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Euler matrix method for linear second-order partial differential equations with complicated conditions

Gamze Yuksel

Department of Mathematics, Faculty of Science, Mugla Sıtkı Koçman University, Mugla 48000, Turkey

Abstract The purpose of this study is to apply the Euler matrix method to linear second order partial differential equations (PDEs) under the most general conditions. Error analysis of the method is presented. By using the residual correction procedure, the absolute error may be estimated. The effectiveness of the method is illustrated in numerical examples. Numerical results are overlapped with the theoretical results.

Keywords Partial differential equations, Euler matrix method, Residual correction procedure

1. Introduction

There are some well-known numerical methods such as finite difference methods, finite element methods, polynomial approximate methods, spectral methods, Galerkin, and collocation methods to numerically solve PDEs [1-2]. However, recently various approximate methods are discussed in the literature such as the differential transform method, Legendre-wavelet method, Chebyshev-tau method and Adomian decomposition method [3-13]. In this paper, we have developed a matrix method which is based on Euler polynomials. The method was given by error estimation and error analysis.

Let Ω be a rectangular region, $\Omega = \{(x, y) : 0 \le x, y \le b \le \infty\}$ and $\partial\Omega$ is the boundary of Ω. In general form, for all $(x, y) \in \Omega$, linear partial differential equations with variable coefficients follow as,

$$P(x,y)\frac{\partial^2 u}{\partial x^2} + Q(x,y)\frac{\partial^2 u}{\partial x \partial y} + R(x,y)\frac{\partial^2 u}{\partial y^2} + S(x,y)\frac{\partial u}{\partial x} + T(x,y)\frac{\partial u}{\partial y} + V(x,y)u = G(x,y) \quad (1.1)$$

In this paper, we consider (1.1) with the following conditions in three complicated forms [12]:

Case 1: Conditions defined at the points $x = \alpha_k$ and $y = \beta_k$, where $\alpha_k, \beta_k \in \partial\Omega$,

$$\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j}^{k} u^{(i,j)}(\alpha_k, \beta_k) = \lambda_k \quad (1.2)$$

Case 2: Conditions defined at the points $y = \gamma_k$, where $\gamma_k \in \partial\Omega$,

$$\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1} b_{i,j}^{k}(x) u^{(i,j)}(x, \gamma_k) = g_k(x) \quad (1.3)$$

Case 3: Conditions defined at the points $x = \eta_k$, where $\eta_k \in \partial\Omega$,

$$\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}^{k}(y) u^{(i,j)}(\eta_k, y) = h_k(y) \quad (1.4)$$

Here $P(x,y), Q(x,y), R(x,y), S(x,y), T(x,y), V(x,y)$ and $G(x,y)$ are functions defined in Ω. $g_k(x)$ and $b_{i,j}^k(x)$ are defined in $0 \le x \le b \le \infty$. $h_k(y)$ and $c_{i,j}^k(y)$ are defined in $0 \le x \le b \le \infty$. $a_{i,j}^k$ and λ_k are constants, $t, p, m \in Z^+$. Also $u^{(0,0)}(x,y) = u(x,y)$ and $u^{(i,j)}(x,y) = \dfrac{\partial^{i+j} u(x,y)}{\partial x^i \partial y^j}$ where $i, j = 0,1,2.$

In this study, our purpose is to develop a new matrix method, which is based on the Euler polynomials for solving (1.1). Let consider the approximate solution of (1.1)

$$u(x,y) = \sum_{r=0}^{N} \sum_{s=0}^{N} a_{r,s} E_{r,s}(x,y) \tag{1.5}$$

where $E_{r,s}(x,y) = E_r(x)E_s(y)$ and $a_{r,s}$'s are unknown constants to be determined. Here, $E_r(x)$ and $E_s(y)$ denote the Euler polynomials of degree *r* and *s*, respectively defined by. We choose the collocation points as

$$x_n = y_l = \frac{b}{N}; \quad n = 0,1,\ldots,N, \quad l = 0,1,\ldots,N. \tag{1.6}$$

The rest of this paper is organized as follows. Some necessary definitions and theorems are given in Section 2. The method is constituted in Section 3 and Section 4. A comprehensive error analysis is given in Section 5. To verify the theoretical results given in Section 5, an example is presented in Section 6. In the last Section, a brief summary of the paper are given.

2. Preliminaries and Notations

2.1. Euler polynomials

Euler numbers and polynomials are very useful in classical analysis and numerical mathematics. In many respects, they are closely related to the theory of Bernoulli polynomials and numbers. Euler polynomials and numbers are summarized as follows [14-17]. The classical Euler polynomials $E_n(x)$ is usually defined by means of the following generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi$$

Euler numbers ε_n can be obtained by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \varepsilon_n \frac{t^n}{n!}$$

and the relation between Euler polynomials and Euler numbers is given

$$E_n(1/2) = 2^{-n} \varepsilon_n, \quad n = 0,1,2,\ldots$$

Euler polynomials can be obtained by the following formula recursively;

$$E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) = 2x^n, \quad n = 1,2,\ldots$$

The first four such Euler polynomials are

$$E_0(x) = 1, \; E_1(x) = x - \frac{1}{2}, \; E_2(x) = x^2 - x,$$

$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, E_4(x) = x^4 - 2x^3 + x, \ldots$$

and the first Euler numbers are

$$\varepsilon_0 = 1,\ \varepsilon_1 = 0,\ \varepsilon_2 = -1,\ \varepsilon_3 = 0,\ \varepsilon_4 = 5,\ \varepsilon_5 = 0,\ \varepsilon_6 = -61,\ \varepsilon_7 = 0, \ldots$$

Euler polynomials satisfy the following interesting properties

$$E_n'(x) = nE_{n-1}(x),\ \ n \geq 1$$

$$E_n(x+1) + E_n(x) = 2x^n,\ n \geq 1$$

3. Fundamental Relations

To find the numerical solution of PDEs with Euler matrix method, it is necessary to evaluate the unknown coefficients of the approximate solution. For convenience, the relation (1.5) can be written in the matrix form.

Lemma 3.1. [12] Let consider the approximate solution of (1.1) via by Euler polynomials such that,

$$u(x,y) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s} E_{r,s}(x,y)$$

can be written as a matrix form

$$u(x,y) = \boldsymbol{E}(x)\boldsymbol{F}(y)\overline{\mathbf{A}} \tag{3.1}$$

where

$$\boldsymbol{E} = \begin{bmatrix} E_0(x) & E_1(x) & \cdots & E_N(x) \end{bmatrix}_{1x(N+1)}$$

and

$$\boldsymbol{F} = \begin{bmatrix} E_0(y) & \cdots & E_N(y) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & E_0(y) & \cdots & E_N(y) & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & E_0(y) & \cdots & E_N(y) \end{bmatrix}_{(N+1)x(N+1)^2}$$

and $\overline{\mathbf{A}}$ is unknown coefficients matrix

$$\overline{\mathbf{A}} = \begin{bmatrix} a_{0,0} & a_{0,1} & \ldots & a_{0,N} & a_{1,0} & a_{1,1} & \ldots & a_{1,N} & \ldots & a_{N,0} & a_{N,1} & \ldots & a_{N,N} \end{bmatrix}^T_{(N+1)^2 x1}.$$

Proof: We can prove it easily from the matrix multiplication such as

$$\begin{aligned} u(x,y) = E(x)F(y)\overline{\mathbf{A}} = {} & a_{00}E_0(x)E_0(y) + a_{01}E_0(x)E_1(y) + \ldots + a_{0N}E_0(x)E_N(y) + \\ & a_{10}E_1(x)E_0(y) + a_{11}E_1(x)E_1(y) + \ldots + a_{1N}E_1(x)E_N(y) + \\ & \vdots \\ & a_{N0}E_N(x)E_0(y) + a_{N1}E_N(x)E_1(y) + \ldots + a_{NN}E_N(x)E_N(y) \\ = {} & \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s}E_r(x)E_s(y) \end{aligned}$$

To solve PDEs approximately, an explicit relationship between the solution function $u(x,y)$ and its partial derivatives essentially needed. In the next part we present these relations.

3.1. Matrix Relations for Euler polynomials

The matrix form for derivatives of (3.1),

$$u^{(i,j)}(x,y) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s} E_{r,s}^{(i,j)}(x,y)$$

can be written by Lemma 3.1 as

$$u^{(i,j)}(x,y) = \boldsymbol{E}^{(i)}(x)\boldsymbol{F}^{(j)}(y)\overline{\boldsymbol{A}}. \tag{3.2}$$

We present the following lemma to show the relation between the matrix $\boldsymbol{E}(x)$, $\boldsymbol{F}(y)$ and theirs derivatives.

Lemma 3.2 [12] Let $u(x,y)$ and its $(i+j)th$-ordered partial derivatives be denoted by (3.1) and (3.2), respectively. Then there is relation such as

$$u^{(i,j)}(x,y) = \boldsymbol{E}(x)\left(\boldsymbol{M}^{\mathbf{T}}\right)^{\mathbf{i}} \boldsymbol{F}(y)\left(\overline{\boldsymbol{M}}\right)^{\mathbf{j}} \overline{\mathbf{A}} \tag{3.3}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)x(N+1)} \begin{matrix} \xrightarrow{\text{N even}} \\ \xrightarrow{\text{N odd}} \end{matrix} \quad , \overline{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{M}^{\mathbf{T}} & 0 & \cdots & 0 \\ 0 & \boldsymbol{M}^{\mathbf{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{M}^{\mathbf{T}} \end{bmatrix}_{(N+1)^2 x(N+1)^2}$$

Proof: It is clearly seen that the relation between the matrix $\boldsymbol{E}(x)$ and its derivative $\boldsymbol{E}^{(1)}(x)$ is

$$\boldsymbol{E}^{(1)}(x) = \boldsymbol{E}(x)\boldsymbol{M}^{\mathbf{T}}$$

where $\boldsymbol{M}$ is a $(N+1)\times(N+1)$ matrix such that,

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)x(N+1)} \begin{matrix} \text{N even} \\ \xrightarrow{\text{N odd}} \end{matrix}$$

The second derivative of $\mathbf{E}(x)$ as follows,

$$\mathbf{E}^{(2)}(x) = \mathbf{E}^{(1)}(x)\mathbf{M}^{\mathbf{T}} = \mathbf{E}(x)(\mathbf{M}^{\mathbf{T}})^{\mathbf{2}}$$

If we take derivative iteratively, we get the following formula for $i\,th$ derivatives of $\mathbf{E}(x)$:

$$\mathbf{E}^{(i)}(x) = \mathbf{E}^{(i-1)}(x)(\mathbf{M}^{\mathbf{T}}) = \mathbf{E}(x)(\mathbf{M}^{\mathbf{T}})^{i}.$$

Similarly, we can obtain a formula for $j\,th$ derivatives of $\boldsymbol{F}(y)$ as follows,

$$\boldsymbol{F}^{(1)}(y) = \boldsymbol{F}(y)\overline{\boldsymbol{M}}$$
$$\boldsymbol{F}^{(2)}(y) = \boldsymbol{F}^{(1)}(y)\overline{\boldsymbol{M}} = \boldsymbol{F}(y)\left(\overline{\boldsymbol{M}}\right)^{2}$$
$$\vdots$$
$$\boldsymbol{F}^{(j)}(y) = \boldsymbol{F}^{(j-1)}(y)(\overline{\boldsymbol{M}}) = \boldsymbol{F}(y)(\overline{\boldsymbol{M}})^{j}$$

where $\overline{\boldsymbol{M}}$ is a $(N+1)^2 \times (N+1)^2$ block diagonal matrix such that

$$\overline{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{M}^{\mathbf{T}} & 0 & \cdots & 0 \\ 0 & \boldsymbol{M}^{\mathbf{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{M}^{\mathbf{T}} \end{bmatrix}_{(N+1)^2 x (N+1)^2}$$

Finally, substituting $\mathbf{E}^{(i)}(x)$ and $\mathbf{F}^{(j)}(y)$ in (3.2), we obtain the fundamental matrix equation

$$u^{(i,j)}(x,y) = \boldsymbol{E}^{(i)}(x)\boldsymbol{F}^{(j)}(y)\overline{\mathbf{A}} = \boldsymbol{E}(x)\left(\boldsymbol{M}^{\mathbf{T}}\right)^{\mathbf{i}}\boldsymbol{F}(y)\left(\overline{\boldsymbol{M}}\right)^{\mathbf{j}}\overline{\mathbf{A}}.$$

3.2 Matrix Forms of the Conditions

We investigate the conditions in three parts. From Lemma 3.1 and Lemma 3.2, the fundamental matrix relations associated with the condition in Case 1 becomes,

$$\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j}^{k} u^{(i,j)}(\alpha_k,\beta_k) = \left(\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j}^{k}\boldsymbol{E}(\alpha_k)(\boldsymbol{M}^{T})^{i}\boldsymbol{F}(\beta_k)(\overline{\boldsymbol{M}})^{j}\right)\overline{\boldsymbol{A}} = \lambda_k \;, \tag{3.4}$$

Similarly the fundamental matrix relations for Case 2 and Case 3 respectively,

$$\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1} b_{i,j}^{k}(x) u^{(i,j)}(x,\gamma_k) = \left(\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1} b_{i,j}^{k}(x)\boldsymbol{E}(x)(\boldsymbol{M}^{T})^{i}\boldsymbol{F}(\gamma_k)(\overline{\boldsymbol{M}})^{j}\right)\overline{\boldsymbol{A}} = g_k(x), \tag{3.5}$$

and

$$\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}^{k}(y) u^{(i,j)}(\eta_k,y) = \left(\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1} c_{i,j}^{k}(y)\boldsymbol{E}(\eta_k)(\boldsymbol{M}^{T})^{i}\boldsymbol{F}(y)(\overline{\boldsymbol{M}})^{j}\right)\overline{\boldsymbol{A}} = h_k(y). \tag{3.6}$$

Hence, we shall obtain the linear algebraic equations, substituting the collocation points into the fundamental equations in the following section.

4. Method of the Solution

Each term in (1.1) can be given in the matrix equation via (3.3)

$$\begin{aligned} &P(x,y)\boldsymbol{E}(x)(\boldsymbol{M}^{T})^{2}\boldsymbol{F}(y)\overline{\boldsymbol{A}} + Q(x,y)\boldsymbol{E}(x)\boldsymbol{M}^{T}\boldsymbol{F}(y)(\overline{\boldsymbol{M}})\overline{\boldsymbol{A}} + \\ &R(x,y)\boldsymbol{E}(x)\boldsymbol{F}(y)(\overline{\boldsymbol{M}})^{2}\overline{\boldsymbol{A}} + S(x,y)\boldsymbol{E}(x)(\boldsymbol{M}^{T})\boldsymbol{F}(y)\overline{\boldsymbol{A}} + \\ &T(x,y)\boldsymbol{E}(x)\boldsymbol{F}(y)(\overline{\boldsymbol{M}})\overline{\boldsymbol{A}} + V(x,y)\boldsymbol{E}(x)\boldsymbol{F}(y)\overline{\boldsymbol{A}} = \boldsymbol{G}(x,y) \end{aligned} \tag{4.1}$$

By substituting the collocation points (1.6) into (4.1), we obtain the linear algebraic equation

$$\begin{aligned} &P(x_n,y_l)\boldsymbol{E}(x_n)(\boldsymbol{M}^{T})^{2}\boldsymbol{F}(y_l)\overline{\boldsymbol{A}} + Q(x_n,y_l)\boldsymbol{E}(x_n)\boldsymbol{M}^{T}\boldsymbol{F}(y_l)(\overline{\boldsymbol{M}})\overline{\boldsymbol{A}} + \\ &R(x_n,y_l)\boldsymbol{E}(x_n)\boldsymbol{F}(y_l)(\overline{\boldsymbol{M}})^{2}\overline{\boldsymbol{A}} + S(x_n,y_l)\boldsymbol{E}(x_n)(\boldsymbol{M}^{T})\boldsymbol{F}(y_l)\overline{\boldsymbol{A}} + \\ &T(x_n,y_l)\boldsymbol{E}(x_n)\boldsymbol{F}(y_l)(\overline{\boldsymbol{M}})\overline{\boldsymbol{A}} + V(x_n,y_l)\boldsymbol{E}(x_n)\boldsymbol{F}(y_l)\overline{\boldsymbol{A}} = \boldsymbol{G}(x_n,y_l) \\ &n = 0,1,\ldots,N \;,\; l = 0,1,\ldots,N. \end{aligned} \tag{4.2}$$

The fundamental matrix equation of (4.2) follows as

$$\begin{aligned}&\boldsymbol{PE(M^T)^2F\overline{A}+QE(M^T)F(\overline{M})\overline{A}+REF(\overline{M})^2\overline{A}+}\\&\boldsymbol{SE(M^T)F\overline{A}+TEF(\overline{M})\overline{A}+VEF\overline{A}=G}\end{aligned} \tag{4.3}$$

Here, (4.3) corresponds to a system of $(N+1)^2$ linear algebraic equations with unknown Euler coefficients $a_{0,0}, a_{0,1}, \ldots, a_{0,N}, a_{1,0}, a_{1,1}, \ldots, a_{1,N}, \ldots, a_{N,0}, a_{N,1}, \ldots, a_{N,N}$.

Also, we can write (4.3) such that

$$\underbrace{\left\{\begin{aligned}&\boldsymbol{PE(M^T)^2F+QE(M^T)F(\overline{M})+REF(\overline{M})^2+}\\&\boldsymbol{SE(M^T)F+TEF(\overline{M})+VEF}\end{aligned}\right\}}_{\boldsymbol{W}}\boldsymbol{\overline{A}=G} \tag{4.4}$$

$$\boldsymbol{W\overline{A}=G} \tag{4.5}$$

Similarly, by substituting the collocation points (1.6) into (3.4), (3.5) and (3.6) for the complicated conditions, we obtain respectively,

$$\underbrace{\left(\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1}a_{i,j}^{k}\boldsymbol{E}(\alpha_k)(\boldsymbol{M}^T)^i\boldsymbol{F}(\beta_k)(\overline{\boldsymbol{M}})^j\right)}_{\boldsymbol{Z_{1,k}}}\boldsymbol{\overline{A}}=\boldsymbol{\lambda_k}$$

or shortly

$$\boldsymbol{Z_{1,k}\overline{A}=\lambda_k} \tag{4.6}$$

For Case 2,

$$\underbrace{\left(\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1}b_{i,j}^{k}(x_n)\boldsymbol{E}(x_n)(\boldsymbol{M}^T)^i\boldsymbol{F}(\gamma_k)(\overline{\boldsymbol{M}})^j\right)}_{\boldsymbol{Z_{2,k}}}\boldsymbol{\overline{A}}=g_k(x_n),\quad n=0,1,\ldots N$$

or shortly

$$\boldsymbol{Z_{2,k}\overline{A}=g_k} \tag{4.7}$$

For Case 3

$$\underbrace{\left(\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}^{k}(y_l)\boldsymbol{E}(\eta_k)(\boldsymbol{M}^T)^i\boldsymbol{F}(y_l)(\overline{\boldsymbol{M}})^j\right)}_{\boldsymbol{Z_{3,k}}}\boldsymbol{\overline{A}}=h_k(y_l),\ l=0,1,\ldots,N$$

or shortly

$$\boldsymbol{Z_{3,k}\overline{A}=h_k} \tag{4.8}$$

We notice that the conditions associated with PDEs may be given by either of them, or both of them, or all of them. Now, combining the (4.6), (4.7) and (4.8), we can show the matrix equations of conditions in a new matrix form

$$\underbrace{\begin{bmatrix}\boldsymbol{Z_{1,k}}\\\boldsymbol{Z_{2,k}}\\\boldsymbol{Z_{3,k}}\end{bmatrix}}_{\boldsymbol{Z}}\boldsymbol{\overline{A}}=\underbrace{\begin{bmatrix}\boldsymbol{\lambda_k}\\\boldsymbol{g_k}\\\boldsymbol{h_k}\end{bmatrix}}_{\boldsymbol{R}}$$

or shortly

$$\boldsymbol{Z\overline{A}=R}. \tag{4.9}$$

To obtain Euler polynomials solution of (1.1) under conditions (1.2), (1.3) and (1.4), it is formed the augmented matrix from (4.5) and (4.9) as follows,

$$\left[\tilde{W};\tilde{G}\right]=\begin{bmatrix} Z & ; & R \\ W & ; & G \end{bmatrix}.$$

Therefore, the unknown coefficients are obtained as

$$\overline{\mathbf{A}}=\left(\tilde{\tilde{\mathbf{W}}}\right)^{-1}\tilde{\tilde{\mathbf{G}}}$$

where $\left[\tilde{\tilde{\mathbf{W}}};\,\tilde{\tilde{\mathbf{G}}}\right]$ is generated by using the Gauss elimination method and then removing zero rows of augmented matrix $\left[\tilde{\mathbf{W}};\,\tilde{\mathbf{G}}\right]$. The reason of using the gauss elimination for this direct solution that beware of the non-invertible case of the matrix $\tilde{\tilde{\mathbf{W}}}$. When the conditions are added to linear algebraic system, some rows can be same because of the symmetry of collocation points. These terms can be eliminated by Gauss elimination method.

5. Error Analysis

Since the Euler polynomial (1.5) is an approximate solution of (1.1), this solution is substituted in (1.1). The resulting equation must be satisfied approximately, that is, for $x=x_r\in[0,1],\ \ y=y_s\in[0,1]$:

$$E_{N,N}(x_r,y_s)=$$
$$\left|P(x_r,y_s)u_{xx}(x_r,y_s)+Q(x_r,y_s)u_{xy}(x_r,y_s)+R(x_r,y_s)u_{yy}(x_r,y_s)+S(x_r,y_s)u_x(x_r,y_s)\right.$$
$$\left.+T(x_r,y_s)u_y(x_r,y_s)+V(x_r,y_s)u(x_r,y_s)-G(x_r,y_s)\right|\cong 0$$

and

$E_{N,N}(x_r,y_s)\le 10^{-k_i}$ (k_i positive integer).

If max $(10^{k_i})=10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,N}(x_r,y_s)$ at each of the points becomes smaller than the prescribed 10^{-k}. We can also calculate the maximum errors of the method as follows

$$e_{N,N}=\left\|u_{N,N}-u^*\right\|_\infty=\sup\left\{\left|u_{N,N}(x,y)-u^*(x,y)\right|,(x,y)\in\Omega\right\} \tag{5.1}$$

where, u^* is the exact solution of the problem and $u_{N,N}(x,y)$ is the computed results for N.

5.1 Residual correction procedure

First, adding and subtracting the term

$$R(x,y):=A(x,y)\frac{\partial^2u_{N,N}}{\partial x^2}+B(x,y)\frac{\partial^2u_{N,N}}{\partial x\partial y}+C(x,y)\frac{\partial^2u_{N,N}}{\partial y^2}+D(x,y)\frac{\partial u_{N,N}}{\partial x}+E(x,y)\frac{\partial u_{N,N}}{\partial y}+F(x,y)u_{N,N}$$

in (1.1) yields the following differential equation

$$A\frac{\partial^2e}{\partial x^2}+B\frac{\partial^2e}{\partial x\partial y}+C\frac{\partial^2e}{\partial y^2}+D\frac{\partial e}{\partial x}+E\frac{\partial e}{\partial y}+Fe=G-R \tag{5.2}$$

$$\sum_{k=1}^{t}\sum_{i=0}^{1}\sum_{j=0}^{1}a_{i,j}^{k}e^{(i,j)}(\alpha_k,\beta_k)=0 \tag{5.3}$$

$$\sum_{k=1}^{p}\sum_{i=0}^{1}\sum_{j=0}^{1}b_{i,j}^{k}(x)e^{(i,j)}(x,\gamma_k)=0 \tag{5.4}$$

$$\sum_{k=1}^{m}\sum_{i=0}^{1}\sum_{j=0}^{1}c_{i,j}^{k}(y)e^{(i,j)}(\eta_k,y)=0 \tag{5.5}$$

where $e=e_{N,N}:=f-u_{N,N}$. Let $e_{M,M}^{*}$ is solution of (5.2)-(5.5) via Euler matrix method. If $\left\|e_{N,N}-e_{M,M}^{*}\right\|\leq\varepsilon$ is sufficiently small, the absolute error can be estimated by $e_{M,M}^{*}$. Hence, measuring in any norm the error functions $e_{M,M}^{*}$ for different values N, gives the optimal N for the absolute error in that norm [13].

Corallary 5.1: If $u_{N,N}$ is Euler matrix method solution of (1.1), then $u_{N,N}+e_{M,M}^{*}$ is also an approximate solution for (1.1). Moreover, its error function is $e_{N,N}-e_{M,M}^{*}$.

Note that, if $\left\|e_{N,N}-e_{M,M}^{*}\right\|\leq\left\|f-u_{N,N}\right\|$, then the approximate solution $u_{N,N}+e_{M,M}^{*}$ gives better approximation than $u_{N,N}$ in the norm.

6. Numerical Examples:

Example 1. [18] Consider the second-order hyperbolic partial differential equation with variable coefficients on domain $\Omega=\{(x,y):0\leq x,y\leq 1\}$

$$\frac{\partial^2 u}{\partial y^2}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}=0$$

subject to the initial conditions

$$u(x,0)=x$$

$$u_y(x,0)=x^2$$

with the exact solution $u(x,t)=x+x^2\sinh y$. By taking N=5, 7 and 10 we solved the above problem by means of the fundamental matrix equation. In Table 1, we present supremum norms of the absolute errors for N=5, 7 and 10. Moreover, supremum norms of the estimated absolute errors with residual correction procedure are given in the same table. As seen from Table 1, the residual correction procedure estimates the absolute errors well. However, the absolute errors of present method are compared with Chebyshev method in Table 2.

Table 1: Comparison of the maximum errors $\left\|e_{N,N}\right\|_\infty$, estimated absolute errors $\left\|e_{M,M}^{*}\right\|_\infty$ and error functions $\left\|e_{N,N}-e_{M,M}^{*}\right\|_\infty$ for Example 1 (Digits:50)

Present Method	$N=5$ $M=7$	$N=7$ $M=7$	$N=10$ $M=12$
$\left\|e_{N,N}\right\|_\infty$	5.0 E-3	8.0 E-5	5.0 E-7
$\left\|e_{M,M}^{*}\right\|_\infty$	8.0 E-5	1.0 E-6	2.5 E-9
$\left\|e_{N,N}-e_{M,M}^{*}\right\|_\infty$	5.0 E-3	8.0 E-5	6.0 E-7

Table 2: Comparison of the maximum errors $\left\|e_{N,N}\right\|_\infty$ of present method with Chebyshev method for Example 1

Chebyshev Method [19]	$N=7$	$N=10$
$\|e_{N,N}\|_\infty$	6.0 E-5	6.16 E-7

Example 2. [13] Let consider the following elliptic equations with Dirichlet boundary conditions :

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}=2x^2y^2e^{xy}$$

$$u(x,0)=1,\ u(x,1)=e^x$$

$$u(0,y)=1,\ u(1,y)=e^y$$

where the exact solution of the problem is $u(x,y)=e^{xy}$. By taking $N=6,8$ and 10 the problem is solved with present method. In table 3, the absolute errors and the estimated absolute errors are compared with Berstein series solution [13]. As seen from the Table 3, for $N=6,8$ and 10, the present method gives better results than Berstein series solutions. However, $u_{N,N}+e^*_{M,M}$ is a more accurate solution than $u_{N,N}$ since $N=6$ and $N=8$ for the present method.

Table 3: Comparison of the maximum errors $\|e_{N,N}\|_\infty$, estimated absolute errors $\|e^*_{M,M}\|_\infty$ and error functions $\|e_{N,N}-e^*_{M,M}\|_\infty$ for Example 2 with Berstein series solution (Digits:40)

Present Method	$N=6$ $M=10$	$N=8$ $M=10$	$N=10$ $M=10$	Berstein Series Sol.	$N=6$ $M=10$	$N=8$ $M=10$	$N=10$ $M=10$
$\|e_{N,N}\|_\infty$	2.5 E-7	3.5 E-10	3.0 E-13	$\|e_{N,N}\|_\infty$	4.0 E-7	5.0E-10	2.0E-13
$\|e^*_{M,M}\|_\infty$	8.0 E-8	1.6 E-10	7.0 E-15	$\|e^*_{M,M}\|_\infty$	1.0 E-7	3.5E-10	1.8E-13
$\|e_{N,N}-e^*_{M,M}\|_\infty$	2.0 E-7	1.8 E-10	3.0 E-13	$\|e_{N,N}-e^*_{M,M}\|_\infty$	3.6.E-7	5.0E-10	2.5E-13

7. Conclusion

Euler polynomials and Euler numbers have widespread use in many areas in mathematics. In this study Euler Polynomials are used to provide a numerical solution for linear second order PDEs with complicated conditions. The error analysis of the method is presented and also the absolute error can be estimated by residual correction procedure even if f is unknown. One can obtain more accurate results with the aid of Corollary 5.1. Finally, as seen from the example, numerical results are consistent with the theoretical results.

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