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Adams-Spanne type estimates for the commutators of fractional type sublinear operators in generalized Morrey spaces on Heisenberg groups

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Abstract In this paper we give BMO (bounded mean oscillation) space estimates for commutators of fractional type sublinear operators in generalized Morrey spaces on Heisenberg groups. The boundedness conditions are also formulated in terms of Zygmund type integral inequalities.

Keywords Heisenberg group; sublinear operator; fractional integral operator; fractional maximal operator; commutator; BMO space; generalized Morrey space

1 . Introduction and Main Results

Heisenberg groups play an important role in several branches of mathematics, such as quantum physics, Fourier analysis, several complex variables, geometry and topology; see [23] for more details. It is a remarkable fact that the Heisenberg group, denoted by H_n, arises in two aspects. On the one hand, it can be realized as the boundary of the unit ball in several complex variables. On the other hand, an important aspect of the study of the Heisenberg group is the background of physics, namely, the mathematical ideas connected with the fundamental notions of quantum mechanics. In other words, there is its genesis in the context of quantum mechanics which emphasizes its symplectic role in the theory of theta functions and related parts of analysis. Analysis on the groups is also motivated by their role as the simplest and the most important model in the general theory of vector fields satisfying Hörmander's condition. Due to this reason, many interesting works have been devoted to the theory of harmonic analysis on H_n in [6, 8, 9, 19, 20, 23, 26, 27].

We start with some basic knowledge about Heisenberg group in generalized Morrey spaces and refer the reader to [8, 11, 9, 23] and the references therein for more details. The Heisenberg group H_n is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbb{R}^{2n}\times\mathbb{R}$ and the group structure is given by

$$(x,t)\circ(x',t')=\left(x+x',t+t'+2\sum_{j=1}^{n}\left(x_{2j}x'_{2j-1}-x_{2j-1}x'_{2j}\right)\right).$$

Using the coordinates $g=(x,t)$ for points in H_n, the left-invariant vector fields for this group structure are

$$X_{2j-1}=\frac{\partial}{\partial x_{2j-1}}+2x_{2j}\frac{\partial}{\partial t}, \qquad j=1,\ldots,n,$$

$$X_{2j}=\frac{\partial}{\partial x_{2j}}+2x_{2j-1}\frac{\partial}{\partial t}, \qquad j=1,\ldots,n.$$

These vector fields generate the Lie algebra of H_n and the commutators of the vector fields $(X_1,\ldots,X_{2n})$ satisfy the relation

$$\left[X_j, X_{n+j}\right] = -4X_{2n+1}, \qquad j = 1,\ldots,n,$$

with all other brackets being equal to zero.
The inverse element of $g=(x,t)$ is $g^{-1}=(-x,-t)$ and we denote the identity (neutral) element of H_n as $e=(0,0)\in \mathbb{R}^{2n+1}$. The Heisenberg group is a connected, simply connected nilpotent Lie group. One-parameter Heisenberg dilations δ_r : $H_n \rightarrow H_n$ are given by $\delta_r(x,t)=(rx,r^2t)$ for each real number $r>0$. The Haar measure on H_n also coincides with the usual Lebesgue measure on $\mathbb{R}^{2n+1}$. These dilations are group automorphisms and Jacobian determinant of δ_r with respect to the Lebesgue measure is equal to r^Q, where $Q=2n+2$ is the homogeneous dimension of H_n. We denote the measure of any measurable set $\Omega\subset H_n$ by $|\Omega|$. Then

$$|\delta_r(\Omega)| = r^Q|\Omega|, \qquad d(\delta_r x) = r^Q dx.$$

The homogeneous norm on H_n is defined as follows

$$\|x\|_{H_n} = \|(x_1,\ldots,x_{2n},x_{2n+1})\|_{H_n} = \left[\left(\sum_{j=1}^{2n} x_j^2\right)^2 + x_{2n+1}^2\right]^{1/4},$$

and the Heisenberg distance is given by

$$d(g,h) = d(g^{-1}h,0) = |g^{-1}h|.$$

This distance d is left-invariant in the sense that $d(g,h)=|g^{-1}h|$ remains unchanged when g and h are both left-translated by some fixed vector on H_n. Moreover, d satisfies the triangular inequality (see [15], page 320)

$$d(g,h) \le d(g,x) + d(x,h), \qquad g,x,h\in H_n.$$

Using this norm, we define the Heisenberg ball

$$B(g,r) = \{h\in H_n : |g^{-1}h| < r\}$$

with center $g=(x,t)$ and radius r and denote by $B^C(g,r)=H_n\setminus B(g,r)$ its complement, and we denote by $B_r = B(e,r) = \{h\in H_n : |h|<r\}$ the open ball centered at e, the identity (neutral) element of H_n, with radius r. The volume of the ball $B(g,r)$ is $c_Q r^Q$, where c_n is the volume of the unit ball B_1:

$$c_Q = |B(e,1)| = \frac{2\pi^{n+\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{(n+1)\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)}.$$

For more details about Heisenberg group, one can refer to [8].
In the study of local properties of solutions to of second order elliptic partial differential equations (PDEs), together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(H_n)$ play an important role, see [10, 16]. They were introduced by Morrey in 1938 [18]. For the properties and applications of classical Morrey spaces, see [4, 5, 13] and the references therein. We recall its definition on a Heisenberg group as

$$L_{p,\lambda}(H_n)=\left\{ f : \|f\|_{L_{p,\lambda}(H_n)} = \sup_{g\in H_n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(g,r))} < \infty \right\},$$

where $f \in L_p^{loc}(H_n)$, $0 \le \lambda \le Q$ and $1 \le p < \infty$.

Note that $L_{p,0} = L_p(H_n)$ and $L_{p,Q} = L_\infty(H_n)$. If $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on H_n . It is known that $L_{p,\lambda}(H_n)$ is an expansion of $L_p(H_n)$ in the sense that $L_{p,0} = L_p(H_n)$.

We also denote by $WL_{p,\lambda} \equiv WL_{p,\lambda}(H_n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(H_n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(H_n)} = \sup_{g\in H_n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(g,r))} < \infty,$$

where $WL_p(B(g,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned}\|f\|_{WL_p(B(g,r))} &\equiv \left\| f\chi_{B(g,r)} \right\|_{WL_p(H_n)} \\ &= \sup_{\tau>0} \tau \left| \{ h \in B(g,r) : | f(h) | > \tau \} \right|^{1/p} \\ &= \sup_{0<\tau\le|B(g,r)|} \tau^{1/p} \left(f\chi_{B(g,r)} \right)^{*}(\tau) < \infty.\end{aligned}$$

Here g^* denotes the non-increasing rearrangement of a function g .

Note that

$$WL_p(H_n)=WL_{p,0}(H_n),\ L_{p,\lambda}(H_n)\subset WL_{p,\lambda}(H_n) \text{ and } \|f\|_{WL_{p,\lambda}(H_n)} \le \|f\|_{L_{p,\lambda}(H_n)}.$$

Let $|B(g,r)|$ be the Haar measure of the ball $B(g,r)$. Let f be a given integrable function on a ball $B(g,r) \subset G$. The fractional maximal function $M_\alpha f$, $0 \le \alpha < Q$, of f is defined by the formula

$$M_\alpha f(g) = \sup_{r>0} | B(g,r) |^{-1+\frac{\alpha}{Q}} \int_{B(g,r)} | f(h) | \, dh.$$

In the case of $\alpha = 0$, the fractional maximal function $M_\alpha f$ coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ (see [8, 23]) and is closely related to the fractional integral

$$\overline{T}_\alpha f(g) = \int_{H_n} \frac{f(h)}{|g^{-1}h|^{Q-\alpha}} dh \qquad 0 < \alpha < Q.$$

The operators M_α and $\overline{T}_\alpha$ play an important role in real and harmonic analysis (see [7, 8, 23, 26]).

The classical Riesz potential I_α is defined on $\mathbb{R}^n$ by the formula

$$I_\alpha f = (-\Delta)^{-\frac{\alpha}{2}} f, \qquad 0 < \alpha < n,$$

where Δ is the Laplacian operator. It is known that

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \equiv \overline{T}_\alpha f(x),$$

where $\gamma(\alpha)=\pi^{\frac{n}{2}}2^{\alpha}\dfrac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$. The Riesz potential on the Heisenberg group is defined in terms of the sub-Laplacian $\mathcal{L}=\Delta_{H_n}$.

Definition 1 *For* $0<\alpha<Q$ *the Riesz potential* I_α *is defined by on the Schwartz space* $S(H_n)$ *by the formula*

$$I_\alpha f(g)=\mathcal{L}^{-\frac{\alpha}{2}}f(g)\equiv\int_0^\infty e^{-r\mathcal{L}}f(g)r^{\frac{\alpha}{2}-1}dr,$$

where

$$e^{-r\mathcal{L}}f(g)=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_{H_n}K_r(h,g)f(h)d(h)$$

is the semigroups of the operator $\mathcal{L}$.

In [26], relations between the Riesz potential and the heat kernel on the Heisenberg group are studied. The following assertion $[[26],\text{Theorem } 4.2]$ yields an expression for I_α, which allows us to discuss the boundedness of the Riesz potential.

Theorem 1 *Let* $q_s(g)$ *be the heat kernel on* H_n. *If* $0\leq\alpha<Q$, *then for* $f\in S(H_n)$

$$I_\alpha f(g)=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty s^{\frac{\alpha}{2}-1}q_s(\cdot)ds*f(g).$$

The Riesz potential I_α satisfies the estimate $[[26],\text{Theorem } 4.2]$

$$|I_\alpha f(g)|\lesssim\overline{T}_\alpha f(g)$$

which provides a suitable estimate for the Riesz potential on the Heisenberg group. It is well known that, see [8, 23] for example, $\overline{T}_\alpha$ is bounded from $L_p(H_n)$ to $L_q(H_n)$ for all $p>1$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}>0$, and $\overline{T}_\alpha$ is also of weak type $\left(1,\frac{Q}{Q-\alpha}\right)$(i.e. Hardy-Littlewood Sobolev inequality).

Spanne (published by Peetre [21]) and Adams [1] have studied boundedness of the fractional integral operator $\overline{T}_\alpha$ on $L_{p,\lambda}(\mathbb{R}^n)$. This result has been reproved by Chiarenza and Frasca [3], and also studied in [12].

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Recall that the concept of the generalized Morrey space $M_{p,\varphi}\equiv M_{p,\varphi}(H_n)$ on Heisenberg group has been introduced in [11].

Definition 2 *[11] Let* $\varphi(g,r)$ *be a positive measurable function on* $H_n \times (0,\infty)$ *and* $1 \le p < \infty$. *We denote by* $M_{p,\varphi} \equiv M_{p,\varphi}(H_n)$ *the generalized Morrey space, the space of all functions* $f \in L_p^{loc}(H_n)$ *with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{g\in H_n, r>0} \varphi(g,r)^{-1} |B(g,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(g,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(H_n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{loc}(H_n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{g\in H_n, r>0} \varphi(g,r)^{-1} |B(g,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(g,r))} < \infty.$$

According to this definition, we recover the Morrey space $L_{p,\lambda}$ and weak Morrey space $WL_{p,\lambda}$ under the choice $\varphi(g,r) = r^{\frac{\lambda-Q}{p}}$:

$$L_{p,\lambda} = M_{p,\varphi}|_{\varphi(g,r)=r^{\frac{\lambda-Q}{p}}}, \qquad WL_{p,\lambda} = WM_{p,\varphi}|_{\varphi(g,r)=r^{\frac{\lambda-Q}{p}}}.$$

In [11], Guliyev et al. prove the Spanne type boundedness of Riesz potentials I_α, $\alpha \in (0,Q)$ from one generalized Morrey space $M_{p,\varphi_1}(H_n)$ to another $M_{q,\varphi_2}(H_n)$, where $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, Q is the homogeneous dimension of H_n and from the space $M_{1,\varphi_1}(H_n)$ to the weak space $WM_{1,\varphi_2}(H_n)$, where $1 < q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{Q}$. They also prove the Adams type boundedness of the Riesz potentials I_α, $\alpha \in (0,Q)$ from $M_{p,\varphi^{\frac{1}{p}}}(H_n)$ to another $M_{q,\varphi^{\frac{1}{q}}}(H_n)$ for $1 < p < q < \infty$ and from the space $M_{1,\varphi}(H_n)$ to the weak space $WM_{1,\varphi^{\frac{1}{q}}}(H_n)$ for $1 < q < \infty$.

For a locally integrable function b on H_n, suppose that the commutator operator $T_{b,\alpha}$, $\alpha \in (0,Q)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(H_n)$ with compact support and $x \notin suppf$

$$|T_{b,\alpha} f(g)| \le c_0 \int_{H_n} |b(g) - b(h)| \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh, \tag{1.1}$$

where c_0 is independent of f and g.

The condition (1.1) is satisfied by many interesting operators in harmonic analysis, such as fractional maximal operator, fractional Marcinkiewicz operator, fractional integral operator and so on (see [17], [22] for details).

Let T be a linear operator. For a locally integrable function b on H_n, we define the commutator $[b,T]$ by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for any suitable function f.

Let b be a locally integrable function on H_n, then for $0<\alpha<Q$, we define the linear commutator generated by fractional integral operator and b and the sublinear commutator of the fractional maximal operator as follows, respectively (see also [17]).

$$[b,\overline{T}_{\alpha}]f(g)\equiv b(g)\overline{T}_{\alpha}f(g)-\overline{T}_{\alpha}(bf)(g)=\int_{H_n}[b(g)-b(h)]\frac{f(h)}{|g^{-1}h|^{Q-\alpha}}dh,$$

$$M_{b,\alpha}(f)(g)=\sup_{r>0}|B(g,r)|^{-1+\frac{\alpha}{Q}}\int_{B(g,r)}|b(g)-b(h)||f(h)|dh.$$

Now, we will examine some properties related to the space of functions of Bounded Mean Oscillation, BMO, introduced by John and Nirenberg [14] in 1961. This space has become extremely important in various areas of analysis including harmonic analysis, PDEs and function theory. BMO-spaces are also of interest since, in the scale of Lebesgue spaces, they may be considered and appropriate substitute for L_{∞}. Appropriate in the sense that are spaces preserved by a wide class of important operators such as the Hardy-Littlewood maximal function, the Hilbert transform and which can be used as an end point in interpolating L_p spaces.

Let us recall the defination of the space of $BMO(H_n)$ (see, for example, [8, 17, 24]).

Definition 3 *Suppose that* $b\in L_1^{loc}(H_n)$, *let*

$$\|b\|_*=\sup_{g\in H_n,r>0}\frac{1}{|B(g,r)|}\int_{B(g,r)}|b(h)-b_{B(g,r)}|dh<\infty, \tag{1.2}$$

where

$$b_{B(g,r)}=\frac{1}{|B(g,r)|}\int_{B(g,r)}b(h)dh.$$

Define

$$BMO(H_n)=\{b\in L_1^{loc}(H_n):\|b\|_*<\infty\}.$$

Endowed with the norm given in (1.2), $BMO(H_n)$ becomes Banach space provided we identify functions which differ a.e. by constant; clearly, $\|b\|_*=0$ for $b(h)=c$ a.e. in H_n.

Remark 1 *Note that* $L_{\infty}(H_n)$ *is contained in* $BMO(H_n)$ *and we have*

$$\|b\|_*\le 2\|b\|_{\infty}.$$

Moreover, BMO contains unbounded functions, in fact the function $\log|h|$ on H_n, is in BMO but it is not bounded, so $L_{\infty}(H_n)\subset BMO(H_n)$.

Remark 2 (1) *The John-Nirenberg inequality [14]: there are constants* C_1, $C_2>0$, *such that for all* $b\in BMO(H_n)$ *and* $\beta>0$

$$|\{g\in B:|b(g)-b_B|>\beta\}|\le C_1|B|e^{-C_2\beta/\|b\|_*},\quad \forall B\subset H_n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{g\in H_n, r>0}\left(\frac{1}{|B(g,r)|}\int_{B(g,r)} |b(h)-b_{B(g,r)}|^p\, dh\right)^{\frac{1}{p}} \tag{1.3}$$

for $1<p<\infty$.

(3) Let $b\in BMO(H_n)$. Then there is a constant $C>0$ such that

$$\left|b_{B(g,r)}-b_{B(g,\tau)}\right| \le C\|b\|_* \ln\frac{\tau}{r} \text{ for } 0<2r<\tau, \tag{1.4}$$

where C is independent of b, g, r and τ.

Inspired by [11], in this paper, provided that $b\in BMO(H_n)$ and $T_{b,\alpha}$, $\alpha\in(0,Q)$ satisfying condition (1.1) is a sublinear operator, we find the sufficient conditions on the pair (φ_1,φ_2) which ensures the Spanne type boundedness of the commutator operators $T_{b,\alpha}$ from $M_{p,\varphi_1}(H_n)$ to $M_{q,\varphi_2}(H_n)$, where $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$. We also find the sufficient conditions on φ which ensures the Adams type boundedness of the commutator operators $T_{b,\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(H_n)$ to another $M_{q,\varphi^{\frac{1}{q}}}(H_n)$ for $1<p<q<\infty$. In all the cases the conditions for the boundedness of $T_{b,\alpha}$ are given in terms of Zygmund-type (supremal-type) integral inequalities on (φ_1,φ_2) and φ which do not assume any assumption on monotonicity of φ_1,φ_2 and φ in r. Our main results can be formulated as follows.

Theorem 2 *(Spanne type result) Let* $1<p<\infty$, $0<\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$ *and* $b\in BMO(H_n)$. *Let* $T_{b,\alpha}$ *be a sublinear operator satisfying condition (1.1) and bounded from* $L_p(H_n)$ *to* $L_q(H_n)$. *Let also, the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^\infty\left(1+\ln\frac{\tau}{r}\right)\frac{\operatorname*{essinf}_{\tau<s<\infty}\varphi_1(g,s)s^{\frac{Q}{p}}}{\tau^{\frac{Q}{q}+1}}dt \le C\varphi_2(g,r), \tag{1.5}$$

Then, the operator $T_{b,\alpha}$ is bounded from $M_{p,\varphi_1}(H_n)$ to $M_{q,\varphi_2}(H_n)$. Moreover

$$\|T_{b,\alpha}f\|_{M_{q,\varphi_2}} \lesssim \|b\|_*\|f\|_{M_{p,\varphi_1}}$$

From Theorem 2 we get the following new result.

Corollary 1 *Let* $1<p<\infty$, $0<\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $b\in BMO(H_n)$ *and the pair* (φ_1,φ_2) *satisfies condition (1.5). Then, the operators* $M_{b,\alpha}$ *and* $[b,\overline{T}_\alpha]$ *are bounded from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

Theorem 3 *(Adams type result) Let* $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$, $b\in BMO(H_n)$ *and let* $\varphi(g,\tau)$ *satisfies the conditions*

$$\sup_{r<\tau<\infty}\tau^{-\frac{Q}{p}}\left(1+\ln\frac{\tau}{r}\right)^{p}\operatorname*{essinf}_{t<s<\infty}\varphi(g,s)s^{\frac{Q}{p}}\leq C\varphi(g,r), \tag{1.6}$$

and

$$\int_r^\infty\left(1+\ln\frac{\tau}{r}\right)\tau^{\alpha}\varphi(g,\tau)^{\frac{1}{p}}\frac{d\tau}{\tau}\leq Cr^{-\frac{\alpha p}{q-p}}, \tag{1.7}$$

where C does not depend on $g\in H_n$ and $r>0$. Let also $T_{b,\alpha}$ be a sublinear operator satisfying condition (1.1) and the condition

$$\left|T_{b,\alpha}\left(f\chi_{B(g,r)}\right)(g)\right|\lesssim r^{\alpha}M_b f(g) \tag{1.8}$$

holds for any ball $B(g,r)$.

Then the operator $T_{b,\alpha}$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}(H_n)$ to $M_{q,\varphi^{\frac{1}{q}}}(H_n)$. Moreover, we have

$$\left\|T_{b,\alpha}f\right\|_{M_{q,\varphi^{\frac{1}{q}}}}\leq\|b\|_*\|f\|_{M_{p,\varphi^{\frac{1}{p}}}}.$$

From Theorem 3, we get the following new result.

Corollary 2 *Let* $1<p<\infty$, $0<\alpha<\frac{Q}{p}$, $p<q$, $b\in BMO(H_n)$ *and let also* $\varphi(x,\tau)$ *satisfies conditions (1.6) and (1.7). Then the operators* $M_{b,\alpha}$ *and* $[b,\overline{T}_\alpha]$ *are bounded from* $M_{p,\varphi^{\frac{1}{p}}}(H_n)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(H_n)$.

At last, throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence. By $A\lesssim B$ we mean that $A\leq CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2. Some Lemmas

To prove the main results (Theorems 2 and 3), we need the following lemmas. Firstly, for the proof of Spanne type results, we need following Lemma 1.

Lemma 1 *(Our main lemma) Let* $1<p<\infty$, $0<\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $b\in BMO(H_n)$, *and* $T_{b,\alpha}$ *is a sublinear operator satisfying condition (1.1) and bounded from* $L_p(H_n)$ *to* $L_q(H_n)$. *Then, the inequality*

$$\left\|T_{b,\alpha}f\right\|_{L_q(B(g,r))}\lesssim\|b\|_*r^{\frac{Q}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{\tau}{r}\right)\tau^{-\frac{Q}{q}-1}\|f\|_{L_p(B(g,\tau))}d\tau \tag{2.1}$$

holds for any ball $B(g,r)$ and for all $f\in L_p^{loc}(H_n)$.

Proof. Let $1<p<\infty$, $0<\alpha<\frac{n}{p}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. For an arbitrary ball $B=B(g,r)$ we set $f=f_1+f_2$, where $f_1=f\chi_{2B}$, $f_2=f\chi_{(2B)^C}$ and $2B=B(g,2r)$. Then we have

$$\left\|T_{b,\alpha}f\right\|_{L_q(B)}\leq\left\|T_{b,\alpha}f_1\right\|_{L_q(B)}+\left\|T_{b,\alpha}f_2\right\|_{L_q(B)}.$$

From the boundedness of $T_{b,\alpha}$ from $L_p(H_n)$ to $L_q(H_n)$ (see, for example, [8, 24]) it follows that:

$$\begin{aligned}\left\|T_{b,\alpha}f_1\right\|_{L_q(B)}&\leq\left\|T_{b,\alpha}f_1\right\|_{L_q(H_n)}\\&\lesssim\|b\|_*\|f_1\|_{L_p(H_n)}=\|b\|_*\|f\|_{L_p(2B)}.\end{aligned}$$

It is known that $g\in B$, $h\in(2B)^C$, which implies $\frac{1}{2}\left|h^{-1}w\right|\leq\left|g^{-1}h\right|\leq\frac{3}{2}\left|h^{-1}w\right|$. Then for $g\in B$, we have

$$\left|T_{b,\alpha}f_2(g)\right|\lesssim\int_{(2B)^C}\left|b(h)-b(g)\right|\frac{|f(h)|}{\left|g^{-1}h\right|^{Q-\alpha}}dh$$

Hence we get

$$\begin{aligned}\left\|T_{b,\alpha}f_2\right\|_{L_q(B)}&\lesssim\left(\int_B\left(\int_{(2B)^C}\left|b(h)-b(g)\right|\frac{|f(y)|}{\left|g^{-1}h\right|^{Q-\alpha}}dh\right)^q dg\right)\\&\lesssim\left(\int_B\left(\int_{(2B)^C}\left|b(h)-b(g)\right|\frac{|f(h)|}{\left|g^{-1}h\right|^{Q-\alpha}}dh\right)^q dg\right)^{\frac{1}{q}}\\&+\left(\int_B\left(\int_{(2B)^C}\left|b(h)-b(g)\right|\frac{|f(y)|}{\left|g^{-1}h\right|^{Q-\alpha}}dh\right)^q dg\right)^{\frac{1}{q}}\\&=J_1+J_2.\end{aligned}$$

We have the following estimation of J_1. When $\frac{1}{\mu}+\frac{1}{p}=1$, by the Fubini's theorem

$$J_1 \approx r^{\frac{Q}{q}} \int_{(2B)^C} |b(h)-b_B| \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh$$

$$\approx r^{\frac{Q}{q}} \int_{(2B)^C} |b(h)-b_B| |f(h)| \int_{|g^{-1}h|}^{\infty} \frac{d\tau}{\tau^{Q+1-\alpha}} dh$$

$$\approx r^{\frac{Q}{q}} \int_{2r}^{\infty} \int_{2r\le |g^{-1}h| \le \tau} |b(h)-b_B| |f(h)| dh \frac{d\tau}{\tau^{Q+1-\alpha}}$$

$$\lesssim \square r^{\frac{Q}{q}} \int_{2r}^{\infty} \int_{B(g,\tau)} |b(h)-b_B| |f(h)| dh \frac{d\tau}{\tau^{Q+1-\alpha}}$$

is valid. Applying the Hölder's inequality and by (1.3), (1.4), we get

$$J_1 \lesssim r^{\frac{Q}{q}} \int_{2r}^{\infty} \int_{B(g,\tau)} |b(h)-b_{B(g,\tau)}| |f(h)| dh \frac{d\tau}{\tau^{Q+1-\alpha}}$$

$$+ r^{\frac{Q}{q}} \int_{2r}^{\infty} |b_{B(g,r)}-b_{B(g,\tau)}| \int_{B(g,\tau)} |f(h)| dh \frac{d\tau}{\tau^{Q+1-\alpha}}$$

$$\lesssim \square r^{\frac{Q}{q}} \int_{2r}^{\infty} \left\|\left(b(\cdot)-b_{B(g,\tau)}\right)\right\|_{L_\mu(B(g,\tau))} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{Q+1-\alpha}}$$

$$+ r^{\frac{Q}{q}} \int_{2r}^{\infty} |b_{B(g,r)}-b_{B(g,\tau)}| \|f\|_{L_p(B(g,\tau))} |B(g,\tau)|^{1-\frac{1}{p}} \frac{d\tau}{\tau^{Q+1-\alpha}}$$

$$\lesssim \|b\|_* r^{\frac{Q}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{\tau}{r}\right) \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}.$$

In order to estimate J_2 note that

$$J_2 = \left\|\left(b(\cdot)-b_{B(g,\tau)}\right)\right\|_{L_q(B(g,\tau))} \int_{(2B)^C} \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh.$$

By (1.3), we get

$$J_2 \lesssim \|b\|_* r^{\frac{Q}{q}} \int_{(2B)^C} \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh.$$

On the other hand, by the Fubini's theorem, we have

$$\int_{(2B)^C} \frac{|f(w)|}{|g^{-1}w|^{Q-\alpha}} dw \approx \int_{(2B)^C} |f(w)| \int_{|g^{-1}w|}^{\infty} \frac{d\tau}{\tau^{Q+1-\alpha}} dw$$

$$\approx \int_{2r}^{\infty} \int_{2r\le |g^{-1}w| \le \tau} |f(w)| dw \frac{d\tau}{\tau^{Q+1-\alpha}}$$

$$\lesssim \int_{2r}^{\infty} \int_{B(g,\tau)} |f(w)| dw \frac{d\tau}{\tau^{Q+1-\alpha}}.$$

Applying the Hölder's inequality, we get

$$\int_{(2B)^C} \frac{|f(w)|}{|g^{-1}w|^{Q-\alpha}} dw$$

$$\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}. \tag{2.2}$$

Thus, by (2.2)

$$J_2 \lesssim \|b\|_* r^{\frac{Q}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}.$$

Summing up J_1 and J_2, for all $p \in (1,\infty)$ we get

$$\|T_{b,\alpha} f_2\|_{L_q(B)} \lesssim \|b\|_* r^{\frac{Q}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{\tau}{r}\right) \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}. \tag{2.3}$$

Finally, we have the following

$$\|T_{b,\alpha} f\|_{L_q(B)} \lesssim \|b\|_* \|f\|_{L_p(2B)} + \|b\| r^{\frac{Q}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{\tau}{r}\right) \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}.$$

On the other hand, we have

$$\|f\|_{L_p(2B)} \approx r^{\frac{Q}{q}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}$$

$$\leq r^{\frac{Q}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(g,\tau))} \frac{d\tau}{\tau^{\frac{Q}{q}+1}}, \tag{2.4}$$

which completes the proof of Lemma 1 by (2.4).

Secondly, for the proof of Adams type results, we need some lemmas and theorems about the estimates of sublinear commutator of fractional maximal operator in generalized Morrey spaces on Heisenberg groups.

Lemma 2 *Let* $1 < p < \infty$, $0 \leq \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in BMO(H_n)$. *Then the inequality*

$$\|M_{b,\alpha} f\|_{L_q(B(g,r))} \lesssim \|b\|_* r^{\frac{Q}{q}} \sup_{\tau > 2r} \left(1+\ln\frac{\tau}{r}\right) \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(g,\tau))}$$

holds for any ball $B(g,r)$ and for all $f \in L_p^{loc}(H_n)$.

Proof. Let $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$. For an arbitrary ball $B=B(g,r)$ we set $f=f_1+f_2$, where $f_1=f\chi_{2B}$, $f_2=f\chi_{(2B)^C}$ and $2B=B(g,2r)$. Hence,

$$\|M_{b,\alpha}f\|_{L_q(B)}\le\|M_{b,\alpha}f_1\|_{L_q(B)}+\|M_{b,\alpha}f_2\|_{L_q(B)}.$$

From the boundedness of $M_{b,\alpha}$ from $L_p(H_n)$ to $L_q(H_n)$ (see, for example, [2, 8, 24]) it follows that:

$$\|M_{b,\alpha}f_1\|_{L_q(B)}\le\|M_{b,\alpha}f_1\|_{L_q(H_n)}$$
$$\lesssim\|b\|_*\|f_1\|_{L_p(H_n)}=\|b\|_*\|f\|_{L_p(2B)}.$$

Let h be an arbitrary point in B. If $B(h,\tau)\cap(2B)^C\ne\varnothing$, then $\tau>r$. Indeed, if $w\in B(h,\tau)\cap(2B)^C$, then $\tau>|h^{-1}w|\ge|g^{-1}w|-|g^{-1}h|>2r-r=r$. On the other hand, $B(h,\tau)\cap(2B)^C\subset B(g,2\tau)$. Indeed, for $w\in B(h,\tau)\cap(2B)^C$ we have $|g^{-1}w|\le|h^{-1}w|+|g^{-1}h|<\tau+r<2\tau$. Hence,

$$M_{b,\alpha}f_2(h)=\sup_{\tau>0}\frac{1}{|B(h,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(h,\tau)\cap(2B)^C}|b(w)-b(h)||f(w)|dw$$
$$\le2^{Q-\alpha}\sup_{\tau>r}\frac{1}{|B(g,2\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,2\tau)}|b(w)-b(h)||f(w)|dw$$
$$=2^{Q-\alpha}\sup_{\tau>2r}\frac{1}{|B(g,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,\tau)}|b(w)-b(h)||f(w)|dw.$$

Therefore, for all $h\in B$ we have

$$M_{b,\alpha}f_2(h)\le2^{Q-\alpha}\sup_{\tau>2r}\frac{1}{|B(g,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,\tau)}|b(w)-b(h)||f(w)|dw. \tag{2.5}$$

Then

$$\|M_{b,\alpha}f_2\|_{L_q(B)}\lesssim\left(\int_B\left(\sup_{\tau>2r}\frac{1}{|B(g,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,\tau)}|b(w)-b(h)||f(w)|dw\right)^q dg\right)^{\frac{1}{q}}$$
$$\le\left(\int_B\left(\sup_{\tau>2r}\frac{1}{|B(g,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,\tau)}|b(w)-b_B||f(w)|dw\right)^q dg\right)^{\frac{1}{q}}$$

$$+\left(\int_B\left(\sup_{\tau>2r}\frac{1}{|B(g,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,\tau)}|b(h)-b_B||f(w)|dw\right)^q dg\right)^{\frac{1}{q}}$$

$$=J_1+J_2.$$

Let us estimate J_1.

$$J_1=r^{\frac{Q}{q}}\sup_{\tau>2r}\frac{1}{|B(g,\tau)|^{1-\frac{\alpha}{Q}}}\int_{B(g,\tau)}|b(w)-b_B||f(w)|dw$$

$$\approx r^{\frac{Q}{q}}\sup_{\tau>2r}\tau^{\alpha-Q}\int_{B(g,\tau)}|b(w)-b_B||f(w)|dw.$$

Applying the Hölder's inequality, by (1.3), (1.4) and $\frac{1}{\mu}+\frac{1}{p}=1$ we get

$$J_1 \lesssim r^{\frac{Q}{q}}\sup_{\tau>2r}\tau^{\alpha-Q}\int_{B(g,\tau)}|b(w)-b_{B(g,\tau)}||f(w)|dw$$

$$+r^{\frac{Q}{q}}\sup_{\tau>2r}\tau^{\alpha-Q}|b_{B(g,r)}-b_{B(g,\tau)}|\int_{B(g,\tau)}|f(w)|dw$$

$$\lesssim r^{\frac{Q}{q}}\sup_{\tau>2r}\tau^{\alpha-\frac{Q}{p}}\|(b(\cdot)-b_{B(g,\tau)})\|_{L_\mu(B(g,\tau))}\|f\|_{L_p(B(g,\tau))}$$

$$+r^{\frac{Q}{q}}\sup_{\tau>2r}t^{\alpha-Q}|b_{B(g,r)}-b_{B(g,\tau)}|\|f\|_{L_p(B(g,\tau))}|B(g,\tau)|^{1-\frac{1}{p}}$$

$$\lesssim\|b\|_*r^{\frac{Q}{q}}\sup_{\tau>2r}\left(1+\ln\frac{\tau}{r}\right)t^{-\frac{Q}{q}}\|f\|_{L_p(B(g,\tau))}.$$

In order to estimate J_2 note that

$$J_2=\|(b(\cdot)-b_{B(g,\tau)})\|_{L_q(B(g,\tau))}\sup_{\tau>2r}t^{\alpha-Q}\int_{B(g,\tau)}|f(w)|dw.$$

By (1.3), we get

$$J_2 \lesssim\|b\|_*r^{\frac{Q}{q}}\sup_{\tau>2r}t^{\alpha-Q}\int_{B(g,\tau)}|f(w)|dw.$$

Thus, by (2.2)

$$J_2 \lesssim\|b\|_*r^{\frac{Q}{q}}\sup_{\tau>2r}t^{-\frac{Q}{q}}\|f\|_{L_p(B(g,\tau))}.$$

Summing up J_1 and J_2, for all $p\in(1,\infty)$ we get

$$\|M_{b,\alpha}f_2\|_{L_q(B)}\lesssim\|b\|_*r^{\frac{Q}{q}}\sup_{\tau>2r}t^{-\frac{Q}{q}}\left(1+\ln\frac{\tau}{r}\right)\|f\|_{L_p(B(g,\tau))}. \tag{2.6}$$

Finally, we have the following

$$\left\|M_{b,\alpha} f_2\right\|_{L_q(B)} \lesssim \|b\|_* \|f\|_{L_p(2B)} + \|b\|_* r^{\frac{Q}{q}} \sup_{\tau>2r} t^{-\frac{Q}{q}} \left(1+\ln\frac{\tau}{r}\right) \|f\|_{L_p(B(g,\tau))}$$

$$\lesssim \|b\|_* r^{\frac{Q}{q}} \sup_{\tau>2r} t^{-\frac{Q}{q}} \left(1+\ln\frac{\tau}{r}\right) \|f\|_{L_p(B(g,\tau))},$$

which completes the proof.

Similarly to Lemma 2 the following lemma can also be proved.

Lemma 3 *Let* $1<p<\infty$, $b \in BMO(H_n)$ *and* M_b *is bounded on* $L_p(H_n)$. *Then the inequality*

$$\|M_b f\|_{L_p(B(g,r))} \lesssim \|b\|_* r^{\frac{Q}{q}} \sup_{\tau>2r} \left(1+\ln\frac{\tau}{r}\right) \tau^{-\frac{Q}{p}} \|f\|_{L_p(B(g,\tau))}$$

holds for any ball $B(g,r)$ and for all $f \in L_p^{loc}(H_n)$.

The following theorem is true.

Theorem 4 *Let* $1<p<\infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $b \in BMO(H_n)$ *and let* (φ_1,φ_2) *satisfies the condition*

$$\sup_{r<t<\infty} t^{\alpha-\frac{Q}{p}} \left(1+\ln\frac{t}{r}\right) \operatorname*{essinf}_{t<\tau<\infty} \varphi_1(g,\tau)\tau^{\frac{Q}{p}} \le C\varphi_2(g,r),$$

where C does not depend on g and r. Then the operator $M_{b,\alpha}$ is bounded from $M_{p,\varphi_1}(H_n)$ to $M_{q,\varphi_2}(H_n)$. Moreover

$$\|M_{b,\alpha} f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}}.$$

Proof. By Theorem 4.1 in [11] and Lemma 2, we get

$$\|M_{b,\alpha} f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \sup_{g\in H_n, r>0} \varphi_2(g,r)^{-1} \sup_{\tau>r} \left(1+\ln\frac{\tau}{r}\right) \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(g,\tau))}$$

$$\lesssim \|b\|_* \sup_{g\in H_n, r>0} \varphi_1(g,r)^{-1} r^{-\frac{Q}{p}} \|f\|_{L_p(B(g,r))} = \|b\|_* \|f\|_{M_{p,\varphi_1}}.$$

In the case of $\alpha=0$ and $p=q$, we get the following corollary by Theorem 4.

Corollary 3 *Let* $1<p<\infty$, $b \in BMO(H_n)$ *and* (φ_1,φ_2) *satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \left(1+\ln\frac{t}{r}\right) \operatorname*{essinf}_{t<\tau<\infty} \varphi_1(g,\tau)\tau^{\frac{Q}{p}} \le C\varphi_2(g,r),$$

where C does not depend on g and r. Then the operator M_b is bounded from $M_{p,\varphi_1}(H_n)$ to $M_{p,\varphi_2}(H_n)$. Moreover

$$\|M_b f\|_{M_{p,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}}.$$

3. Proofs of the main results

3.1. Proof of Theorem 2.

Proof. To prove Theorem 2, we will use the following relationship between essential supremum and essential infimum

$$\left(\operatorname*{essinf}_{x\in E} f(x)\right)^{-1} = \operatorname*{essup}_{x\in E} \frac{1}{f(x)}, \tag{3.1}$$

where f is any real-valued nonnegative function and measurable on E (see [25], page 143). Indeed, since $f \in M_{p,\varphi_1}$, by (3.1) and the non-decreasing, with respect to τ, of the norm $\|f\|_{L_p(B(g,\tau))}$, we get

$$\begin{aligned}
&\frac{\|f\|_{L_p(B(g,\tau))}}{\operatorname*{essinf}_{0<\tau<s<\infty} \varphi_1(g,s) s^{\frac{Q}{p}}} \le \operatorname*{essup}_{0<\tau<s<\infty} \frac{\|f\|_{L_p(B(g,\tau))}}{\varphi_1(g,s) s^{\frac{Q}{p}}} \\
&\le \operatorname*{essup}_{0<s<\infty} \frac{\|f\|_{L_p(B(g,s))}}{\varphi_1(g,s) s^{\frac{Q}{p}}} \le \|f\|_{M_{p,\varphi_1}}.
\end{aligned} \tag{3.2}$$

For $1<p<\infty$, since (φ_1,φ_2) satisfies (1.5) and by (3.2), we have

$$\begin{aligned}
&\int_r^\infty \left(1+\ln\frac{\tau}{r}\right)\|f\|_{L_p(B(g,\tau))} \tau^{-\frac{Q}{q}} \frac{d\tau}{\tau} \\
&\le \int_r^\infty \left(1+\ln\frac{\tau}{r}\right) \frac{\|f\|_{L_p(B(g,\tau))}}{\operatorname*{essinf}_{\tau<s<\infty} \varphi_1(g,s) s^{\frac{Q}{p}}} \frac{\operatorname*{essinf}_{\tau<s<\infty} \varphi_1(g,s) s^{\frac{Q}{p}}}{\tau^{\frac{Q}{q}}} \frac{d\tau}{\tau} \\
&\le C\|f\|_{M_{p,\varphi_1}} \int_r^\infty \left(1+\ln\frac{\tau}{r}\right) \frac{\operatorname*{essinf}_{\tau<s<\infty} \varphi_1(g,s) s^{\frac{Q}{p}}}{\tau^{\frac{Q}{q}}} \frac{d\tau}{\tau} \\
&\le C\|f\|_{M_{p,\varphi_1}} \varphi_2(g,r).
\end{aligned} \tag{3.3}$$

Then by (2.1) and (3.3), we get

$$\begin{aligned}
&\|T_{b,\alpha} f\|_{M_{q,\varphi_2}} = \sup_{g\in H_n, r>0} \varphi_2(g,r)^{-1} |B(g,r)|^{-\frac{1}{q}} \|T_{b,\alpha} f\|_{L_q(B(g,r))} \\
&\lesssim \|b\|_* \sup_{g\in H_n, r>0} \varphi_2(g,r)^{-1} \int_r^\infty \left(1+\ln\frac{\tau}{r}\right)\|f\|_{L_p(B(g,\tau))} \tau^{-\frac{Q}{q}} \frac{d\tau}{\tau} \\
&\lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}}.
\end{aligned}$$

This completes the proof of Theorem 2.

3.2. Proof of Theorem 3.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{Q}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $p < q$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$. For an arbitrary ball $B = B(g,r)$ we set $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f\chi_{(2B)^C}$ and $2B = B(g,2r)$. Then we have

$$\|T_{b,\alpha} f\|_{L_q(B)} \leq \|T_{b,\alpha} f_1\|_{L_q(B)} + \|T_{b,\alpha} f_2\|_{L_q(B)}.$$

For $g \in B$ we have

$$|T_{b,\alpha} f_2(g)| \lesssim \int_{(2B)^C} |b(h) - b(g)| \frac{|f(h)|}{|g^{-1}h|^{Q-\alpha}} dh.$$

Analogously to Section 2, for all $p \in (1,\infty)$ and $g \in B$ we get

$$|T_{b,\alpha} f_2(x)| \lesssim \|b\|_* \int_r^\infty \left(1 + \ln\frac{\tau}{r}\right) \tau^{\alpha - \frac{Q}{p} - 1} \|f\|_{L_p(B(g,\tau))} d\tau. \tag{3.4}$$

Then from conditions (1.7), (1.8) and inequality (3.4) we get

$$\begin{aligned} |T_{b,\alpha} f(g)| &\lesssim \|b\|_* r^\alpha M_b f(g) + \|b\|_* \int_r^\infty \left(1 + \ln\frac{\tau}{r}\right) \tau^{\alpha - \frac{Q}{p} - 1} \|f\|_{L_p(B(g,\tau))} d\tau \\ &\lesssim \|b\|_* r^\alpha M_b f(g) + \|b\|_* \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \int_r^\infty \left(1 + \ln\frac{\tau}{r}\right) \tau^\alpha \varphi(g,\tau)^{\frac{1}{p}} \frac{d\tau}{\tau} \\ &\lesssim \|b\|_* r^\alpha M_b f(g) + \|b\|_* r^{-\frac{\alpha p}{q-p}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned} \tag{3.5}$$

Hence choosing $r = \left(\frac{\|f\|_{M_{p,\varphi^{\frac{1}{p}}}}}{M_b f(g)} \right)^{\frac{q-p}{\alpha q}}$ for every $g \in H_n$, we have

$$|T_{b,\alpha} f(g)| \lesssim \|b\|_* (M_b f(g))^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1 - \frac{p}{q}}.$$

Consequently the statement of the theorem follows in view of the boundedness of the commutator of the maximal operator M_b in $M_{p,\varphi^{\frac{1}{p}}}(H_n)$ provided by Corollary 3 in virtue of condition (1.6).

Therefore, we have

$$\|T_{b,\alpha} f\|_{M_{q,\varphi^{\frac{1}{q}}}} = \sup_{g\in H_n,\tau>0} \varphi(g,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|T_{b,\alpha} f\|_{L_q(B(g,\tau))}$$

$$\lesssim \|b\|_* \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{g\in H_n,\tau>0} \varphi(g,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|M_b f\|_{L_p(B(g,\tau))}^{\frac{p}{q}}$$

$$= \|b\|_* \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{g\in H_n,\tau>0} \varphi(g,\tau)^{-\frac{1}{p}} \tau^{-\frac{Q}{p}} \|M_b f\|_{L_p(B(g,\tau))} \right)^{\frac{p}{q}}$$

$$= \|b\|_* \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|M_b f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}}$$

$$\lesssim \|b\|_* \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}.$$

Remark 3 *In the case of* $\varphi(g,r) = r^{\lambda-Q}$, $0<\lambda<Q$ *from Theorem 3 we get the following Adams type result ([1]) for the commutators of fractional maximal and integral operators.*

Corollary 4 *Let* $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $0<\lambda<Q-\alpha p$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q-\lambda}$ *and* $b\in BMO(H_n)$. *Then, the operators* $M_{b,\alpha}$ *and* $[b,\overline{T}_\alpha]$ *are bounded from* $L_{p,\lambda}(H_n)$ *to* $L_{q,\lambda}(H_n)$.

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