* $f(z)$ *given by (1.1) be in the class* $S_\Sigma^*\left(\dfrac{1+(1-2\alpha)z}{1-z}\right) = S_\Sigma^*(\alpha)$, $0 \le \alpha < 1$. *Then,*

$$|a_2| \le \frac{2(1-\alpha)}{2-\beta} \quad and \quad |a_3| \le \begin{cases} \frac{1-\alpha}{3-2\beta}, & \text{if } \alpha \le 1-\alpha_0, \\ \frac{4(1-\alpha)^2}{(2-\beta)^2}, & \text{if } \alpha > 1-\alpha_0, \end{cases}$$

where $\alpha_0 = \frac{(2-\beta)^2}{8(3-2\beta)}$.

Moreover

$$|a_3 - a_2^2| \le \frac{1-\alpha}{3-2\beta}.$$

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