Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2016, 3(6):188-199**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

2-D Problem of Generalized Thermoelastic Medium with Voids under the Effect of Gravity: Comparison of Different Theories

Mohamed IA Othman, Ezaira RM Edeeb

Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt

Abstract The purpose of this paper is to study the 2-D problem of generalized thermoelastic medium with voids under the effect of gravity within the framework of the Green-Lindsay, Lord–Shulman and classical coupled theories. The normal mode analysis is used in our problem. Numerical results with comparisons between theories are illustrated graphically. Comparisons are made between the three theories in the presence and absence of gravity and also with and without voids.

Keywords gravity; Lord–Shulman; Green-Lindsay; thermoelasticity; normal mode analysis, voids. UDC 537.6, 539.3

1. Introduction

The effect of mechanical and thermal distribution of an elastic body is studied within the framework of the theory of thermoelasticity. The generalized theory of thermoelasticity is one of the modified versions of classical uncoupled and coupled theory of thermoelasticity that has been developed in order to remove the paradox of physical impossible phenomena of infinite velocity of thermal signals in the classical coupled thermoelasticity theory stated by Biot [1]. The coupled CD theory of thermo-elasticity was extended by including the thermal relaxation times into the constitutive equation by Lord and Shulman [2] and Green and Lindsay [3]. These theories eliminate the paradox of an infinite velocity of the heat propagation and were termed the generalization theories of thermoelasticity, there are the following differences between the two theories:

i. The Lord-Shulman L-S theory involves one relaxation time of the thermo-elastic τ_0, while the Green and Lindsay G-L theory takes into account two relaxation times τ_0, υ_0.
ii. In the L-S theory, the energy equation involves the first and second derivatives of the strain with respect to time, whereas the corresponding equation in the G-L theory needs only the first derivative of this strain with respect to time.
iii. In the linear case, according to the approach of the G-L theory, the heat cannot propagate with a finite speed unless the stresses depend on the temperature, velocity, whereas according to the L-S theory, the heat can propagate with a finite speed even though the stresses are independent of the temperature, velocity.
iv. The two theories are structurally different from one another, and one cannot be obtained as a particular case of the other.

Theory of linear elastic materials with voids is an important development of the classical theory of elasticity; this theory deals with materials which have a distribution of small voids, where the volume of void is included among the kinematics variables and investigate various types of geological and biological materials since the classical theory of elasticity is not sufficient. The theory reduces to the classical theory in the limiting case of the volume of void tending to zero. Cowin and Nunziato [4] developed a theory of linear elastic materials with voids to study mathematically the mechanical behavior of porous solids. Puri and Cowin [5] studied the behavior of plane waves in a linear elastic material with voids. Iesan [6] developed the linear theory of thermoelastic materials with voids. Dhaliwal and Wang [7] developed a heat flux dependent theory of thermoelasticity with voids. Ciarletta and Scarpetta [8] discussed some results on thermoelasticity of dielectric materials with voids. Othman et al. [9] studied a 2-D problem of a rotating thermoelastic solid with voids under thermal loading due to laser pulse and initial stress type III. Othman and Edeeb [10, 11] investigated the 2-D problem of a rotating thermoelastic solid with voids, thermal loading due to laser pulse and two-temperature under three theories. The effect of gravity on the wave propagation in an elastic solid medium was first considered by Bromwich [12] treating the force of gravity as a type of body force. Othman et al. [13] studied the effect of gravity on the generalized thermoelastic medium with temperature dependent properties for different theories. Ailawalia, and Narah [14] studied the effect of rotation in generalized thermoelastic solid under the influence of gravity with an overlying infinite thermoelastic fluid.

In the present work, we have formulated the generalized thermoelastic medium with voids for three theories under the influence of gravity and solve for the components of displacement, stresses, temperature distribution. The normal mode method was used to obtain the exact expression for the considered variables. Comparisons are carried out between the considered variables as calculated from the generalized thermoelastic pours medium based on the L-S, G-L and CD theories in the absence and presence of gravity. In addition, a comparison made between the three theories with and without voids.

2. Formulation of the Problem and Basic Equations

We consider a homogeneous isotropic elastic body with voids in a half-space $z \geq 0$ under the effect of a constant gravitational field of acceleration g. We are interested in plane strain in the $xz-$plane with displacement components u_1, u_3 such that $u_1 = u_1(x,z,t),\ u_3 = u_3(x,z,t)$.

Case 1:

The basic governing equations of a linear thermoelastic medium with voids under the effect of gravity based on the L-S, G-L and CD theories are

The stress-strain relation written as:

$$\sigma_{ij} = [\lambda e_{kk} + b\phi - \beta(1+\upsilon\frac{\partial}{\partial t})T]\delta_{ij} + \mu e_{ij}, \tag{1}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

The dynamical equations of an elastic medium are given by

$$\mu\nabla^2 u_1 + (\lambda+\mu)\frac{\partial e}{\partial x} + b\frac{\partial \phi}{\partial x} - \beta(1+\upsilon\frac{\partial}{\partial t})\nabla\frac{\partial T}{\partial x} + \rho\,\mathrm{g}\frac{\partial u_3}{\partial x} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{3}$$

$$\mu\nabla^2 u_3 + (\lambda+\mu)\frac{\partial e}{\partial z} + b\frac{\partial \phi}{\partial z} - \beta(1+\upsilon\frac{\partial}{\partial t})\nabla\frac{\partial T}{\partial z} - \rho\, g\frac{\partial u_1}{\partial x} = \rho\frac{\partial^2 u_3}{\partial t^2}, \tag{4}$$

The equation of voids is

$$\alpha\nabla^2\phi - b\,e - \xi\phi - \omega_0 b\frac{\partial\phi}{\partial t} + m(1+\upsilon\frac{\partial}{\partial t})T = \rho\chi\frac{\partial^2\phi}{\partial t^2}, \tag{5}$$

The heat conduction equation,

$$K\nabla^2 T = \rho C_E(1+\tau_0\frac{\partial}{\partial t})\dot{T} + \beta T_0(1+n_0\tau_0\frac{\partial}{\partial t})\dot{e} + mT_0(1+n_0\tau_0\frac{\partial}{\partial t})\dot{\phi}. \tag{6}$$

Where, σ_{ij} are the components of stress tensor, e_{ij} are the components of strain, λ, μ are the Lame' constants, $\beta = (3\lambda+2\mu)\alpha_t$ such that α_t is the coefficient of thermal expansion, δ_{ij} is the Kronecker delta,

$\alpha, b, \xi, \omega_0, m, \chi$ are the material constants due to the presence of voids, ρ is the density, C_E is the specific heat at constant strain, n_0 is a parameter, τ_0, υ are the thermal relaxation times, K is the thermal conductivity, T_0 is the reference temperature is chosen so that $|(T - T_0)/T_0| < 1$ ϕ is the change in the volume fraction field.

For a two dimensional problem in *xz*-plane, Eq. (1) can be written as:

$$\sigma_{ij} = [\lambda(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}) + b\phi - \beta(1+\upsilon\frac{\partial}{\partial t})T]\,\delta_{ij} + \mu(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}), \qquad \text{i, j} = 1,3. \tag{7}$$

For the purpose of numerical evaluation, we introduce dimensions variables

$$(x', z') = \frac{\omega_1^*}{c_0}(x, z), \qquad (u_1', u_3') = \frac{\omega_1^*}{c_1}(u_1, u_3),\ \sigma_{ij}' = \frac{\sigma_{ij}}{\mu}\sigma_{ij},\ \phi' = \frac{\omega_1^* \chi}{c_1^2}\phi,\ T' = \frac{T}{T_0},\ t' = \omega_1^* t, \qquad g' = \frac{g}{c_1 \omega_1^*},$$

$$\upsilon' = \omega_1^* \upsilon,\quad \tau_0' = \omega_1^* \tau_0,\quad c_1^2 = \frac{\lambda + 2\mu}{\rho},\quad \omega_1^* = \frac{\rho C_E c_1^2}{K}.$$

Using the above dimensions quantities, Eqs. (3)-(6) become

$$\nabla^2 u_1 + A_1\frac{\partial e}{\partial x} + A_2\frac{\partial \phi}{\partial x} - A_3(1+\upsilon\frac{\partial}{\partial t})\frac{\partial T}{\partial x} + A_4\frac{\partial u_3}{\partial x} = A_5\frac{\partial^2 u_1}{\partial t^2}, \tag{8}$$

$$\nabla^2 u_3 + A_1\frac{\partial e}{\partial z} + A_2\frac{\partial \phi}{\partial z} - A_3(1+\upsilon\frac{\partial}{\partial t})\frac{\partial T}{\partial z} - A_4\frac{\partial u_1}{\partial x} = A_5\frac{\partial^2 u_3}{\partial t^2}, \tag{9}$$

$$\nabla^2\phi - A_6 e - A_7\phi - A_8\frac{\partial \phi}{\partial t} + A_9(1+\upsilon\frac{\partial}{\partial t})T = A_{10}\frac{\partial^2 \phi}{\partial t^2}, \tag{10}$$

$$\varepsilon_1\nabla^2 T - A_{11}(1+\mathrm{n}_0\tau_0\frac{\partial}{\partial t})\frac{\partial \phi}{\partial t} = (1+\tau_0\frac{\partial}{\partial t})\frac{\partial T}{\partial t} + \varepsilon_2(1+\mathrm{n}_0\tau_0\frac{\partial}{\partial t})\frac{\partial e}{\partial t}. \tag{11}$$

where, $A_1 = \frac{\lambda+\mu}{\mu},\ A_2 = \frac{b c_1^2}{\mu\omega_1^{*2}\chi},\ A_3 = \frac{\beta T_0}{\mu},\ A_4 = \frac{\rho g c_1^2}{\mu},\ A_5 = \frac{\rho c_1^2}{\mu},\ A_6 = \frac{b\chi}{\alpha},\ A_7 = \frac{\xi c_1^2}{\alpha\omega_1^{*2}},$

$$A_8 = \frac{\omega_0 c_1^2}{\alpha\omega_1^*},\ A_9 = \frac{m T_0 \chi}{\alpha},\ A_{10} = \frac{\rho c_1^2\chi}{\alpha},\ A_{11} = \frac{m c_1^2}{\rho C_E\omega_1^{*2}\chi},\quad \varepsilon_1 = \frac{K\omega_1^*}{\rho c_1^2 C_E},\quad \varepsilon_2 = \frac{\beta}{\rho C_E}.$$

We define displacement potentials R and Q which relate to displacement components u_1 and u_3 as,

$$u_1 = \frac{\partial R}{\partial x} + \frac{\partial Q}{\partial z}, \qquad u_3 = \frac{\partial R}{\partial z} - \frac{\partial Q}{\partial x}, \tag{12}$$

$$e = \nabla^2 R, \qquad (\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}) = \nabla^2 Q. \tag{13}$$

Using Eq. (13) in Eqs. (8)-(11), we obtain:

$$(S_1\nabla^2 - A_5\frac{\partial^2}{\partial t^2})\mathrm{R} - A_4\frac{\partial}{\partial x}\mathrm{Q} + A_2\phi - A_3(1+\upsilon\frac{\partial}{\partial t})T = 0, \tag{14}$$

$$A_4\frac{\partial}{\partial x}R + (\nabla^2 - A_5\frac{\partial^2}{\partial t^2})\mathrm{Q} = 0, \tag{15}$$

$$-A_6\nabla^2 R + (\nabla^2 - A_7 - A_8\frac{\partial}{\partial t} - A_{10}\frac{\partial^2}{\partial t^2})\phi + A_9(1+\upsilon\frac{\partial}{\partial t})T = 0, \tag{16}$$

$$\varepsilon_2\frac{\partial^2}{\partial t^2}\nabla^2 R - A_{11}(1+n_0\tau_0\frac{\partial}{\partial t})\frac{\partial \phi}{\partial t} + \varepsilon_1\nabla^2 T - (1+\tau_0\frac{\partial}{\partial t})\frac{\partial T}{\partial t}. \tag{17}$$

The components of stress tensor are

$$\sigma_{xx} = A_{12}[\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}] + 2\frac{\partial u_1}{\partial x} + A_{13}\phi - A_{14}T, \tag{18}$$

$$\sigma_{zz} = A_{12}[\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}] + 2\frac{\partial u_3}{\partial z} + A_{13}\phi - A_{14}T, \tag{19}$$

$$\sigma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}. \tag{20}$$

Where $A_{12} = \frac{\lambda}{\mu}$, $A_{13} = \frac{bc_1^2}{\mu\omega_1^{*2}\chi}$, $A_{14} = \frac{\beta T_0}{\mu}(1+\upsilon\omega)$, $S_1 = 1 + A_1$.

3. Normal Mode Analysis

The solution of the considered physical variable decomposed in terms of normal modes as the following form

$$\{R, Q, \phi, T, \sigma_{ij}\}(x,z,t) = \{R^*, Q^*, \phi^*, T^*, \sigma_{ij}^*\}(z)\exp[i(\omega t + cx)]. \tag{21}$$

Where $R^*, Q^*, \phi^*, T^*, \sigma_{ij}^*$ are the amplitudes of the functions $R, Q, \phi, T, \sigma_{ij}$, ω is the complex time constant, $i = \sqrt{-1}$ and c is the wave number in the x -direction.

Using (21) in Eqs. (14)-(17), we obtain

$$(D^2 - S_2)R^* - S_3Q^* + S_4\phi^* - S_5T^* = 0, \tag{22}$$

$$S_6R^* + (D^2 - S_7)Q^* = 0, \tag{23}$$

$$-A_6(D^2 - c^2)R^* + (D^2 - S_8)\phi^* + S_9T^* = 0, \tag{24}$$

$$S_{10}(D^2 - c^2)R^* - S_{11}\phi^* + (D^2 - S_{12})T^* = 0. \tag{25}$$

Where, $S_2 = \frac{S_1c^2 - A_5\omega^2}{S_1}$, $S_3 = \frac{iA_4c}{S_1}$, $S_4 = \frac{A_2}{S_1}$, $S_5 = \frac{A_3(1+i\upsilon\omega)}{S_1}$, $S_6 = icA_4$, $S_7 = c^2 - A_5\omega^2$,

$S_8 = c^2 + A_7 + iA_8\omega - A_{10}\omega^2$, $S_9 = A_9(1+i\upsilon\omega)$, $S_{10} = \frac{-\varepsilon_2(i\omega + n_0\tau_0\omega^2)}{\varepsilon_1}$, $S_{11} = \frac{A_{11}}{\varepsilon_1}$, $S_{12} = \frac{\varepsilon_1c^2 + i\omega - \tau_0\omega^2}{\varepsilon_1}$.

Eliminating Q^*, ϕ^* and T^* between Eqs. (22)-(25), we get the following ordinary differential equation of the eighth order which satisfied with R^*:

$$[D^8 - B_1D^6 + B_2D^4 - B_3D^2 + B_4]R^*(z) = 0. \tag{26}$$

Where $B_1 = S_{12} + S_8 + S_7 + S_2 - A_6S_4 - S_5S_{10}$,

$$B_2 = S_8S_{12} + S_9S_{11} + S_7S_{12} + S_7S_8 + S_2S_{12} + S_2S_8 + S_2S_7 + S_6S_3 - A_6S_4S_{12} - A_6S_4c^2 + S_4S_9S_{10}$$
$$- A_6S_4S_7 - A_6S_5S_{11} - S_5S_{10}c^2 - S_5S_8S_{10} - S_5S_7S_{10},$$

$$B_3 = S_7S_8S_{12} + S_7S_9S_{11} + S_2S_8S_{12} + S_2S_9S_{11} + S_2S_7S_{12} + S_2S_7S_8 + S_3S_6S_{12} + S_3S_6S_8 + A_6S_4S_{10}c^2 + S_4S_9S_{10}c^2$$
$$- A_6S_4S_7S_{12} - A_6S_4S_7c^2 + S_4S_7S_9S_{10} - A_6S_5S_{11}c^2 - S_5S_8S_{10}c^2 - A_6S_5S_7S_{11} - S_5S_7S_{10}c^2 - S_5S_7S_8S_{10},$$

$$B_4 = S_2S_7S_8S_{12} + S_2S_7S_9S_{11} + S_3S_6S_8S_{12} + S_3S_6S_9S_{11} - A_6S_4S_7S_{12}c^2 + S_4S_7S_9S_{10}c^2$$
$$- A_6S_5S_7S_{11}c^2 - S_5S_7S_8S_{10}c^2.$$

In a similar manner, we get

$$[D^8 - B_1D^6 + B_2D^4 - B_3D^2 + B_4]\{R^*(z), Q^*(z), \phi^*(z), T^*(z)\} = 0. \tag{27}$$

Equation (27) factored as

$$[(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)(D^2 - k_4^2)]\{R^*(z), Q^*(z), \phi^*(z), T^*(z)\} = 0. \tag{28}$$

Where $k_n^2 (n = 1,2,3,4)$ are the roots of the Eq. (27), $D = \frac{d}{dz}$.

The solution of Eq. (27) bound as $z \to \infty$, is given by:

$$R^* = \sum_{n=1}^{4} M_n e^{-k_n z}, \tag{29}$$

$$Q^* = \sum_{n=1}^{4} H_{1n} M_n e^{-k_n z}, \tag{30}$$

$$\phi^* = \sum_{n=1}^{4} H_{2n} M_n e^{-k_n z}, \tag{31}$$

$$T^* = \sum_{n=1}^{4} H_{3n} M_n e^{-k_n z}. \tag{32}$$

Where, M_n $(n = 1,2,3,4)$ are constants.

To obtain the components of the displacement vector, from (29) and (33) in (12)

$$u_1^* = \sum_{n=1}^{4} H_{4n} M_n e^{-k_n z}, \tag{33}$$

$$u_3^* = \sum_{n=1}^{4} H_{5n} M_n e^{-k_n z}, \tag{34}$$

From Eqs. (31)-(34) in (18)-(20) to obtain the components of the stresses

$$\sigma_{xx}^* = \sum_{n=1}^{4} H_{6n} M_n e^{-k_n z}, \tag{35}$$

$$\sigma_{zz}^* = \sum_{n=1}^{4} H_{7n} M_n e^{-k_n z}, \tag{36}$$

$$\sigma_{xz}^* = \sum_{n=1}^{4} H_{8n} M_n e^{-k_n z}. \tag{37}$$

Where $H_{1n} = \dfrac{-S_6}{(k_n^2 - S_7)}$, $H_{2n} = \dfrac{-[-A_6 S_5 (k_n^2 - c^2) + S_9 (k_n^2 - S_2 - S_3 H_{1n}]}{[S_5 (k_n^2 - S_8) + S_4 S_9]}$, $H_{3n} = \dfrac{k_n^2 - S_2 - S_3 H_{1n} + S_4 H_{2n}}{S_5}$,

$H_{4n} = ic - k_n H_{1n}$, $H_{5n} = -(k_n + icH_{1n})$, $H_{6n} = A_{12}(icH_{4n} - k_n H_{5n}) + 2icH_{4n} + A_{13} H_{2n} - A_{14} H_{3n}$,

$H_{7n} = A_{12}(icH_{4n} - k_n H_{5n}) - 2k_n H_{5n} + A_{13} H_{2n} - A_{14} H_{3n}$, $H_{8n} = (-k_n H_{4n} + icH_{5n})$.

4. Boundary Conditions

In this section, we need to consider the boundary conditions at $z = 0$, in order to determine the constants M_n $(n = 1,2,3,4)$.

(1) The mechanical boundary condition

$$\sigma_{zz} = -P_1 e^{i(\omega t + cx)}, \qquad \sigma_{xz} = 0, \qquad \frac{\partial \phi}{\partial z} = 0. \tag{38}$$

(2) The thermal boundary condition that the surface of the half-space is subjected to

$$T = P_2 e^{i(\omega t + cx)}. \tag{39}$$

Where the magnitude of the applied force in the half-space is P_1, and P_2 is the applied constant temperature to the boundary.

Using the expressions of the variables into the above boundary conditions (38), (39), we obtain

$$\sum_{n=1}^{4} H_{7n} M_n = -P_1, \tag{40}$$

$$\sum_{n=1}^{4} H_{8n} M_n = 0, \tag{41}$$

$$\sum_{n=1}^{4} - k_n H_{2n} M_n = 0, \tag{42}$$

$$\sum_{n=1}^{4} H_{3n} M_n = P_2. \tag{43}$$

Invoking boundary conditions (40)-(43) at the surface $z = 0$ of the plate, we obtain a system of four equations. After applying the inverse of matrix method, we get the values of the four constants M_n $(n = 1, 2, 3, 4)$.

Case 2:

The solution of wave propagation of a generalized thermoelastic medium without voids under the effect of gravity is: by putting $\alpha, b, \xi, \omega_0, m, \chi$ equal to zero, then the basic governing Eqs. (1) and (3)-(6) of a linear thermoelastic medium without voids under the effect of gravity can be written as:

$$\mu\, \nabla^2 u_1 + (\lambda + \mu)\frac{\partial e}{\partial x} - \beta(1+\upsilon\frac{\partial}{\partial t})\nabla\frac{\partial T}{\partial x} + \rho\, g\,\frac{\partial u_3}{\partial x} = \rho\,\frac{\partial^2 u_1}{\partial t^2}, \tag{44}$$

$$\mu\, \nabla^2 u_3 + (\lambda + \mu)\frac{\partial e}{\partial z} - \beta(1+\upsilon\frac{\partial}{\partial t})\nabla\frac{\partial T}{\partial z} - \rho\, g\,\frac{\partial u_1}{\partial x} = \rho\,\frac{\partial^2 u_3}{\partial t^2}, \tag{45}$$

$$K\nabla^2 T = \rho C_E(1+\tau_0\frac{\partial}{\partial t})\frac{\partial T}{\partial t} + \beta T_0(1+n_0\tau_0\frac{\partial}{\partial t})\frac{\partial e}{\partial t}, \tag{46}$$

$$\sigma_{ij} = [\lambda(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}) - \beta(1+\upsilon\frac{\partial}{\partial t})T]\delta_{ij} + \mu(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}), \qquad i, j = 1, 3. \tag{47}$$

The dimensions of Eqs. (44)-(46) have the form

$$\nabla^2 u_1 + E_1\frac{\partial e}{\partial x} - E_2(1+\upsilon\frac{\partial}{\partial t})\frac{\partial T}{\partial x} + E_3\frac{\partial u_3}{\partial x} = E_4\frac{\partial^2 u_1}{\partial t^2}, \tag{48}$$

$$\nabla^2 u_3 + E_1\frac{\partial e}{\partial z} - E_2(1+\upsilon\frac{\partial}{\partial t})\frac{\partial T}{\partial z} - E_3\frac{\partial u_1}{\partial x} = E_4\frac{\partial^2 u_3}{\partial t^2}, \tag{49}$$

$$\varepsilon_3\nabla^2 T = (1+\tau_0\frac{\partial}{\partial t})\frac{\partial T}{\partial t} + \varepsilon_4(1+n_0\tau_0\frac{\partial}{\partial t})\frac{\partial e}{\partial t}\ . \tag{50}$$

Where $E_1 = \dfrac{\lambda+\mu}{\mu}$, $\quad E_2 = \dfrac{\beta T_0}{\mu}$, $\quad E_3 = \dfrac{\rho g c_1^2}{\mu}$, $\quad E_4 = \dfrac{\rho c_1^2}{\mu}$, $\quad \varepsilon_3 = \dfrac{K\omega_1^*}{\rho c_1^2 ce}$, $\quad \varepsilon_4 = \dfrac{\beta}{\rho ce}$.

Using Eq. (13) in Eqs. (48)-(50), we obtain:

$$(F_1\nabla^2 - E_4\frac{\partial^2}{\partial t^2})R - E_3\frac{\partial}{\partial x}Q - E_2(1+\upsilon\frac{\partial}{\partial t})T = 0, \tag{51}$$

$$E_3\frac{\partial}{\partial x}R + (\nabla^2 - E_4\frac{\partial^2}{\partial t^2})Q = 0, \tag{52}$$

$$-\varepsilon_3\frac{\partial^2}{\partial t^2}\nabla^2 R + \varepsilon_4\nabla^2 T - (1+\tau_0\frac{\partial}{\partial t})\frac{\partial T}{\partial t} = 0. \tag{53}$$

The components of stress tensor are

$$\sigma_{xx} = E_5[\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}] + 2\frac{\partial u_1}{\partial x} - E_6(1+i\,\upsilon\omega)T, \tag{54}$$

$$\sigma_{zz} = E_5[\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}] + 2\frac{\partial u_3}{\partial z} - E_6(1+i\,\upsilon\omega)T, \tag{55}$$

$$\sigma_{xz} = [\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}]. \tag{56}$$

Where $E_5 = \frac{\lambda}{\mu}$, $E_6 = \frac{\beta T_0}{\mu}(1 + i\,\upsilon\omega)$, $F_1 = 1 + E_1$

Using (21) in Eqs. (51)-(53), we obtain

$$(\mathrm{D}^2 - F_2)R^* - F_3 Q^* - F_4 T^* = 0, \tag{57}$$

$$F_5 R^* + (\mathrm{D}^2 - F_6)\, Q^* = 0, \tag{58}$$

$$E_7(\mathrm{D}^2 - c^2)R^* + (\mathrm{D}^2 - E_8)T^* = 0, \tag{59}$$

where, $F_2 = \frac{F_1 c^2 - E_4\omega^2}{F_1}$, $F_3 = \frac{icE_3}{F_1}$, $F_4 = \frac{E_2(1+i\,\upsilon\omega)}{F_1}$, $F_5 = icE_3$, $F_6 = c^2 - E_4\omega^2$,

$$F_7 = \frac{-\varepsilon_4(i\,\omega - n_0\tau_0\omega^2)}{\varepsilon_3}, \quad F_8 = \frac{\varepsilon_3 c^2 + i\,\omega + \tau_0\omega^2}{\varepsilon_3}.$$

Eliminating Q^* and T^* between Eqs. (57)-(59), we get the following ordinary differential equation of sixth order which satisfied with R^*

$$[\mathrm{D}^6 - \mathrm{I}_1\mathrm{D}^4 + \mathrm{I}_2\mathrm{D}^2 - \mathrm{I}_3]R^*(\mathrm{z}) = 0. \tag{60}$$

Where $\mathrm{I}_1 = F_8 + F_6 + F_2 - F_4F_7$, $\mathrm{I}_2 = F_6F_8 + F_2F_8 + F_2F_6 + F_3F_5 - F_4F_6F_7 - F_4F_7c^2$,

$$\mathrm{I}_3 = F_2F_6F_8 + F_2F_8 - F_4F_6F_7c^2.$$

In a similar manner, we get

$$[\mathrm{D}^6 - \mathrm{I}_1\mathrm{D}^4 + \mathrm{I}_2\mathrm{D}^2 - \mathrm{I}_3]\{R^*(\mathrm{z}), Q^*(\mathrm{z}), T^*(\mathrm{z})\} = 0. \tag{61}$$

Equation (61) factored as

$$[(\mathrm{D}^2 - k_1^2)(\mathrm{D}^2 - k_2^2)(\mathrm{D}^2 - k_3^2)]\{R^*(\mathrm{z}), Q^*(\mathrm{z}), T^*(\mathrm{z})\} = 0. \tag{62}$$

Where $k_{\mathrm{n}}^2(\mathrm{n} = 1,2,3)$ are the roots of the Eq. (61).

The solution of Eq. (61) bound as $z \to \infty$, is given by:

$$R^* = \sum_{\mathrm{n}=1}^{3} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}, \tag{63}$$

$$Q^* = \sum_{\mathrm{n}=1}^{3} L_{1\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}, \tag{64}$$

$$T^* = \sum_{\mathrm{n}=1}^{3} L_{2\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}, \tag{65}$$

where M_{n} $(\mathrm{n} = 1,2,3)$ are some constants.

To obtain the components of the displacement vector, from (63) and (64) in (12)

$$u_1^* = \sum_{\mathrm{n}=1}^{3} L_{3\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}, \tag{66}$$

$$u_3^* = \sum_{\mathrm{n}=1}^{3} L_{4\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}. \tag{67}$$

From Eqs. (63)-(65) in (54)-(56) to obtain the components of the stress vector

$$\sigma^*_{xx} = \sum_{\mathrm{n}=1}^{3} L_{5\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}, \tag{68}$$

$$\sigma^*_{zz} = \sum_{\mathrm{n}=1}^{3} H_{6\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}, \tag{69}$$

$$\sigma^*_{xz} = \sum_{\mathrm{n}=1}^{3} H_{7\mathrm{n}} M_{\mathrm{n}} e^{-k_{\mathrm{n}} z}. \tag{70}$$

Where $L_{1\mathrm{n}} = \dfrac{-F_5}{(k_{\mathrm{n}}^2 - F_6)}$, $L_{2\mathrm{n}} = \dfrac{-F_7 (k_{\mathrm{n}}^2 - c^2)}{(k_{\mathrm{n}}^2 - S_8)}$, $L_{3\mathrm{n}} = \mathrm{i}c + k_{\mathrm{n}} L_{1\mathrm{n}}$, $L_{4\mathrm{n}} = -(k_{\mathrm{n}} + \mathrm{i}c L_{1\mathrm{n}})$,

$L_{5n} = E_5 (\mathrm{i}c L_{3\mathrm{n}} - k_{\mathrm{n}} L_{4\mathrm{n}}) + 2\mathrm{i}c L_{3\mathrm{n}} - E_6 L_{2\mathrm{n}}$, $H_{6n} = E_5 \; cL \; (-k_3 \, \mathrm{i}L \;_{\mathrm{n}} - \; k \; L \;_{\mathrm{n}} - E) L_4$

$L_{7n} = (-k_n L_{3n} + icL_{4n})$.

5. Boundary Conditions

In this section, we need to consider the boundary conditions at $z = 0$, in order to determine the constants M_{n} $(\mathrm{n} = 1, 2, 3)$.

(1) The mechanical boundary conditions

$$\sigma_{zz} = -P_1 e^{i(\omega t + cx)}, \qquad \sigma_{xz} = 0, \tag{71}$$

(2) The thermal boundary condition that the surface of the half-space subjected to

$$T = P_2 e^{i(\omega t + cx)}. \tag{72}$$

Where P_1 is the magnitude of the applied force in of the half-space and P_2 is the applied constant temperature to the boundary.

Using the expressions of the variables into the above boundary conditions (71), (72), we obtain

$$\sum_{\mathrm{n}=1}^{3} L_{6\mathrm{n}} M_{\mathrm{n}} = -P_1, \tag{73}$$

$$\sum_{\mathrm{n}=1}^{3} L_{7\mathrm{n}} M_{\mathrm{n}} = 0, \tag{74}$$

$$\sum_{\mathrm{n}=1}^{3} L_{2\mathrm{n}} M_{\mathrm{n}} = P_2. \tag{75}$$

Invoking boundary conditions (73)-(75) at the surface $z = 0$ of the plate, we obtain a system of three equations. After applying the inverse of matrix method, we can get the values of the three constants M_{n} $(\mathrm{n} = 1, 2, 3)$.

6. Numerical results and discussion

In order to illustrate the obtained theoretical results in the preceding section, following Dhaliwal and Singh [15] the magnesium material chosen for purposes of numerical evaluations. The constants of the problem taken as

$\lambda = 2.14 \times 10^{10}\ N/.m^2$, $\mu = 3.278 \times 10^{10}\ N/m^2$, $K = 1.7 \times 10^2 W/m\,\mathrm{deg}$, $\alpha_t = 1.78 \times 10^{-5} N/m^2$, $T_0 = 298\,K$,

$\rho = 1.74 \times 10^3\ Kg/m^3$, $C_E = 1.04 \times 10^3\ J/Kg\,\mathrm{deg}$, $\beta = 2.68 \times 10^6 N/m^2\,\mathrm{deg}$, $\omega_1^* = 3.58 \times 10^{11}/s$.

The voids parameters are

$\chi = 1.753 \times 10^{-15} m^2$, $\xi = 1.475 \times 10^{10} N/m^2$, $b = 1.13849 \times 10^{10} N/m^2$, $\alpha = 3.688 \times 10^{-5} N$,

$m = 2 \times 10^6 N/m^2\,\mathrm{deg}$, $\omega_0 = 0.0787 \times 10^{-3} N/m^2 s$.

The comparisons carried out for

$x = 0.5,\quad t = 0.03,\quad c = 0.2,\quad \omega = \zeta_0 + i\,\zeta_1,\quad \zeta_0 = -0.6,\ \zeta_1 = -\quad p_1 = 0.1,\quad p_2 = 2,\quad \tau_0 = 0.05\ s,\quad \upsilon = 0.5\ s,$
$0 \le z \le 7.$

The computations were carried out at $t = 0.03$. The numerical technique, outlined above, was used for the distribution of the real part of the displacement u_3, the stresses σ_{zz}, σ_{xz} and the change in the volume fraction field ϕ with the distance z , for the problem under consideration. All the considered variables depend not only on the variables t, z and x, but also on the thermal relaxation times τ_0 and υ. The results are shown in Figs. 1-6. The graphs show the six curves predicted by three different theories of thermoelasticity (CD, L-S, and G-L). In these figures, the solid lines represent the solution in the CD theory, the dashed lines represent the solution with the G-L theory, and the dashed-dotted lines represent the solution with the L-S theory. Here, all the variables are taken in non-dimensional forms and we consider four cases:

(1) Equations of the CD theory, when $n_0 = 0,\ \tau_0 = \upsilon = 0.$

(2) Lord and Shulman L-S theory when $n_0 = 1,\ \upsilon = 0,\ \tau_0 > 0,$

(3) Green and Lindsay G-L theory when $n_0 = 0,\ \upsilon > \tau_0 > 0.$

(4) The three theories in the absence of a gravity field from the above mentioned by taking $g = 0.$

Figs. 1-3 show comparisons among the considered variables in the absence and presence of the gravity effect $(g = 0,\ g = 9.8).$

Fig. 1 shows that the distribution of the vertical displacement u_3 always begins from a positive value with gravity, in the context of the three theories CD, L-S, and G-L, it decreases in the range $0 \le z \le 1.5,$ then increases in the range $4 \le z \le 7,$ while increases in the range $1.5 \le z \le 4.$ However without gravity it begins from zero, and increases in the range $0 \le z \le 1.5,$ then, decreases in the range $1.5 \le z \le 3.5.$ Fig. 2 depicts the distribution of the change in the volume fraction field with and without ϕ gravity under the three theories. The values of ϕ with gravity are decreasing in the range $0 \le z \le 1.8,$ then increasing in the range $1.8 \le z \le 4.5.$ The values of ϕ without gravity is greater than that with gravity in the range $0 \le z \le 1.8,$ while the vice versa in the range $1.8 \le z \le 6.$ Fig. 3 shows that the distribution of the stress component σ_{xz}, begins from zero in the context of the three theories, and satisfies the boundary conditions at $z = 0.$ The values of σ_{xz}, with gravity are greater than that without gravity.

Figs. 4-6 show the comparisons among the considered variables in the absence and presence of voids in the case of material with gravity. Fig. 4 exhibits that the distribution of the vertical displacement u_3 begins from positive values in the presence and absence of voids. In the context of the three theories, we notice that the values of u_3 with voids are greater than that without voids in the range $0 \le z \le 1.5,$ while the vice versa in the range $1.5 \le z \le 5.5,$ then converges to zero. Fig. 5 determines the distribution of stress component σ_{zz}, in the context the three theories in the presence and absence of voids. It explained that the distribution of σ_{zz} increases with the increase of the values of voids in the range $0 \le z \le 2,$ then decreases with the increase of the void in the range $2 \le z \le 6.$ Fig. 6 demonstrates that the distribution of the stress component σ_{xz}, in the context of the three theories begins from zero and satisfies the boundary conditions at $z = 0$ with and without voids. It is noticed that the value of σ_{xz} with voids is greater than that without voids in the range $0 \le z \le 3.7,$ and the vice versa in the range $3.7 \le z \le 7.$

7. Conclusion

By comparing the figures that were obtained for the three thermoelastic theories, important phenomena are observed:

1. The values of all physical quantities converge to zero with increasing distance z, and all functions are continuous.

2. The normal mode analysis technique has been used is applied to a wide range of problems in thermodynamics and thermoelasticity.
3. All the physical quantities satisfy the boundary conditions.
4. The gravity has a significant effect on the variation of the considered physical quantities, since they make great changes in the behavior of the functions, also the same observation of the absence and presence of the voids in the thermoelastic solid.

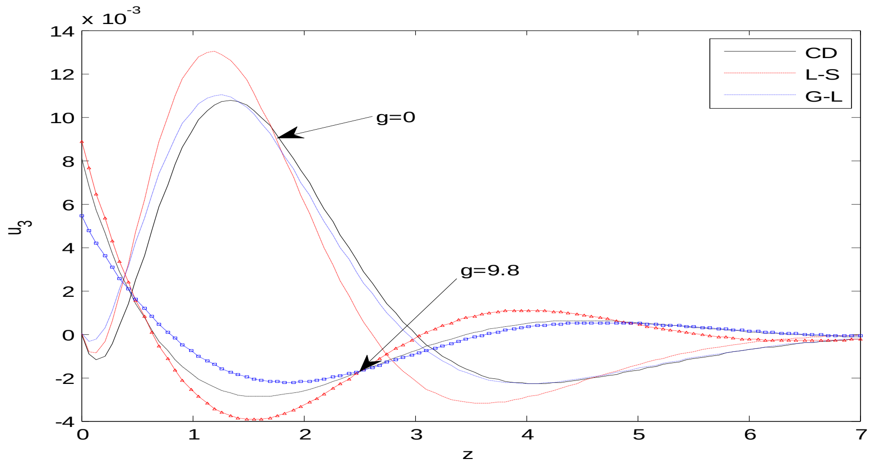

Figure 1: The displacement component u_3 distribution against z with and without gravity

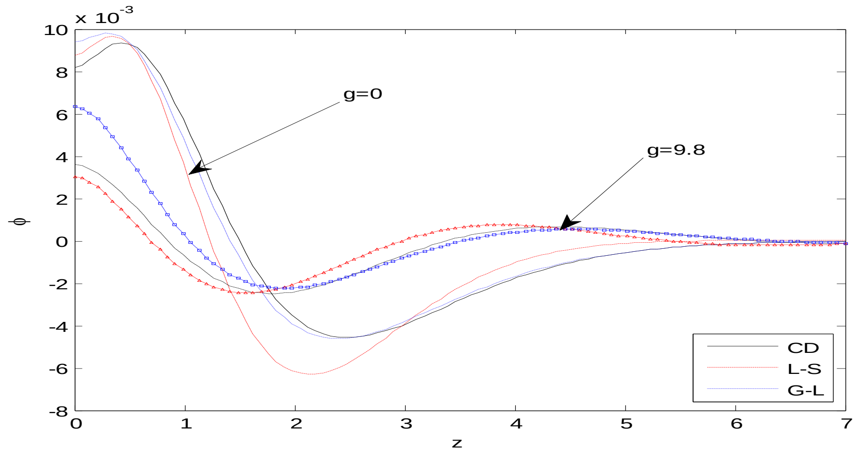

Figure 2: The displacement of volume fraction field ϕ against z with and without gravity

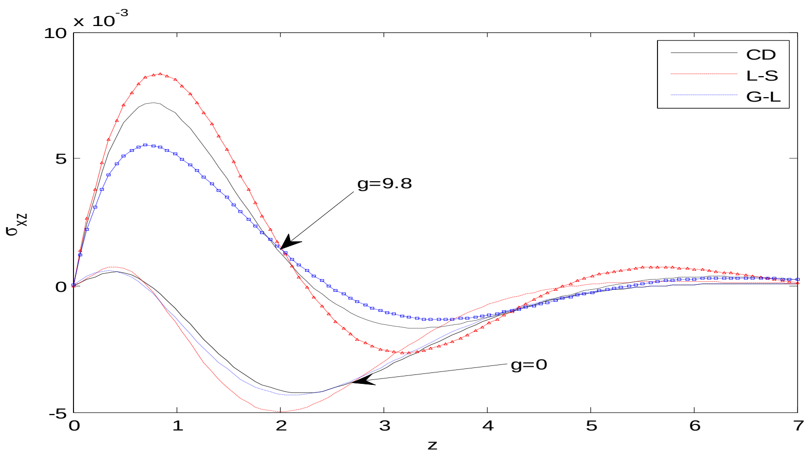

Figure 3: The displacement of the stress tensor σ_{xz} against z with and without gravity

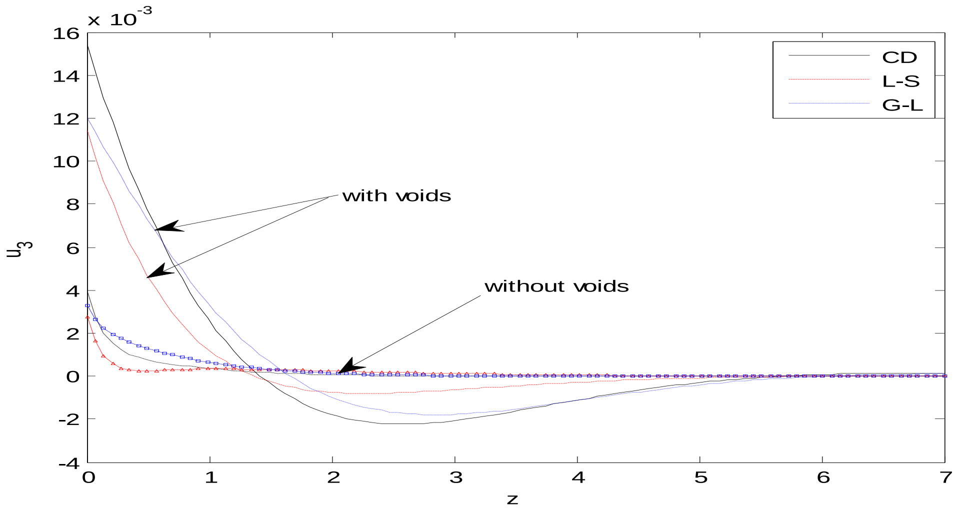

Figure 4: The displacement component u_3 distribution against z with and without voids

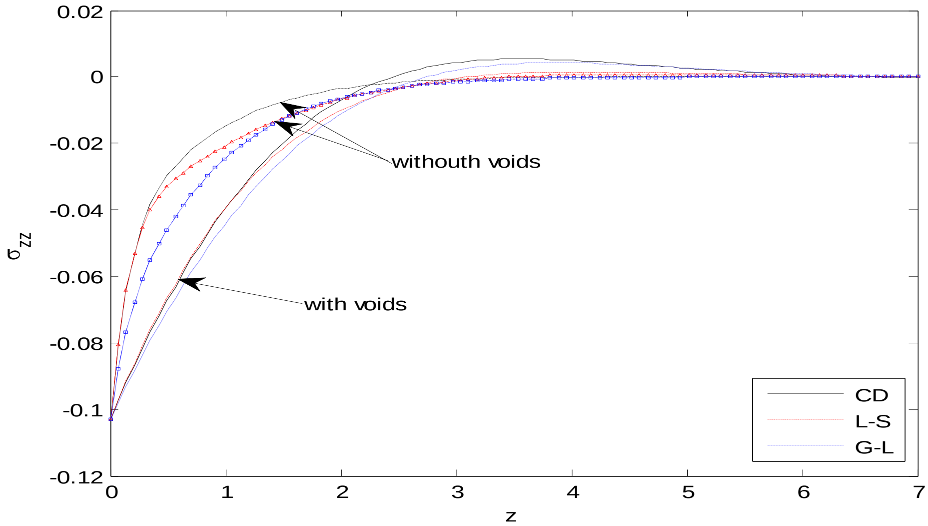

Figure 5: The displacement of the stress tensor σ_{zz} against z with and without voids

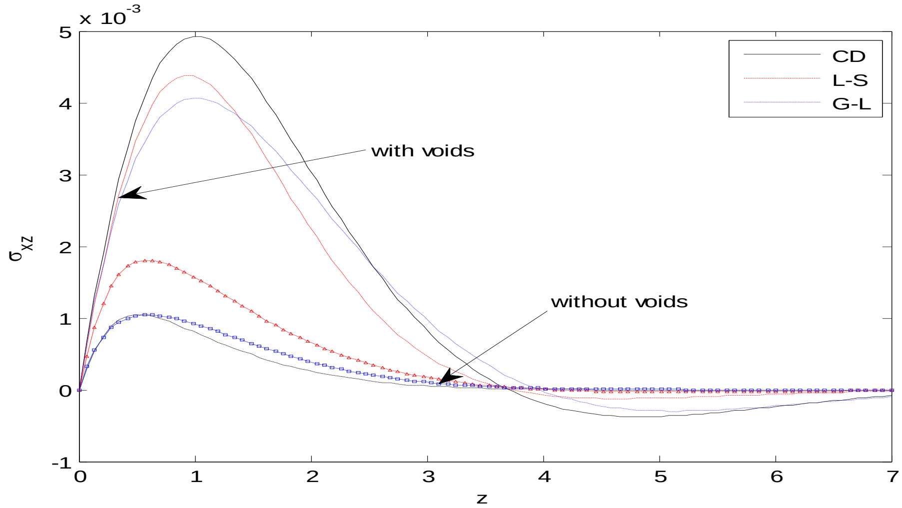

Figure 6: The displacement of the stress tensor σ_{xz} against z with and without void

References

[1]. Biot, M. (1956). Thermoelasticity and irreversible thermodynamics. Journal of Applied Physics. 27, 240-253.

[2]. Lord, H.W. & Shulman, Y. (1967). A generalized dynamical theory of thermo- elasticity. Mechanical Journal of Physical Solids, 15, 299-309.

[3]. Green, A.E. & Lindsay, K.A. (1972). Thermoelasticity. J. Elasticity, 2, 1-7.

[4]. Cowin, S.C. & Nunziato, J.W. (1983). Linear elastic materials with voids. Journal of Elasticity 13, 125-147.

[5]. Puri, P. & Cowin, S.C. (1985) Plane waves in linear elastic materials with voids. Journal of Elasticity. 15, 167-183.

[6]. Iesan, D. (1986). A theory of thermoelastic materials with voids. Acta Mechanica 60, 67-89.

[7]. Dhaliwal, R.S. & Wang, J. (1995) A heat-flux dependent theory of thermo-elasticity with voids. Acta Mechanica. 110, 33-39.

[8]. Ciarletta, M. & Scarpetta, E. (1995) Some results on thermoelasticity for dielectric materials with voids. Journal of Applied Mathematics and Mechanics, 75, 707-714.

[9]. Othman, M.I.A., Zidan, M.E.M. & Hilal, M.I.M. (2015). 2-D problem of a rotating thermoelastic solid with voids under thermal loading due to laser pulse and initial stress type III. Journal of Thermal Stresses. 38, 835-853.

[10]. Othman, M.I.A. & Edeeb, E.R.M. (2016). 2-D problem of a rotating thermo-elastic solid with voids and thermal loading due to laser pulse under three theories. Journal of Computational and Theoretical Nanoscience. 13, 294-305.

[11]. Othman, M.I.A. & Edeeb, E.R.M. (2016). Two-temperature generalized rotation-thermoelastic medium with voids and initial stress: comparison of different theories, Journal of Scientific and Engineering Research, 3, 10-25.

[12]. Bromwich, T.J. (1898). On the influence of gravity on elastic waves and in particular on the vibrations of an elastic globe. The Proceedings of the London Mathematical Society 30, 98-120.

[13]. Othman, M.I.A., Elmaklizi, J.D. & Saied, S.M. (2013). Generalized thermo-elastic medium with temperature dependent properties for different theories under the effect of gravity field. International Journal of Thermophysics. 34, 521-537.

[14]. Ailawalia, P. & Narah, N.S. (2009). Effect of rotation in generalized thermo-elastic solid under the influence of gravity with an overlying infinite thermo-elastic fluid. Applied of Mathematics and Mechanics. 30, 1505-1518.

[15]. Dhaliwal, R.S. & Singh, A. (1980). Dynamic coupled thermoelasticity. Hindustan Publication Corp, New Delhi, India.