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Loop Argument approach for a Navier–Stokes Equations in a Strip with Excluded Pressure Term

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Abstract In this article, we improve on previous results in Anthony (2016)[2] where phase-portrait analysis was used to obtain bounds for Navier-Stokes Equations in uniformly local spaces; this concerned situation where the pressure term had been removed artificially. This paper focused on using a new approach termed Loop Argument method to handle a more general situation than the phase-portrait one that applies only on deliberate pressure omission in the momentum equation.

Keywords Navier – Stokes Equations, phase-portrait analysis, Loop Argument criteria

1. Introduction

This paper is concerned with the use of *Loop Argument criteria* on Navier Stokes Equations in model situation where a standard method has been used to eliminate pressure rather than merely omitting the pressure in the system. The uniformly local spaces is the phase-space used for obtaining bounds for the solutions.

We consider the following Navier-Stokes system:

$$\begin{cases} \partial_t u + (u, \nabla) - \Delta u + \nabla p = f \\ divu = 0, \ \ u|_{\partial\Omega} = 0, \ \ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

is considered in $\Omega = \mathbb{R} \times [-1,1]$.

The global in time estimate for $2D$ Navier – Stokes equations was first obtained for bounded domains in the works of Ladyzhenkiya(1972)[1]. Later on, the unbounded domain case was treated by Abergel (1979)[2] and Babin(1992)[3], and the forcing term was required to lie in some weighted space. However the dimension estimates of the attractor for more general forces.

We know that, based on energy estimate, we can obtain energy solutions for (1.1) by multiplying through by u and integrating over the domain Ω and use the fact that the nonlinear term disappears:

$$\big((u, \nabla), u\big) = \int_{x\in\Omega} (u(x), \nabla) u(x). u(x) dx \equiv 0 \tag{1.2}$$

for every divergence free vector field with Dirichlet boundary conditions.

The situation is completely different when the domain Ω is unbounded because the space of square integrable (divergence free) vector field is not a convenient phase space to work with as we are unable to multiply u because doing so the integral will not make sense. In an unbounded domain, the quest for estimates is intended to have the assumption that $u \in L^2(\Omega) \Rightarrow u(x) \to 0 \ as \ |x| \to \infty$ which is too restrictive a decay condition. So under this choice of the phase space many hydro dynamical objects like Poiseuille flows (infinite energy), Kolmogorov flows etc. cannot be considered in the circumstances; because of the above restrictions hold on our model equation (1.1) we are unable to consider constant solutions space periodic solutions etc. which will hinder us from capturing physically relevant solutions.

Overcoming the above obstacles is a work in Zelik (2007)[4] where the weighted energy theory was fully developed for $2D$ Navier – Stokes problem in a strip $\Omega = \mathbb{R} \times [-1,1]$. In this paper, we want to neglect the pressure term of Navier – Stokes system by not adopting any specific method available to excluding pressure. We work in uniformly local spaces and use the phase portrait method to determine bounds for the amended Navier – Stokes system.

2. Preliminaries

2.1. Uniform and Weighted Energy Spaces

In this section, we introduce and briefly discuss the weighted and uniformly local spaces which are the main technical tools to deal with infinite-energy solutions, see Zelik (2007)[4] for more detailed exposition. These tools will help us to obtain estimates for our equations (1.1) in unbounded domain $\Omega = \mathbb{R}[-1,1]$.We explain the space as follows: Let us define $B^1_{x0} \ = 1$ - a unit rectangle centered at $(x_0,0)$ represented as:

$$B^1_{x0} \ = \left(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right) \times (-1,1), x_0 \in \Re \qquad 2.1.1$$

Let us briefly state the definition and basic properties of weight functions and weighted functional spaces as presented by Zelik (2003)[5], Anthony and Zelik (2014)[6], Triebel (1978)[7] and the references therein. Which will be systematically used throughout this project (see also Efendiev and Zelik (2002)[8] for more details). We start with the class of admissible weight functions.

Definition 2.1: *A function $\phi \ \in C_{loc}(\mathbb{R})$ is weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:*

$$\phi(x+y) \le C_\phi \phi(x) e^{\mu|y|}, \phi(x) > 0, \qquad (2.1.2)$$

For a $x, y \ \in \Omega = \mathbb{R}$

We now introduce a class of weighted Sobolev spaces in a regular unbounded domains associated with weights introduced above. We need only the case where $\Omega = \mathbb{R} \times [-1,1]$ is a strip which obviously have regular boundary. One would like to ask why we need weighted Sobolev Spaces; recall that the uniformly local spaces encountered some deficiencies in that they are not differentiable when the supremum is involved but the weighted energy spaces resolve this problem.

Definition 2.2:

$$L^p_\phi(\Omega) = \{u \in L^p_{loc}(\Omega), \|u\|^p_{L^p_\phi} = \int \phi^p(x)|u(x)|^p dx < \infty\}$$

And

$$L^p_{b,\phi}(\Omega) = \{u \in L^p_{loc}(\Omega), \|u\|^p_{L^p_{b,\phi}} = \sup_{x_0 \in \Re}(\phi(x)\|u\|_{L^p(B^1_{x_0})})\}$$

$x, y \ \in \Omega = \mathbb{R}$

The uniformly local space $L^p_{b,\phi}(\Omega)$ consists of all functions $u \in L^p_{loc}(\Omega)$ for which the following norm is finite

$$\|u\|^p_{L^p_b} = \sup_{x_0 \in \Re}\|u\|_{L^p(B^1_{x_0})} < \infty$$

.

If $u \in L^\infty \Rightarrow u \in L^2_b$, and $\|u\|_{L^2_b} \le C\|u\|_{L^\infty}$. This is because all functions that are bounded in L^∞ are also bounded in L^2_b but the reverse is not true.

$$\|u\|_{L^p(B^1_{x_0})} \le |B^1_{x_0}| \|u\|_{L^\infty(B^1_{x_0})} \le C\|u\|_{L^\infty(\Omega)}$$

Similarly, the uniformly local Sobolev spaces $H^s_b(\Omega)$ consist of all functions $u \in H^s_{loc}(\Omega)$ for which the following norm is finite:

$$\|u\|_{H^s_b(\Omega)} = \sup_{x_0 \in \Re}\|u\|_{H^s(B^1_{x_0})} < \infty$$

Where H^s is the space of all distributions whose derivative up to order s is in L^2. The following Lemma establishes the relationship between the spaces L^2_ϕ and L^2_b.

Lemma 2.2. *Let ϕ be a weight function of exponential growth rate, where $\phi_{x_0}(x) = \phi_{x_0}(x - x_0)$, satisfying*

$\int \phi^2 dx < \infty$ *then the following inequalities hold*

$$\|u\|_{L^2_\phi} \le C_1 \|u\|^2_{L^2_b} \bullet \int \phi^2 dx \tag{2.1.3}$$

$$\|u\|^2_{L^2_\phi} \le C_2 \sup_{x_0 \in \Re} \|u\|^2_{L^2_\phi x_0} \tag{2.1.4}$$

Where C_1 *and* C_2 *depend only on* C_ϕ *and* τ .

Lemma 2.3. *Let* ϕ_ε *be a weight function defined by (2.1.2). Then, for all l and of exponential growth rate, the map* T_{ϕ_ε} is an isomorphism between $W^{l,p}(\Omega)$ *and* $W_\phi^{l,p}(\Omega)$ *and the following inequalities hold:*

$$C_1 \|\phi u\|^2_{W^{l,p}} \le \|u\|^2_{W^{l,p}} \le C_2 \|\phi u\|^2_{W^{l,p}} \tag{2.1.5}$$

Where C_1 and C_2 are independent of ε but may depend on l and p.

We define the following special weights for use in the sequel:

$$\phi_\varepsilon(x) = (1 + \varepsilon^2 |x|^2)^{-\frac{1}{2}} . \tag{2.1.5*}$$

This weight, satisfies the following property:

$$\left|\phi_\varepsilon{}'(x)\right| \le C\varepsilon\phi_\varepsilon(x)^2 < C\varepsilon\phi_\varepsilon(x)$$

2.2. Estimates for Navier – Stokes Equations without Pressure

For the sake of clarity, we wish to repeat some details of Anthony (2016)[9] before we transit into the loop argument criteria. We want to prove that any sufficiently regular solution of the Navier – Stokes problem; (1.1) in a cylinder satisfies the uniformly local estimate:

$$\|u(t)\|_{L^2_b(\Omega)} \le Q\left(\|u(0)\|_{L^2_b(\Omega)}\right) + Q\left(\|f\|_{L^2_b(\Omega)}\right) \tag{2.2.1}$$

for some monotone function Q. The first difficulty here is that, in contrast to the case of usual energy solutions, the function $t \to \|u(t)\|^2_{L^2_b(\Omega)}$ is not differentiable due to the presence of supremum in the definition of L^2_b -norm. This does not allow us to obtain estimate (2.2.1) directly by multiplying the equation by the appropriate factor and use Gronwall's inequality. Instead, following the general strategy in, we deduce the weighted energy estimates as an intermediate step; multiplying the equation by $\phi^2 u$ where ϕ is a proper weight function. If we succeed to verify the analogue of (2.2.1) in all weighted spaces $L^2_{\phi x_0}(\mathbb{R})$ uniformly with respect to all shifts $x_0 \in \mathbb{R}$, the desired uniformly local estimate will obtained by taking the supremum with respect to $x_0 \in \mathbb{R}$ and using **Lemma 2.2**. Thus, we need to multiply equation (1.1) by $\phi^2 u$ where $\phi = \phi(x)$ is an appropriate weight function in x_1 direction. But the nonlinear term will still remain unresolved since it will not disappear as in the bounded case. In fact it will produce a cubic nonlinearity $\phi' u^3$. Note that the cubic term is not clear how to control the cubic term in order to produce an a priori estimate. Another setback is the fact that $\phi^2 u$ is not divergence free so the pressure p does not disappear in the weighted energy equality and $(\phi\phi' p, u)$ zill pose a problem closely related with finding a reasonable extension of the Helmholtz projector (to divergence free vector fields) to uniformly local spaces. In summary, we have at least two hurdles to overcome in order to close our estimates: to estimate the cubic term $\phi' u^3$produced by the nonlinear term and $(\phi' p, u)$ when we multiply the momentum equation (1.1) by $\phi^2 u$ and integrate over the domain Ω. Let us put this in perspective to have a clearer view of the terms when we multiply (1.1) by $\phi^2_\epsilon(x_1)u$ $(where\ \phi_\epsilon(x_1)$ is defined as in (2.1.5*) and ϵ is a small positive constant to be determined later) and integrate over Ω to obtain:

$$(\partial_t u, \phi^2 u) + \left((u,\nabla)u, \phi^2 u\right) - (\nu\Delta u, \phi^2 u) + (\Delta p, \phi\phi^2 u) = (f, \phi^2 u) \tag{2.2.2}$$

and hence,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} + \left((u,\nabla)u, \phi^2 u\right) + \nu\|u\|^2_{L^2_\phi} + (\Delta p, \phi^2 u) = (f, \phi^2 u) - \nu(u\phi\phi'.\nabla u) \tag{2.2.3}$$

Next, we try and resolve the nasty terms in (2.2.3) i.e. the second and fourth terms on the LHS; this is to help simplify them as much as possible. Using 2D coordinates; we now seek to estimate the non – linear term as follows:

$-\sum_{i,j}^{2}\int_{\Omega}u_j\partial_j u_i.\phi^2 u_i dx = -\sum_{i,j}^{2}\left(u_i u_j\phi^2\partial_j u_i + u_i^2\phi^2\partial_j u_i + u_i^2\partial_j u_i\phi^2\right)dx$

applying the divergent – free condition and collecting like terms we obtain

$2\sum_{i,j}^{2}\int_{\Omega}u_i u_j\phi^2\partial_j u_i dx = -\sum_{i,j}^{2}\int_{\Omega}u_i^2 u_j\partial_j\phi^2 dx$

Simplifying the above sum integral with careful consideration that ϕ is applied in the x_1 direction we obtain:

$$-\sum_{i,j}^{2}\int_{\Omega}u_i^2 u_j\partial_j\phi^2 dx = -\int_{\Omega}\left(u_1^2 u_1\partial_{dx_1}\phi^2 + u_2^2 u_1\partial_{dx_1}\phi^2\right)dx = -\frac{1}{2}\left(\partial_{dx_1}\phi^2, u_1(u_1^2+u_2^2)\right) \le \left(\phi|\phi'|, |u|^3\right) \tag{2.2.4}$$

Next, we have the pressure term to resolve but it will simply not disappear because the equation:

$\int_{\Omega}\nabla p.\phi^2 u dx = -\int_{\Omega}p\nabla(\phi^2 u)dx = -\int_{\Omega}p[\phi^2\nabla u + u\nabla\phi^2]dx = 2(\phi\phi' p, u)$

But the application of the divergence free condition did not help exclude the term with pressure $(\phi\phi' p, u)$ – a difficult piece to estimate.

Just for the moment, we shall proceed with other terms of Navier – stokes equation without the term containing the pressure and compute the estimate with the impression that we shall return later with good theory that will enable us overcome the difficulty posed by $(\phi\phi' p, u)$ and eventually close our estimate for Navier - Stokes equation in an unbounded domain $\Omega \subset R^2$.

Now, recall (2.2.3) using (2.2.4) and ignoring the term $(\phi\phi' p, u)$ we have:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 + \left(\phi|\phi'|, |u|^3\right) + \nu\|u\|_{L_\phi^2}^2 + \le (f, \phi u) - \nu(u\phi\phi'.\nabla u) \tag{2.2.5}$$

The non – linear term is then estimated using corollary 2.1 and Poincare inequality to obtain the following:

$$(\phi\phi', |u|^3) \le (\epsilon\phi^3, |u|^3)_{L^1} \le \epsilon\|u\|_{L_\phi^3}^3 \le C\epsilon\|u\|_{L_\phi^2}^2 \tag{2.2.6}$$

Let us tidy up these bits of the Navier – Stokes equation and write (2.2.5) using (2.2.6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C\epsilon\|u\|_{L_\phi^2}^2\|\nabla u\|_{L_\phi^2} + \nu\|u\|_{L_\phi^2}^2 + \le (f, \phi u) - \nu(u\phi\phi'.\nabla u) \tag{2.2.7}$$

By Cauchy Schwartz inequality on the RHS of (2.2.7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C\epsilon\|u\|_{L_\phi^2}^2\|\nabla u\|_{L_\phi^2} + \nu\|\nabla u\|_{L_\phi^2}^2 \le \|f\|_{L_\phi^2}\|u\|_{L_\phi^2} - \nu\epsilon\|u\|_{L_\phi^2}\|\nabla u\|_{L_\phi^2} \tag{2.2.8}$$

By Young's inequality on (2.2.8) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C\epsilon^2\|u\|_{L_\phi^2}^4 - \frac{\nu\|u\|_{L_\phi^2}^2}{4} + \nu\|u\|_{L_\phi^2}^2 \le \frac{\|f\|_{L_\phi^2}^2}{2\gamma} + \frac{\gamma\|u\|_{L_\phi^2}^2}{2} + \frac{\nu\epsilon^2\|u\|_{L_\phi^2}^2}{2} + \frac{\nu\|\nabla u\|_{L_\phi^2}^2}{2} \tag{2.2.9}$$

This simplifies to:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C\epsilon^2\|u\|_{L_\phi^2}^4 - \frac{\nu\|\nabla u\|_{L_\phi^2}^2}{4} + \frac{\nu\|\nabla u\|_{L_\phi^2}^2}{2} \le \frac{\|f\|_{L_\phi^2}^2}{2\gamma} + \frac{\gamma\|u\|_{L_\phi^2}^2}{2} + \frac{\nu\epsilon^2\|u\|_{L_\phi^2}^2}{2}$$

(2.2.10)

Where γ is a positive constant to be determined later. Applying Poincare inequality on the second term of the RHS of (2.2.10) and assuming $\epsilon \ll 1$ is small enough; the equation reduces to:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C\epsilon^2\|u\|_{L_\phi^2}^4 + \frac{\nu\|\nabla u\|_{L_\phi^2}^2}{4} \le \frac{\|f\|_{L_\phi^2}^2}{2\gamma} + \frac{C\gamma\|\nabla u\|_{L_\phi^2}^2}{2} \tag{2.2.11}$$

Take that $\frac{\nu}{4} \ge \frac{C\gamma}{2}$ for the purpose of achieving a positive linear term that will afford us not only global existence of solution but also dissipative.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C\epsilon^2\|u\|_{L_\phi^2}^4 + \frac{3}{4}C\gamma\|\nabla u\|_{L_\phi^2}^2 \le \frac{\|f\|_{L_\phi^2}^2}{2\gamma} \le \epsilon^{-1}C_1\|f\|_{L_\phi^2}^2 \tag{2.2.12}$$

Where estimate (2.1.1) from lemma 2.1 and $\int\phi_\epsilon^2(x)dx = \frac{C}{\gamma}$ are used in order to obtain the RHS of (2.2.12) and constant C and C_1 are independent of $\epsilon \ge 0$. We concisely get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 - C_2\epsilon^2\|u\|_{L_\phi^2}^4 + C_3\gamma\|\nabla u\|_{L_\phi^2}^2 \le \frac{\|f\|_{L_\phi^2}^2}{2\gamma} \le \epsilon^{-1}C_1\|f\|_{L_\phi^2}^2 \tag{2.2.13}$$

We take $\|u\|_{L_\phi^2}^2 = y(t)$ to obtain the following differential inequality:

$$y'(t) + C_3\gamma y(t) \le \epsilon^{-1}C_1\|f\|_{L_\phi^2}^2 + C_2\epsilon^2 y^2(t) \tag{2.2.14}$$

By change of variable $z = \epsilon y$ and upon using the fact that $\|u(0)\|_{L_\phi^2}^2 \le \|u(0)\|_{L_\phi^2}^2\int\phi\left(\frac{w}{\epsilon}\right)\frac{dz}{\epsilon} = \|u(0)\|_{L_\phi^2}^2\frac{\pi}{\epsilon} < \infty$; we have, first that

$y(0) = \|u(0)\|^2_{L^2_\phi} \leq \frac{C}{\epsilon}\|u(0)\|^2_{L^2_\phi}$ (2.2.15)

Now, recall (3.14) and write it as: $y'(t) + \gamma y(t) \leq \frac{C_f}{\epsilon} + \epsilon^2 y^2(t)$ so that for the initial data $y(0) \leq C\epsilon^{-1}\|u(0)\|^2_{L^2_\phi}$ and by the above scaling, with its initial condition$z(0) \leq C\|u(0)\|^2_{L^2_b}$.

We shall seek to solve (2.2.14) to prove that it has global bounds for solutions because of the positive linear term on the LHS of (2.2.14). we state a Lemma involving a two stage proof of the estimate for (2.2.14): The first part of the proof considers the case of a simple ODE, while the next considers a system of ODE:

3. Results: Loop Argument Criteria and Analysis of a Priori Estimates for the NSE

Theorem 4.1 Let y(t) = $y_\epsilon\ (t)$ be absolutely continuous and satisfy for every small ϵ the following differential inequality:

$y'(t) + \frac{\gamma}{2}y(t) \leq \epsilon^{-1}C_f$

$+ y(t)(\epsilon^2 y(t) - \frac{\gamma}{2})$ (3.1)

$y(0) \leq \epsilon^{-1}C_0$ (3.2)

For some C_f, C_0 $independent\ of\ \epsilon$. Then $y(t)\ is\ globally\ bounded\ for\ all\ t \geq 0$:

$y(t) \leq \epsilon^{-1}(C_0 + \frac{C_f}{\gamma/2})$ (3.3)

If $\epsilon\ is\ small\ enough$:

$\epsilon \leq C(C_f + C_0)$ (3.4)

And constant $C\ is\ independent\ of\ \epsilon \to 0$

Proof:

Unfortunately, the argument based on the above transparent phase-portrait analysis works only in model situation where the pressure term is neglected. By this reason, we present below an alternative proof of the Lemma which works in general situation as well; to this end we reformulate (3.2) as follows:

$z'(t) + \frac{\gamma}{2}z(t) + \leq C_f + z(t)\left(\epsilon z(t) - \frac{\gamma}{2}\right)$ (3.5)

Now, we observe that if we do know, a priori, that the term

$\epsilon z(t) - \frac{\gamma}{2} \leq 0$ (3.6)

Then the estimate is gotten by dropping it out since the inequality (3.5) remains satisfied in the circumstances. We have

$z'(t) + \frac{\gamma}{2}z(t) \leq C_f.$

This by Gronwall inequality gives the following estimates:

$z(t) \leq C_0 e^{-\gamma/2 t} + \frac{C_f}{\frac{\gamma}{2}}$ (3.7)

And for all $t \geq 0, z(t)$ is bounded above by

$z(t) \leq C_0 + C_f$ (3.8)

We emphasize that estimate (3.8) is justified only if ϵ is small enough that (3.6) is true. However, inserting (3.8) into (3.6), we may expect that (3.6) will be true if

$\epsilon \leq \frac{\frac{\gamma}{2}}{(C_0 + C_f)}.$

We want to show that it is indeed true for t$\in \mathbb{R}$ if ϵ satisfies a slightly stronger assumption:

$\epsilon \leq \frac{\frac{\gamma}{4}}{(C_0 + C_f)}$ (3.9*)

Let the time moment

$T^* = sup\{T \geq 0\ for\ which\ \epsilon z(t) \leq \frac{\gamma}{4}\ holds, for\ all\ t \in [0, T]\}$ (3.9)

i.e. $T^*$is the largest time T for which the estimate $\epsilon z(t) \leq \frac{\gamma}{4}$ for all $t \in [0, T]$. If $T^* = \infty$, (3.6) is obviously true and the Lemma is proved. So let us assume that $T^* < \infty$. Since $z(t)$ is continuous then (3.9) implies that $\epsilon z(t) \leq \frac{\gamma}{2}$ for some small $\delta > 0$ therefore for all $t \leq T^* + \delta$ (3.6) holds. If (3.6) holds for $t \in [0, T^* + \delta]$ then estimate (3.8) holds for $t \in [0, T^* + \delta]$ as well. But the validity of (3.8) plus our choice of ϵ (3.9*) implies that $\epsilon z(t) \leq \frac{\gamma}{4}$ for all $t \in [0, T^* + \delta]$ and this gives a contradiction with (3.9). Hence we conclude that $T^* = \infty$ and that $z(t)$ is bounded for all time t with estimate (3.8). That finishes the proof of the Theorem.

We are now ready to complete the derivation of the main estimate (2.2.1): To this end, we apply **Theorem 4.1** to equation (2.2.14). Then, taking $C_f = C||\mathrm{f}||^2_{L^2_b}, and\ C_0 = C||\mathrm{u}_0||^2_{L^2_b}$, we have due to (2.2.18)

$$||u(t)||^2_{L^2_{\emptyset_\epsilon}} \leq \epsilon^{-1}C(||f||^2_{L^2_b} + ||u_0||^2_{L^2_b}) \tag{3.10}$$

for all ϵ satisfying (2.2.19). Note also that the shifted weight $\emptyset_\epsilon(x - x_0)$ satisfies estimate (2.1.2) and (2.1.4) uniformly with respect to $x_0 \in \mathbb{R}$. By this reason, starting with the weight $\emptyset_\epsilon(x - x_0)$ from the very beginning, we end up with:

$$||u(t)||^2_{L^2_{\emptyset_\epsilon(.-x_0)}} \leq \epsilon^{-1}C(||f||^2_{L^2_b} + ||u_0||^2_{L^2_b} \tag{3.11}$$

Where the constant C is independent of ϵ and x_0 and using formula (2.1.4) of **Lemma 2.1** we end up with:

$$||u(t)||^2_{L^2_b} \leq \epsilon^{-1}C(||f||^2_{L^2_b} + ||u_0||^2_{L^2_b}).$$

Finally, fixing the largest possible $\epsilon = C \leq C(||f||^2_{L^2_b} + ||u_0||^2_{L^2_b})^{-1}$, we end up with the desired uniformly local estimate (2.2.1).

Thus we have shown how to derive the desired a priori estimate for the NSE in uniformly local spaces in a more general situation without pressure.

The phase portrait analysis was found slightly deficient to handle a more complex situation hence the motivation for the use of loop argument criteria. Existence, uniqueness and further regularity of solutions may be obtained in a standard way by the Galerkin approximation, see Anthony(2014)[10] for more exposition . We must emphasize here, however, that this method applies only to model situation where pressure is formally omitted.

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