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Research Article

A circulant and block-diagonal splitting method for solving Toeplitz systems

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Abstract By the principle of using sufficiently the property of circulant matrix and based on the technique of matrix iterative method, we set up a new circulant and block-diagonal splitting method for solving the Toeplitz systems. Moreover, we present a successive overrelaxation acceleration scheme for the proposed splitting iteration. Theoretical analysis shows that if given reasonable restrictions for the parameter of the Toeplitz matrix, the new splitting method is convergent.

Keywords Toeplitz systems; circulant matrix; iterative method; SOR method; convergence

1. Introduction

Considering the Toeplitz system

$$Ax = b, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is a nonsingular Toeplitz matrix with form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{-(n-2)} & a_{-(n-3)} & a_{-(n-4)} & \cdots & a_0 & a_1 \\ a_{-(n-1)} & a_{-(n-2)} & a_{-(n-3)} & \cdots & a_{-1} & a_0 \end{pmatrix},$$

i.e., the elements of A are constant along its diagonals. The algorithms for solving the Toeplitz systems are called Toeplitz solvers.

Toeplitz systems arise in a variety of applications in mathematics, scientific computing and engineering, for instance, image restoration problems in image processing, numerical differential equations and integral equations, time series and control theory, etc. These applications have motivated both mathematics and engineering to develop sufficient methods for solving Toeplitz systems.

The properties of Toeplitz matrices and the numerical methods for solving Toeplitz systems have been investigated by many authors (see [5, 7, 8, 9, 10, 12, 13]). Some current developments and applications in using iterative methods for solving block Toplitz systems are summarized by Jin in [11]. Also, some preconditioners for Toeplitz systems are proposed by some authors (cf. [2, 3, 4, 10]).

In order to solve the Toeplitz system (1) using iterative methods, in [12], the matrix A possesses a circulant and skew-circulant splitting:

$$A = \widetilde{C} + \widetilde{S}$$

with

$$\tilde{C}=\begin{pmatrix} a_0 & \frac{a_1+a_{-(n-1)}}{2} & \frac{a_2+a_{-(n-2)}}{2} & \cdots & \frac{a_{n-1}+a_{-1}}{2} \\ \frac{a_{-1}+a_{n-1}}{2} & a_0 & \frac{a_1+a_{-(n-1)}}{2} & \cdots & \frac{a_{n-2}+a_{-2}}{2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{a_{-(n-2)}+a_2}{2} & \frac{a_{-(n-3)}+a_3}{2} & \frac{a_{-(n-4)}+a_4}{2} & \cdots & \frac{a_1+a_{-(n-1)}}{2} \\ \frac{a_{-(n-1)}+a_1}{2} & \frac{a_{-(n-2)}+a_2}{2} & \frac{a_{-(n-3)}+a_3}{2} & \cdots & a_0 \end{pmatrix},$$

$$\tilde{S}=\begin{pmatrix} 0 & \frac{a_1-a_{-(n-1)}}{2} & \frac{a_2-a_{-(n-2)}}{2} & \cdots & \frac{a_{n-1}-a_{-1}}{2} \\ \frac{a_{-1}-a_{n-1}}{2} & 0 & \frac{a_1-a_{-(n-1)}}{2} & \cdots & \frac{a_{n-2}-a_{-2}}{2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{a_{-(n-2)}-a_2}{2} & \frac{a_{-(n-3)}-a_3}{2} & \frac{a_{-(n-4)}-a_4}{2} & \cdots & \frac{a_1-a_{-(n-1)}}{2} \\ \frac{a_{-(n-1)}-a_1}{2} & \frac{a_{-(n-2)}-a_2}{2} & \frac{a_{-(n-3)}-a_3}{2} & \cdots & 0 \end{pmatrix}.$$

Here, $\tilde{C}$ is a circulant matrix and $\tilde{S}$ is a skew-circulant matrix. And he also gave a CSCS iterative method. Theoretical analysis has shown that the convergence of the iterative method depends on the circulant and skew-circulant matrices.

In [11], another splitting method of Toeplitz system (1) has been given by

$$A=\tilde{B}+\tilde{D}$$

with

$$\tilde{B}=\begin{pmatrix} a_0 & a_1 & \cdots & a_{\frac{n}{2}-1} & \xi & a_{-(\frac{n}{2}-1)} & \cdots & a_{-2} & a_{-1} \\ a_{-1} & a_0 & \cdots & a_{\frac{n}{2}-2} & a_{\frac{n}{2}-1} & \xi & \cdots & a_{-3} & a_{-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{-(\frac{n}{2}-1)} & a_{-(\frac{n}{2}-2)} & \cdots & a_0 & a_1 & a_2 & \cdots & a_{\frac{n}{2}-1} & \xi \\ \xi & a_{-(\frac{n}{2}-1)} & \cdots & a_{-1} & a_0 & a_1 & \cdots & a_{\frac{n}{2}-2} & a_{\frac{n}{2}-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1 & a_2 & \cdots & \xi & a_{-(\frac{n}{2}-1)} & a_{-(\frac{n}{2}-2)} & \cdots & a_{-1} & a_0 \end{pmatrix},$$

$$\tilde{D}=\begin{pmatrix} 0 & \cdots & 0 & a_{\frac{n}{2}}-\xi & \cdots & a_{n-1}-a_{-1} \\ 0 & \cdots & 0 & 0 & \cdots & a_{n-2}-a_{-2} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a_{\frac{n}{2}}-\xi \\ a_{-(\frac{n}{2})}-\xi & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{-(n-1)}-a_1 & \cdots & a_{-\frac{(n)}{2}}-\xi & 0 & \cdots & 0 \end{pmatrix}.$$

Here, $\tilde{B}$ is a circulant matrix and $\tilde{D}$ is a block skew-diagonal matrix. Thus the matrix $\tilde{D}$ can be written as follows.

$$\tilde{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

where D_+ is an $\frac{n}{2} \times \frac{n}{2}$ strictly lower Toeplitz matrix and D_- is an $\frac{n}{2} \times \frac{n}{2}$ strictly upper Toeplitz matrix.

In [8], the normal/skew-Hermitian splitting method has been considered for circulant-plus-diagonal systems.
While in [1], Bai et. al gave an Hermitian/skew-Hermitian splitting for any non-Hermitian positive definite systems and they also proposed the HSS iterative method. Such iterative method converges unconditionally to the exact solution of the Toeplitz systems (1) with the bound on convergence speed about the same as that of the conjugate gradient method when applied to the Hermitian part of the coefficient matrix A. Moreover, the upper bound of the contraction factor is dependent only on the spectrum of the Hermitian part of A, and it is independent on the spectrum of the skew-Hermitian part of A.

In this paper, we investigate the iterative methods for solving Toeplitz system (1). By the principle of using sufficiently the property of circulant matrix and based on the technique of matrix splittings, we set up a new circulant and block-diagonal splitting method for solving the Toeplitz systems. Theoretical analysis shows that if given reasonable restrictions for the parameter of the Toeplitz matrix, the new iterative method derived by the matrix splitting is convergent. Numerical results show that the new iterative method converges faster than the CSCS iterative method given in [12].

The arrangement of this paper is as follows. In Section 2, we introduce a new circulant and block-diagonal splitting of Toeplitz matrices. Based on this splitting the iterative method for solving Toeplitz systems is derived. In Section 3, we discuss the convergence condition and the determination of the optimal parameters. In Section 4, the corresponding SOR iterative method is defined. In Section 5, two simple numerical experiments are given.

2. A new circulant and block-diagonal splitting method

In this section, we propose a new circulant and block-diagonal splitting method for Toeplitz matrix.

When A is an even order Toeplitz matrix, then $n = 2m$ for some $m > 1$, and the matrix A can be splitted to a 2×2 block matrix

$$A = \begin{pmatrix} D & B \\ C & D \end{pmatrix},$$

where D, B and C are all $m \times m$ Toeplitz matrices defined by

$$D = \begin{pmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_{-1} & a_0 & \cdots & a_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{-m+1} & a_{-m+2} & \cdots & a_0 \end{pmatrix},$$

$$B = \begin{pmatrix} a_m & a_{m+1} & \cdots & a_{2m-1} \\ a_{m-1} & a_m & \cdots & a_{2m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_m \end{pmatrix}, C = \begin{pmatrix} a_{-m} & a_{-m+1} & \cdots & a_{-1} \\ a_{-m-1} & a_{-m} & \cdots & a_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{-2m+1} & a_{-2m+2} & \cdots & a_{-m} \end{pmatrix}.$$

So we can transform A to

$$\hat{A} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} A = \begin{pmatrix} C & D \\ D & B \end{pmatrix}.$$

It can be splitted to a circulant matrix and a block diagonal matrix as

$$\hat{A} = M + N$$

with

$$M = \begin{pmatrix} \tau & a_{-\frac{n}{2}+1} & \cdots & a_{-1} & a_0 & a_1 & \cdots & a_{\frac{n}{2}-1} \\ a_{\frac{n}{2}-1} & \tau & \cdots & a_{-2} & a_{-1} & a_0 & \cdots & a_{\frac{n}{2}-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\ a_1 & a_2 & \cdots & \tau & a_{-\frac{n}{2}+1} & a_{-\frac{n}{2}+2} & \cdots & a_0 \\ a_0 & a_1 & \cdots & a_{\frac{n}{2}-1} & \tau & a_{-\frac{n}{2}+1} & \cdots & a_{-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\ a_{-\frac{n}{2}+1} & a_{-\frac{n}{2}+2} & \cdots & a_0 & a_1 & a_2 & \cdots & \tau \end{pmatrix},$$

$$N = \begin{pmatrix} a_{-\frac{n}{2}} - \tau & \cdots & 0 & 0 & \cdots & 0 \\ a_{-(\frac{n}{2}+1)} - a_{\frac{n}{2}-1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{-(n-1)} - a_1 & \cdots & a_{-\frac{n}{2}} - \tau & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{\frac{n}{2}} - \tau & \cdots & a_{n-1} - a_{-1} \\ 0 & \cdots & 0 & 0 & \cdots & a_{n-2} - a_{-2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a_{\frac{n}{2}} - \tau \end{pmatrix},$$

i.e.,

$$M = circ(\tau, a_{-(\frac{n}{2})}, \ldots, a_{-1}, a_0, \ldots, a_{\frac{n}{2}-1}),$$

$$N = \begin{pmatrix} L & 0 \\ 0 & U \end{pmatrix},$$

where L is an $\frac{n}{2} \times \frac{n}{2}$ lower-triangular Toeplitz matrix and U is an $\frac{n}{2} \times \frac{n}{2}$ upper-triangular Toeplitz matrix. The spectrum of N can be easily obtained. Typically we can let

$$a_{\frac{n}{2}} - \tau > 0, \quad a_{-(\frac{n}{2})} - \tau > 0,$$

i.e., $\tau < min\{a_{\frac{n}{2}}, a_{-(\frac{n}{2})}\}$, so that N is positive definite and $\rho(N) > 0$.

While, when A is an odd order Toeplitz matrix, then $n = 2m+1$ for some $m > 1$. We can also split the matrix A to

$$A=\begin{pmatrix} & & a_m & & \\ D & & \vdots & & B \\ a_{-m} & \cdots & a_0 & \cdots & a_m \\ C & & \vdots & & D \\ & & a_{-m} & & \end{pmatrix},$$

where D, B and C are all $m\times m$ Toeplitz matrices like the even order case. So we can transform A to

$$\hat{A}=\begin{pmatrix} 0 & I_m \\ I_{m+1} & 0 \end{pmatrix}A=\begin{pmatrix} & & a_{-1} & & \\ C & & \vdots & & D \\ & & a_{-m} & & \\ & & a_m & & \\ D & & \vdots & & B \\ a_{-m} & \cdots & a_0 & \cdots & a_m \end{pmatrix}.$$

It can be splid to $\hat{A}=M+N$, where

$$M=\begin{pmatrix} \tau & a_{-m} & \cdots & a_{-1} & a_0 & a_1 & \cdots & a_{m-1} \\ a_{m-1} & \tau & \cdots & a_{-2} & a_{-1} & a_0 & \cdots & a_{m-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\ a_1 & a_2 & \cdots & \tau & a_{-m} & a_{-m+1} & \cdots & a_0 \\ a_0 & a_1 & \cdots & a_{m-1} & \tau & a_{-m} & \cdots & a_{-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\ a_{-m} & a_{-m+1} & \cdots & a_0 & a_1 & a_2 & \cdots & \tau \end{pmatrix},$$

$$N=\begin{pmatrix} a_{-m-1}-\tau & \cdots & 0 & 0 & \cdots & 0 \\ a_{-m-2}-a_{m-1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{-2m}-a_1 & \cdots & a_{-m-1}-\tau & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_m-\tau & \cdots & a_{2m}-a_{-1} \\ 0 & \cdots & 0 & 0 & \cdots & a_{2m-1}-a_{-2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a_m-\tau \end{pmatrix},$$

i.e.,

$$M=circ(\tau,a_{-m},\ldots,a_{-1},a_0,a_1,\ldots,a_{m-1}),\ N=\begin{pmatrix} L & 0 \\ 0 & U \end{pmatrix}.$$

Here L is a $m\times m$ lower-triangular Toeplitz matrix and U is a $(m+1)\times(m+1)$ upper-triangular Toeplitz matrix. The spectrum of N can be easily got.

Now, we have proved the matrix $\hat{A}$ is transformed from the Toeplitz matrix A, i.e.,

$$\hat{A}=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}A=PA,$$

it can be seen as an preconditioner of A, and the Toeplitz system can be transformed to

$$PAx = Pb.$$

We just take A as $\hat{A}$. Based on the splitting above, we can split Toeplitz matrix A to the new form

$$A = (\alpha I + M) - (\alpha I - N) = (\alpha I + N) - (\alpha I - M).$$

With the splitting, we can define the circulant/block-diagonal splitting method, as presented in the following:
Given an initial guess x^0, for $k = 0,1,2,\ldots$ compute

$$x^{k+\frac{1}{2}} = (\alpha I + M)^{-1}(\alpha I - N)x^k + (\alpha I + M)^{-1}b, \tag{1}$$

$$x^{k+1} = (\alpha I + N)^{-1}(\alpha I - M)x^{k+\frac{1}{2}} + (\alpha I + N)^{-1}b.$$

Evidently, the two-half step of the iteration alternates between the circulant matrix M and the block diagonal matrix N, analogously to the CSCS iterative method proposed by Ng in [12]. The role of M and N can be interchanged. Since circulant matrix can be diagonalized by the discrete Fourier matrix F and lower (upper) triangular matrix can be diagonalized easily, i.e.,

$$M = F^* \Lambda F,$$

$$N = T^* \cdot diag(a_{-\frac{n}{2}} - \tau, \ldots, a_{-\frac{n}{2}} - \tau, a_{\frac{n}{2}} - \tau, \ldots, a_{\frac{n}{2}} - \tau)T,$$

where Λ is the diagonal matrix holding the eigenvalue of M and the eigenvalue of N are $a_{-(\frac{n}{2})} - \tau$ and $a_{\frac{n}{2}} - \tau$.

3. Convergence analysis

Now we study the convergence of the iterative method (1).
We first give an lemma for circulant matrix. It has been proved in [6].

Lemma 3.1 *If $M \in C^{n\times n}$ is a circulant, then it is diagonalized by matrix F. More precisely,*

$$M = F^* \Lambda F,$$

where,

$$\Lambda = \Lambda_M = diag(p_\gamma(1), p_\gamma(\omega), \ldots, p_\gamma(\omega^{n-1})).$$

The eigenvalues of M are therefore

$$\lambda_j = p_\gamma(\omega^{j-1}) = \phi_\gamma(\frac{2\pi(j-1)}{n}),\ j = 1,2,\ldots,n,$$

where,

$$\phi_\gamma(\theta) = c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \ldots + c_{n-1} e^{(n-1)i\theta}.$$

Using this Lemma, we can get some properties in special situations.

Lemma 3.2 *Let $M \in C^{n\times n}$ be a circulant. Then M is Hermitian if and only if its eigenvalues are real, and M is skew-Hermitian if and only if its eigenvalues are pure imaginary numbers.*

Lemma 3.3 *Let $M \in C^{n\times n}$ be a circulant. M is Hermitian positive definite if and only if its eigenvalues are positive.*

For the convergence property of the iterative method (1), we give a general expressional mode for the two-step iterative method. The following lemma has been proved in [14].

Lemma 3.4 *For* $A \in C^{n\times n}$*, let* $A = M_i - N_i$, $i = 1,2$*, be two splittings and* $x^0 \in C^n$ *be a given initial vector. If* x^k *is defined by a two step iteration sequence*

$$M_1 x^{k+\frac{1}{2}} = N_1 x^k + b$$

$$M_2 x^{k+1} = N_2 x^{k+\frac{1}{2}} + b, k = 0,1,2,\ldots,$$

then

$$x^{k+1} = M_2^{-1}N_2M_1^{-1}N_1x^k + M_2^{-1}(N_2M_1^{-1} + I)b, k = 0,1,2,\ldots.$$

Moreover, if the spectral radius of the iteration matrix $M_2^{-1}N_2M_1^{-1}N_1$ less than 1, then the iteration sequence $\{x^k\}$ converges to the unique solution $x^* \in C^n$ of the system of linear equations $Ax = b$ for all initial vector $x^0 \in C^n$.

Now, using the lemmas above, we give the convergence of our new iterative method.

Theorem 3.5 *For* $A \in C^{n\times n}$*, let* $A = M + N$ *be the circulant/block-diagonal splitting. Then the iteration matrix* $G(\alpha)$ *is*

$$G(\alpha) = (\alpha I + N)^{-1}(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)$$

and its spectrum radius $\rho(G(\alpha))$ is bounded by

$$\sigma(\alpha) \equiv \max_{\lambda_j \in \lambda_M} \frac{|\alpha - \lambda_j|}{|\alpha + \lambda_j|} \cdot \max_{j=1,2} \frac{|\alpha - \mu_j|}{|\alpha + \mu_j|},$$

where $\mu_1 = a_{-(\frac{n}{2})} - \tau$, $\mu_2 = a_{\frac{n}{2}} - \tau$.

If we make τ such that $\lambda_j > 0$, $\mu_j > 0$, then it holds

$$\rho(G(\alpha)) \le \sigma(\alpha) < 1, \forall \alpha > 0,$$

i.e., the iterative method is convergent.

Proof. By Lemma 3.4 the iteration matrix is

$$G(\alpha) = (\alpha I + N)^{-1}(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N).$$

By the similarity invariance of the matrix spectrum, we have

$$\begin{aligned}
\rho(G(\alpha)) &= \rho((\alpha I + N)^{-1}(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)) \\
&= \rho((\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)(\alpha I + N)^{-1}) \\
&\le \|(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)(\alpha I + N)^{-1}\|_2 \\
&\le \|(\alpha I - M)(\alpha I + M)^{-1}\|_2 \cdot \|(\alpha I - N)(\alpha I + N)^{-1}\|_2.
\end{aligned}$$

Since M is a circulant matrix which can diagonalized by Fourier matrix as been shown in Lemma 3.1, and the eigenvalue of N are $a_{-(\frac{n}{2})} - \tau$ and $a_{\frac{n}{2}} - \tau$. Then it follows that

$$\rho(G(\alpha)) \le \max_{\lambda_j \in \lambda_M} \frac{|\alpha - \lambda_j|}{|\alpha + \lambda_j|} \cdot \max_{\mu_j \in \mu_N} \frac{|\alpha - \mu_j|}{|\alpha + \mu_j|}.$$

Hence we get

$$\sigma(\alpha) = \max_{\lambda_j = \lambda_j' + i\lambda_j''} \frac{|\alpha - (\lambda_j' + i\lambda_j'')|}{|\alpha + (\lambda_j' + i\lambda_j'')|} \cdot \max_{\mu_j \in \mu_N} \frac{|\alpha - \mu_j|}{|\alpha + \mu_j|}$$

$$= \max_{\lambda_j = \lambda_j' + i\lambda_j''} \sqrt{\frac{(\alpha - \lambda_j')^2 + (\lambda_j'')^2}{(\alpha + \lambda_j')^2 + (\lambda_j'')^2}} \cdot \max_{\mu_j \in \mu_N} \frac{|\alpha - \mu_j|}{|\alpha + \mu_j|},$$

where i denote the imaginary unit and

$$\lambda_j = \tau + a_{-(\frac{n}{2}-1)}\omega + a_{-(\frac{n}{2}-2)}\omega^2 + \ldots + a_{-1}\omega^{\frac{n}{2}-1} + a_0\omega^{\frac{n}{2}} + \ldots + a_{\frac{n}{2}-1}\omega^{n-1},$$

where $\omega = e^{\frac{i2\pi}{n}}$. Since α is positive, if the real part of λ_j and μ_j are positive, it is easy to see that $\sigma(\alpha)$ is strictly less than 1 and therefore $\rho(G(\alpha)) < 1$, the iterative method is convergent.

Thus the proof is completed.

The theorem indicates that the iterative method is always convergent when the eigenvalues of M have positive real part, i.e., M is Hermitian positive definite. Also it is not easy to determine the value of α in order to minimize the spectral radius of the iteration matrix. Now we concentrate on the case that M is not Hermitian.

Remark 3.6 *If $\gamma_{min}, \gamma_{max}, \eta_{min}, \eta_{max}$ define as follows:*

$$\gamma_{min} = \min_{j=0,1,..,n-1} \{\lambda_j', \mu_1', \mu_2'\}, \quad \gamma_{max} = \max_{j=0,1,..,n-1} \{\lambda_j', \mu_1', \mu_2'\},$$

$$\eta_{min} = \min_{j=0,1,..,n-1} \{\lambda_j'', \mu_1'', \mu_2''\}, \quad \eta_{max} = \max_{j=0,1,..,n-1} \{\lambda_j'', \mu_1'', \mu_2''\},$$

then $\sigma(\alpha)$ can be estimated by

$$\max_{\gamma + i\eta} \frac{(\alpha - \gamma)^2 + (\eta)^2}{(\alpha + \gamma)^2 + (\eta)^2}, \quad \gamma + i\eta \in \Omega = [\gamma_{min}, \gamma_{max}] \times i[\eta_{min}, \eta_{max}].$$

The optimal parameter α^* is chosen such that the above estimate can be minimized. This fact is precisely stated as the following theorem.

Theorem 3.7 *The minimizer of*

$$\max_{\lambda' + i\lambda''} \frac{(\alpha - \lambda')^2 + (\lambda'')^2}{(\alpha + \lambda')^2 + (\lambda'')^2}, \quad \lambda' + i\lambda'' \in \Omega = [\gamma_{min}, \gamma_{max}] \times i[\eta_{min}, \eta_{max}]$$

over all positive α is attained at

$$\alpha^* = \begin{cases} \sqrt{\gamma_{min} \cdot \gamma_{max} - (\eta_{max})^2} & for\ \eta_{max} < \sqrt{\gamma_{min} \cdot \gamma_{max}} \\ \sqrt{(\gamma_{min})^2 + (\eta_{max})^2} & for\ \eta_{max} \ge \sqrt{\gamma_{min} \cdot \gamma_{max}} \end{cases}$$

and the corresponding minimum value is equal to

$$\sigma(\alpha)=\begin{cases}\dfrac{\gamma_{min}+\gamma_{max}-2\sqrt{\gamma_{min}\cdot\gamma_{max}-(\eta_{max})^2}}{\gamma_{min}+\gamma_{max}+2\sqrt{\gamma_{min}\cdot\gamma_{max}-(\eta_{max})^2}} & for\ \eta_{max}<\sqrt{\gamma_{min}\cdot\gamma_{max}} \\ \dfrac{\sqrt{(\gamma_{min})^2+(\eta_{max})^2}-\gamma_{min}}{\sqrt{(\gamma_{min})^2+(\eta_{max})^2}+\gamma_{min}} & for\ \eta_{max}\geq\sqrt{\gamma_{min}\cdot\gamma_{max}}.\end{cases}$$

The proof of this theorem can be found in [1].

4. SOR acceleration

In this section, we first introduce a new block system. It can be shown that the block system is equivalent to

$$(M+N)x=b.$$

Next, the SOR iterative method for solving the block system is introduced. Like in [14], we then analyze the eigenvalue and convergence rate between Jacobi and SOR iterative methods.

Theorem 4.8 *Assume that the spectral radius of $G(\alpha)$ is smaller than 1.*

If x^* is the exact solution of (1), then the vector $\begin{pmatrix} x^* \\ x^* \end{pmatrix}$ satisfies

$$\begin{pmatrix} \alpha I+M & -(\alpha I-N) \\ -(\alpha I-M) & \alpha I+N \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b \\ b \end{pmatrix}. \tag{1}$$

Conversely, if $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$ satisfies (1), then $x^*=y^*$, and x^* is the solution of (1).

Proof. It is sufficient to show that the coefficient matrix in (1) is nonsingular. Indeed, it is nonsingular as

$$\begin{aligned}&\begin{pmatrix} \alpha I+M & -(\alpha I-N) \\ -(\alpha I-M) & \alpha I+N \end{pmatrix} \\ &=\begin{pmatrix} I & 0 \\ -(\alpha I-M)(\alpha I+M)^{-1} & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \alpha I+N \end{pmatrix}\begin{pmatrix} \alpha I+M & -(\alpha I-N) \\ 0 & I-G(\alpha) \end{pmatrix},\end{aligned}$$

where $G(\alpha)$, $\alpha I+N$ and $\alpha I+M$ are nonsingular and $\rho(G(\alpha))<1$ by the condition given.

In order to solve the newly transformed system, we can try to consider the block Jacobi iteration with splitting

$$\begin{aligned}&\begin{pmatrix} \alpha I+M & -(\alpha I-N) \\ -(\alpha I-M) & \alpha I+N \end{pmatrix} \\ &=\begin{pmatrix} \alpha I+M & 0 \\ 0 & \alpha I+N \end{pmatrix}+\begin{pmatrix} 0 & -(\alpha I-N) \\ -(\alpha I-M) & 0 \end{pmatrix}.\end{aligned}$$

As a result, the iterative method is defined as follows.

$$\begin{pmatrix} \alpha I+M & 0 \\ 0 & \alpha I+N \end{pmatrix}\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \alpha I - N \\ \alpha I - M & 0 \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix} + \begin{pmatrix} b \\ b \end{pmatrix}$$

or

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & (\alpha I + M)^{-1}(\alpha I - N) \\ (\alpha I + N)^{-1}(\alpha I - M) & 0 \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix}$$
$$+ \begin{pmatrix} (\alpha I + M)^{-1} b \\ (\alpha I + N)^{-1} b \end{pmatrix}.$$

By Theorem 4.8 and equation (1), the block Guass-Seidel and SOR iterative methods can be introduced as an attempt to improve the convergence of the basic circulant/block-diagonal splitting of Toeplitz systems, that is,

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} (1-\omega)I & \omega(\alpha I + M)^{-1}(\alpha I - N) \\ \omega(1-\omega)(\alpha I + N)(\alpha I - M) & (1-\omega)I + \omega^2 G(\alpha) \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix}$$
$$+ \begin{pmatrix} \omega(\alpha I + M)^{-1} \\ \omega(\alpha I + N)^{-1}(\omega((\alpha I - M)(\alpha I + M)^{-1} + I)) \end{pmatrix} b.$$

Denote that the iteration matrices of the block Jacobi and SOR iterative methods for parameter ω are $J(\alpha)$ and $L_\omega(\alpha)$, respectively. It has been proved in [14] that if μ is a non-zero eigenvalue of $J(\alpha)$ and λ satisfies the relation

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2, \tag{2}$$

then λ is a non-zero eigenvalue of $L_\omega(\alpha)$. On the other hand, if μ satisfies equation (2) and λ is a non-zero eigenvalue of $L_\omega(\alpha)$, then μ is an eigenvalue of $J(\alpha)$. Also $\rho(J(\alpha)) = \rho(G(\alpha)) < 1$. Therefor if the all eigenvalues of $J(\alpha)$ are real, we have the following result.

Theorem 4.9 *The SOR iterative method is convergent iff* $0 < \omega < 2$ *provided that the all eigenvalues of* $J(\alpha)$ *are real.*

Now we would go to the case that the block Jacobi iterative matrix has complex eigenvalues. From equation (2) we can obtain

$$\mu = \frac{1}{\omega}\left(\lambda^{\frac{1}{2}} + \frac{\omega - 1}{\lambda^{\frac{1}{2}}} \right).$$

Assuming that $\lambda^{\frac{1}{2}} = \rho(\cos\theta + i\sin\theta)$ and $\mu = \alpha + i\beta$, where $\lambda^{\frac{1}{2}}$ and μ are expressed in polar form and standard form, respectively, we get

$$\alpha + i\beta = \frac{1}{\omega}\left[\rho(\cos\theta + i\sin\theta) + \frac{\omega-1}{\rho}(\cos\theta - i\sin\theta)\right].$$

Then, by comparing the real parts and the imaginary parts it derives

$$\alpha = \frac{1}{\omega}\left(\rho + \frac{\omega-1}{\rho}\right)\cos\theta, \beta = \frac{1}{\omega}\left(\rho - \frac{\omega-1}{\rho}\right)\sin\theta.$$

Therefore, for $\rho^2 \neq |\omega - 1|$, the point (α, β) lies on the ellipse E_ρ :

$$E_\rho : \frac{\alpha^2}{[\frac{1}{\omega}(\rho + \frac{\omega-1}{\rho})]^2} + \frac{\beta^2}{[\frac{1}{\omega}(\rho - \frac{\omega-1}{\rho})]^2} = 1.$$

Denote that $a = |\frac{1}{\omega}(\rho + \frac{\omega-1}{\rho})|$ and $b = |\frac{1}{\omega}(\rho - \frac{\omega-1}{\rho})|$. Then

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1.$$

If $\rho^2 \geq |\omega - 1|$, then when ρ increases, both a and b increase and therefore E_ρ expands. On the other hand, if $\rho^2 < |\omega - 1|$ and when ρ increases, E_ρ shrinks. Therefor for $\rho^2 \geq |\omega - 1|$, there is a one-to-one correspondence between the circle $C_\rho : |\lambda^{\frac{1}{2}}| = \rho$ and E_ρ. Hence in order to prove the convergence or to find the optimal convergence rate, we can just work on μ (the eigenvalues of the block Jacobi matrix) instead of the iteration matrix of the SOR iterative method.

Then we have the following theorem.

Theorem 4.10 *The SOR iterative method is convergent if for each* $\mu = \alpha + i\beta$ *of the block Jacobi iteration matrix* J_α, *the point* (α, β) *lies on or inside the ellipse* $\alpha^2 + \frac{\beta^2}{D^2} = 1$ *for some* $D > 0$ *and* $0 < \omega < \frac{2}{1+D}$.

The proof of Theorems 4.9 and 4.10 can be found in [14].

5. Numerical examples

In this section, we test the convergence rate of the circulant/block-triangle splitting and the SOR iterative method for the Toeplitz system.

Example 5.11 An $n \times n$ Toeplitz matrix $A_n[f]$ is generated by a function f, i.e., the $(j,k)th$ entry of the Toeplitz matrix is given by a_{j-k} where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \quad k = 0, \pm1, \pm2 \ldots.$$

The function f is called the generating function of the Toeplitz matrix. Two kind of generating functions are tested. They are

(i) $a_j = (1+|j|)^{-p} + i(1+|j|)^{-p}$, $j = 0, \pm1, \pm2, \ldots$

or

(ii) $a_j = (1+j)^{-p}$, $j \geq 0$, and $a_j = (1-j)^{-p}$, $j < 0$.

The parameter $p = 1, 1.1, 0.9$, use the iterative method above, we can solve the Toeplitz system. The number of iteration time is as follows.

All tests are started from the zero vector, performed in MATLAB, and terminated when the current iterate satisfies $\| r^{(k)} \|_2 / \| r^{(0)} \|_2 < 10^{-6}$, where $r^{(k)}$ is the residual of the k th iteration.

The results will be listed in the tables. Where GF denotes generating function, IT1 and IT2 denote respectively the CSCS iterative method and SOR iterative method.

Table 1: The CSCS iterative method and SOR iterative method for the first GF

	$p=1$		$p=1.1$		$p=0.9$	
n	$IT1$	$IT2$	$IT1$	$IT2$	$IT1$	$IT2$
	15	8	15	8	16	8
	17	14	16	12	17	12
	18	15	17	13	19	12
	18	14	17	13	20	13
	19	14	17	15	20	13
	21	16	17	15	22	15

Table 2: The CSCS iterative method and SOR iterative method for the second GF

	$p=1$		$p=1.1$		$p=0.9$	
n	$IT1$	$IT2$	$IT1$	$IT2$	$IT1$	$IT2$
	13	9	11	7	15	6
	14	10	13	9	17	9
	16	11	13	10	17	10
	17	12	14	11	20	11
	17	12	14	11	23	14
	17	13	14	12	23	16

Example 5.12 The $n \times n$ Toeplitz matrix $A_n[f]$ is generated by a function f, where the $(j,k)th$ entry of the Toeplitz matrix is given by a_j defined by

(i) $a_j = \dfrac{1}{j+1}$, $a_{-j} = \dfrac{1}{j+1}$, $j = 0, \pm1, \pm2, \ldots$;

(ii) $a_j = j+1$, $a_{-j} = n - j - 1$, $j = 0, \pm1, \pm2, \ldots$

or

(iii) $a_j = 100(j+1)$, $a_{-j} = -n + j + 1$, $j = 0, \pm1, \pm2, \ldots$.

We compute the Toeplitz equations by the CSCS iterative method, circulant/block-diagonal splitting and SOR iterative method.

The number of iteration time is in Table 3, where IT1, IT2 and IT3 denote respectively the CSCS method, circulant/block-diagonal splitting and SOR iterative method. All tests are started from the zero vector, performed in MATLAB, and terminated when the current iterative method satisfies $\| r^{(k)} \|_2 / \| r^{(0)} \|_2 < 10^{-6}$.

Table 3: The CSCS, circulant/block-diagonal splitting and SOR iterative method

	First GF			**Second GF**			**Third GF**		
n	$IT1$	$IT2$	$IT3$	$IT1$	$IT2$	$IT3$	$IT1$	$IT2$	$IT3$
	6	6	5	9	8	8	11	9	7
	7	6	6	11	10	10	10	10	8
	8	9	7	14	12	11	12	9	8
	9	8	7	13	12	11	12	11	9
	11	9	8	15	13	12	13	11	9
	12	11	10	16	14	13	13	11	10
	13	11	10	15	14	13	13	11	10
	13	12	10	17	14	13	14	11	10
	12	11	10	16	15	13	14	13	10
	13	11	10	18	16	14	14	12	10

References

[1]. Z. Z. Bai, G. H. Golub, M. K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix. Anal. Appl. 24 (2003) 603-626.

[2]. T. Chan, An optimal circulant preconditioner for Toeplitz systems, SIAM J. Sci. Statistic. Comput. 9 (1988) 766-771.

[3]. R. Chan, Circulant preconditioners for Hermitian Toeplitz systems, SIAM J. Matrix. Anal. Appl. 10 (1989) 542-550.

[4]. R. Chan, P. Tang, Fast band Toeplitz preconditioners for Hermitian Toeplitz systems, SIAM J. Sci. Statis. Comput. 15 (1994) 164-171.

[5]. F. Chen, Y. L. Jiang, On HSS and AHSS iteration methods for nonsymmetric positive definite Toeplitz systems, J. Comput. Appl. Math. 234 (2010) 2432-2440.

[6]. P. J. Davis, Circulant Matrices, John Wiley-Sons, New York, 1979.

[7]. C. Q. Gu, Z. L. Tian, On the HSS iteration methods for positive definite Toeplitz linear systems, J. Comput. Appl. Math. 224 (2009) 709-718.

[8]. M. Ho, M. K. Ng, Splitting iterations for circulant-plus-diagonal systems, Numer. Linear Algebra Appl. 12 (2005) 779-792.

[9]. T. Huckle, Circulant and skew circulant matrices for solving Toeplitz matrix problems, SIAM J. Matrix Anal. Appl. 13 (1992) 767-777.

[10]. X. Jin, Band Toeplitz preconditioners for block Toeplitz systems, J. Comput. Appl. Math. 70 (1996) 225-230.

[11]. X. Jin, Developments and Applications of Block Iterative Solvers, Science Press, Beijing/New York, 2002.

[12]. M. K. Ng, Circulant and skew-circulant splitting methods for Toeplitz systems, J. Comput. Appl. Math. 159 (2003) 101-108.

[13]. G. Strang, A proposal for Toeplitz matrix caculations, Stud. Appl. Math. 74 (1986) 171-176.

[14]. D. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.