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An Order Five Explicit Runge-Kutta Method for Direct Solution of General Second Order Odes

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Abstract An illustration of how first order ordinary differential equations (ODEs) Runge-Kutta method can be modified for a second order ODEs was presented. The theory of Nyström method was adopted in the modification of the methods. Through this process a first order ODEs numerical method can be extended to the case in which the approximate solution to a second order ODEs (special or general), as well as first order Initial Value Problems (IVPs) can be calculated. Numerical experiment to illustrate its efficiency and the method can be extended to solve higher order differential equations. The scheme is simple to implement and converges better with the exact solution.

Keywords Runge-Kutta Method, General Second Order Odes

Introduction

There is a vast body of literature addressing the numerical solution of the so called special second order initial value problems (IVP).

$$y'' = f(x,y) \qquad y(x_o) = y \qquad y'(x_o) = \beta \tag{1.1}$$

(see for example(14 and 15) but not so much for the general second order IVP with a dissipative term

$$y'' = f(x,y,y') \qquad y(x_o) = y \qquad y'(x_o) = \beta \tag{1.2}$$

(Different approaches appear in [1-3]).

Although it is possible to integrate a second order IVP by reducing it to first order system and apply one of the method available for such system it seem more natural to provide commercial method in order to integrate the problem directly. The advantage of these approaches lies in the fact that they are able to exploit special information about ODEs and this result in an increase in efficiency (that is, high accuracy at low cost) For instance ,it is well know that Runge-Kutta Nystrom(RKN) method for (1.2) involve a real improvement as compared to standard Runge-Kutta(RK) method for a given number of stages [4].

In this paper, we present an eight-stage Runge-Kutta Nystrom method for direct integration of second order ODEs with the following advantage such as high order and stage order and low implementation cost.

The RKN method is an extension of RK method for first order ODEs of the form

$$y' = f(x,y), \qquad y(x_0) = y_0 \tag{1.3}$$

There are many kinds of RK methods of different Orders, i.e the RK of order 2, 3, 4 etc. The higher the order of the scheme the better the accuracy.

The most popular RK method or classical RK method of order four is good since it has it has local error bounds 0 h^5 which is small enough (h $<$ 1). The classical RK methods of order four for the initial value problem (1.1) is given by

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{1.4}$$

where,

$$k_1 = hf(x, y)$$
$$k_2 = hf(x + \frac{1}{2}h, y + \frac{1}{2}k_1)$$
$$k_3 = hf(x + \frac{1}{2}h, y + \frac{1}{2}k_2)$$
$$k_4 = hf(x + h, y + k_3)$$

h is the step-size vector chosen , usually $h < 1$.[1]

The method (1.2) in Butcher-array form can be written as

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

The RKN method for second order ODEs of the form (1.2) is given by

$$y_{n+1} = y_n + hy'_n + \frac{1}{6}(k_1 + k_2 + k_3) \qquad (1.5)$$

$$y'_{n+1} = y'_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where,

$$k_1 = \frac{1}{2}hf(x_n, y_n, y'_n)$$
$$k_2 = \frac{1}{2}hf\{x_n + \frac{1}{2}h, y_n + \frac{1}{2}h\left(y'_n + \frac{1}{2}k_1\right), y'_n + k_1\}$$
$$k_3 = \frac{1}{2}hf\{x_n + \frac{1}{2}h, y_n + \frac{1}{2}h\left(y'_n + \frac{1}{2}k_1\right), y'_n + k_2\}$$
$$k_4 = \frac{1}{2}hf\{x_n + h, y_n + h\left(y'_n + k_3\right), y'_n + 2k_3\}$$

The method (1.4) in Butcher-array form can be written as

$$\begin{array}{c|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{array}$$

Both the RK and RKN methods could be expanded into Taylor's series [1].

The work is organized as follows: In section 2 we will show how the Butcher's Runge-Kutta methods for the first order differential equations tableau are modified to include second derivative(that is RKN method), this idea will be used in section 3 to illustrate the main derivation of the fifth order explicit Runge-Kutta method for solution of second order ODEs, some numerical experiments are presented in section 4, finally, the conclusion section 5.

Butcher's Runge-kutta methods for the first order differential equations

Butcher [5], defined an s-stage implicit Runge-Kutta methods for the first order differential equations (1.1) in the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} w_i k_i \tag{1.6}$$

Where for i = 1, 2 - - - - s.

$$k_i = f(x_i + \alpha_j h, y_n + h\sum_{j=1}^{s} a_{ij} k_j)$$

The real parameters α_j, k_i, a_{ij} define the method. The method (1.6) in Butcher-array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W^T \end{array}$$

Where a_{ij} = β

An s-stage Runge-Kutta Nyström for direct integration of second order IVPs (1.2) is defined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j \tag{1.7}$$

$$y'_{n+1} = y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij} k_j$$

Where for i = 1, 2 - - - - s.

$$K_i = f(x_i + \alpha_j h, y_n + \alpha_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j, y'_n + h\sum_{j=1}^{i-1} \bar{a}_{ij} k_j)$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}$ define the method, the method (1.7) in Butcher – array form.

$$\begin{array}{c|c|c} \alpha & \bar{A} & A \\ \hline & \bar{b}^T & b \end{array}$$

A = a_{ij} = β^2 Ā = $\bar{a}_{ij}$ =β

β = βe $\bar{b}$ = W b = $W^T\beta$

(see [6-8])

Definition 1 Order and Error Constant of Runge-Kutta Methods

A first and second order ODEs methods are said to be of order p if p is the largest integer for which

$$y(x+h) - y(x) - h\phi(x, y(x), h) = 0(h^{p+1}) \tag{1.8}$$

$$y(x+h) - y(x) - h^2\phi(x, y(x), y'(x), h^2) = 0(h^{p+2}) \tag{1.9}$$

$$y(x+h) - y(x) - h^3\phi(x, y(x), y'(x), y''(x), h^3) = 0(h^{p+3}) \tag{2.0}$$

Holds respectively. Where

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + \dots\dots\dots + \frac{h^s}{s!} y^s(x) \tag{2.1}$$

$\phi(x, y(x), h) = y'(x+h) \equiv f(x, y(x))$, $\phi(x, y(x), y'(x), h^2) = y''(x+h) \equiv f(x, y(x), y'(x))$ and $\phi(x, y(x), y'(x), y''(x), h^3) = y'''(x+h) \equiv f(x, y(x), y'(x), y''(x))$ in the Taylor series expansion about x_o and compare coefficients of $h^k y^k(x_o)$, $y(x_o)$ is the interval value. The coefficients for which p is the largest integer is known as the error constant (See Lambert [9]).

Proposisition 1

An order P method for a G(N) order ODEs extended for a higher order G(N+1) ODEs has order P-1,where G(1),G(2),….,G(N) denote first order, second order, …nth order respectively.

Proof: From definition 1 let $h^k y^k(x_o)$ represent the largest integer for which G(N) equation(1.8) holds, implies $k = p+1+0(h^{p+1})$ then for G(N+1) equation(1.9) $k = p+2+(-1+0(h^{p+1}))$ and for G(N+2) equation (2.0) $k = p+3+(-2+0(h^{p+1}))$ such that $0(h^{p+1}), (-1+0(h^{p+1})) = 0(h^{p+2})$ and $(-2+0(h^{p+1})) = 0(h^{p+3})$ are the order of G(N), G(N+1) and G(N+2) respectively.

Derivation of the Fifth Order Explicit Runge-Kutta Method

Consider the sixth-order eight-stage Runge-Kutta Method for (1.1)

$$y_{n+1} = y_n + \frac{h}{840}(41k_1 + 216k_3 + 27k_4 + 272k_5 + 27k_6 + 216k_7 + 41k_8) \tag{2.2}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{9}h, y_n + h(\frac{1}{9}k_1)$$

$$k_3 = f(x_n + \frac{1}{6}h, y_n + \frac{h}{24}(k_1 + 3k_2)$$

$$k_4 = f(x_n + \frac{1}{3}h, y_n + \frac{h}{6}(k_1 - 3k_2 + 4k_3)$$

$$k_5 = f(x_n + \frac{1}{2}h, y_n + \frac{h}{8}(-5k_1 + 27k_2 - 24k_3 + 6k_4)$$

$$k_6 = f(x_n + \frac{2}{3}h, y_n + \frac{1}{9}(221k_1 - 981k_2 + 867k_3 - 102k_4 + k_5)$$

$$k_7 = f(x_n + \frac{5}{6}h, y_n + \frac{h}{48}(-183k_1 + 678k_2 - 472k_3 - 66k_4 + 80k_5 + 3k_6)$$

$$k_8 = f(x_n + h, y_n + \frac{h}{82}(716k_1 - 2079k_2 + 1002k_3 + 834k_4 - 454k_5 - 9k_6 + 72k_7)$$

0	$\frac{1}{9}$							
$\frac{1}{9}$	$\frac{1}{24}$	$\frac{1}{8}$						
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{-1}{2}$	$\frac{2}{3}$					
$\frac{1}{3}$	$\frac{-5}{8}$	$\frac{27}{8}$	-3	$\frac{3}{4}$				
$\frac{1}{2}$	$\frac{221}{9}$	-109	$\frac{289}{3}$	$\frac{-34}{3}$	$\frac{1}{9}$			
$\frac{2}{3}$	$\frac{-61}{16}$	$\frac{113}{8}$	$\frac{-59}{6}$	$\frac{-11}{8}$	$\frac{5}{3}$	$\frac{1}{16}$		
$\frac{5}{6}$								
1	$\frac{358}{41}$	$\frac{-2079}{82}$	$\frac{501}{41}$	$\frac{417}{41}$	$\frac{-227}{41}$	$\frac{-9}{82}$	$\frac{36}{41}$	
	$\frac{41}{840}$	0	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{34}{105}$	$\frac{9}{280}$	$\frac{9}{35}$	$\frac{41}{840}$

Using (1.7) we obtain in Butcher – array form

$$
\begin{array}{c|cccccccc|cccccccc}
0 & & & & & & & & & & & & & & & \\
\frac{1}{9} & \frac{1}{9} & & & & & & & & & & & & & & \\
\frac{1}{6} & \frac{1}{24} & \frac{1}{8} & & & & & & & \frac{1}{72} & & & & & & \\
\frac{1}{3} & \frac{1}{6} & \frac{-1}{2} & \frac{2}{3} & & & & & & \frac{-1}{36} & \frac{1}{12} & & & & & \\
\frac{1}{2} & \frac{-5}{8} & \frac{27}{8} & -3 & \frac{3}{4} & & & & & \frac{3}{8} & \frac{-3}{4} & \frac{1}{2} & & & & \\
\frac{2}{3} & \frac{221}{9} & -109 & \frac{289}{3} & \frac{-34}{3} & \frac{1}{9} & & & & \frac{-181}{18} & \frac{217}{12} & \frac{-71}{9} & \frac{1}{12} & & & \\
\frac{5}{6} & \frac{-61}{16} & \frac{113}{8} & \frac{-59}{6} & \frac{-11}{8} & \frac{5}{3} & \frac{1}{16} & & & \frac{205}{144} & \frac{-83}{48} & \frac{5}{48} & \frac{13}{24} & \frac{1}{144} & & \\
1 & \frac{358}{41} & \frac{-2079}{82} & \frac{501}{41} & \frac{417}{41} & \frac{-227}{41} & \frac{-9}{82} & \frac{36}{41} & & \frac{-131}{41} & \frac{87}{41} & \frac{343}{82} & -\frac{-675}{164} & \frac{119}{82} & \frac{9}{164} & \\
\hline
 & \frac{41}{840} & 0 & \frac{9}{35} & \frac{9}{280} & \frac{34}{105} & \frac{9}{280} & \frac{9}{35} & \frac{41}{840} & \frac{41}{840} & 0 & \frac{3}{14} & \frac{3}{140} & \frac{17}{105} & \frac{3}{280} & \frac{3}{70}
\end{array}
$$

Putting the array coefficients in equation form(1.7) we obtained a fifth order explicit Runge-Kutta method for direct integration of second order ODEs everywhere on the interval of solution Yakub etal [10] and Chollom and Jackiewicz [11] given by

$$
\begin{aligned}
y_{n+1} &= y'_n + hy''_n + h^2\left(\frac{41}{840}k_1 + 0k_2 + \frac{3}{14}k_3 + \frac{3}{140}k_4 + \frac{17}{105}k_5 + \frac{3}{280}k_6 + \frac{3}{70}k_7 + 0k_8\right) \\
y'_{n+1} &= y'_n + h\left(\frac{41}{840}k_1 + 0k_2 + \frac{9}{35}k_3 + \frac{9}{280}k_4 + \frac{34}{105}k_5 + \frac{9}{280}k_6 + \frac{9}{35}k_7 + \frac{41}{840}k_8\right) \qquad (2.3)
\end{aligned}
$$

where

$$k_1 = f(x_n, y_n, y'_n)$$

$$k_2 = f\left(x_n + \frac{1}{9}h, y_n + \frac{1}{4}hy'_n + h^2(0), y'_n + h\left(\frac{1}{9}k_1\right)\right)$$

$$k_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy'_n + h^2\left(\frac{1}{72}k_1\right), y'_n + h\left(\frac{1}{24}k_1 + \frac{1}{8}k_2\right)\right)$$

$$k_4 = f\left(x_n + \frac{3}{4}h, y_n + \frac{3}{4}hy'_n + h^2\left(-\frac{1}{36}k_1 + \frac{1}{12}k_2\right), y'_n + h\left(\frac{1}{6}k_1 - \frac{1}{2}k_2 + \frac{2}{3}k_3\right)\right)$$

$$
\begin{aligned}
k_5 = f\Bigg(&x_n + \frac{1}{2}h, y_n + hy'_n + \frac{(\frac{1}{2}h)^2}{2}y''_n + h^3\left(-\frac{1}{16}k_1 + \frac{1}{16}k_2\right), y'_n + \frac{1}{2}hy''_n + h^2\left(\frac{3}{8}k_1 - \frac{3}{4}k_2 + \frac{1}{2}k_3\right), \\
&y''_n + h\left(-\frac{5}{8}k_1 + \frac{27}{8}k_2 - 3k_3 + \frac{3}{4}k_4\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
k_6 = f\Bigg(&x_n + \frac{2}{3}h, y_n + \frac{2}{3}hy'_n + h^2\left(-\frac{181}{18}k_1 + \frac{217}{12}k_2 - \frac{71}{9}k_3 + \frac{1}{12}k_4\right), \\
&y'_n + h\left(\frac{221}{9}k_1 - 109k_2 + \frac{289}{3}k_3 - \frac{34}{3}k_4 + \frac{1}{9}k_5\right)\Bigg)
\end{aligned}
$$

$$k_7 = f(x_n + \frac{5}{6}h, y_n + \frac{5}{6}hy'_n + h^2(\frac{205}{144}k_1 - \frac{23}{48}k_2 + \frac{5}{48}k_3 + \frac{13}{24}k_4 + \frac{1}{144}k_5),$$
$$y'_n + h(-\frac{61}{16}k_1 + \frac{113}{8}k_2 - \frac{59}{6}k_3 - \frac{11}{8}k_4 + \frac{5}{3}k_5 + \frac{1}{16}k_6))$$
$$k_5 = f(x_n + h, y_n + hy'_n + h^2(-\frac{131}{41}k_1 + \frac{87}{41}k_2 + \frac{343}{82}k_3 - \frac{675}{164}k_4 + \frac{119}{82}k_5 + \frac{9}{164}k_6),$$
$$y'_n + h(\frac{358}{41}k_1 - \frac{2079}{82}k_2 + \frac{501}{41}k_3 + \frac{417}{41}k_4 - \frac{227}{41}k_5 - \frac{9}{82}k_6 - \frac{36}{41}k_7))$$

Numerical Experiment

Problem 1

$y'' - xy' + 4y = 0 \quad y(0) = 3,\ y'(0) = 0, \quad h = 0.1$

Exact solution is $y(x) = x^4 - 6x^2 + 3$

Table 1: Approximate error of Problem 1

T	RKN	ERKM
0.1	2.E-07	1.E-09
0.2	4.E-07	3.E-09
0.3	6.E-07	3.E-09
0.4	7.E-06	2.E-09

Problem 2 (Theresa and Danny [12]). The classical harmonic oscillator

$\ddot{x}(t) = -\frac{k}{m}x(t)$, $\quad x(0) = 1, \quad \dot{x}(0) = 1$, k=1,m=1 h=0.1, $0 \le t \le 0.4$

Theoretical Solution: $x(t) = \cos x(t) + \sin x(t)$

Table 2: Approximate error of Problem 2

T	NUMEROV (Yusuph[11])	ERKM
0.1	2.E-07	8.E-11
0.2	4.E-07	4.E-10
0.3	6.E-07	1.E-09
0.4	7.E-06	2.E-09

Conclusion

Through the approach presented in this paper, the ERKM method can be extended to solve higher order differential equations. The method requires less work with very little cost (when compared with classical RK) and possesses a gain in efficiency (when compared with RKN).

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