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Research Article

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Validity of Phase Shift Relation for Supersymmetric Partner Potentials

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Abstract A simple analytic relation between the phase shifts produced by two supersymmetric (SUSY) partner potentials remains valid for both shape invariant potential (SIP) and non-SIP. This relationship for the Eckart potential, which is an SIP, has been critically investigated. A possible pitfall in assigning the orbital angular momentum for the supersymmetric partner potential has also been indicated.

Keywords Phase Shift, Supersymmetry, Eckart Potential.

1. Introduction

Supersymmetric Quantum Mechanics (SSQM) [1-2] has been formulated and studied for about three decades which introduced a new conceptual framework for understanding potential problems in Quantum Mechanics using the ideas borrowed from quantum field theory [3]. The concept of Supersymmetry (SUSY) is of great interest, as it provided a new way of relating Hamiltonians with wholly or partially identical energy spectrum. It also explained analytical solvability of a class of potentials, whose partner potentials have the same shape, thus introducing the concept of shape invariance. An exhaustive review of different aspects of SSQM can be found in reported literature [4]. A potential is said to be shape invariant if its SUSY partner potential has the same functional dependence as the original potential, but with different parameters. For any shape invariant potential (SIP), the energy eigenvalue spectra can be obtained algebraically. Using SSQM, from a given Hamiltonian ($H^{(1)}$), a partner Hamiltonian ($H^{(2)}$) can be obtained by deleting the ground state of $H^{(1)}$, the rest of the spectrum of $H^{(1)}$ being identical with that of $H^{(2)}$. The principal result is the energy degeneracy, *viz.*, $E^{(1)}_{n+1}=E^{(2)}_{n}$, ($n$=0,1,2,...), where $E^{(i)}_{n}$ is the energy of the n-th excited (bound) state of $H^{(i)}$ (i=1,2). This procedure can also be extended to the scattering (unbound) states. From the asymptotic form of the scattering wave functions, one can show that the phase shift $\mathrm{d}^{(1)}_{l}(k)$ of the l-th partial wave in $V^{(1)}$ bears a simple relation with the phase shift $\mathrm{d}^{(2)}_{l+1}(k)$ of the (l+1)-th partial wave in $V^{(2)}$ [5]

$$\mathrm{d}^{(2)}_{l+1}(k)=\mathrm{d}^{(1)}_{l}(k)-\tan^{-1}\left(\frac{g_0}{k}\right), \qquad (1)$$

where k is the wave number of the incident wave and $g_0=\sqrt{\frac{2mB_0}{\hbar^2}}$, B_0 being the binding energy of the ground state in $V^{(1)}$.

Now if, $V^{(2)}$ of the partner Hamiltonian $H^{(2)}$ has the same functional shape (but with different parameters a_2 and a_1) as that of the potential $V^{(1)}$ of the original Hamiltonian $H^{(1)}$, *i.e.*

$$V^{(2)}(r;a_1)=V^{(1)}(r;a_2)+R(a_1)$$
$$\text{where} \quad a_2 = f(a_1) \qquad (2)$$

(the remainder $R(a_1)$ being independent of r), then the potential is said to be a 'shape invariant potential (SIP)'.

Then the entire bound state energy spectrum and corresponding energy eigen functions of $H^{(1)}$ can be obtained algebraically [4]. It has been demonstrated that all text book potentials, for which exact analytical solutions are possible, are indeed shape invariant. So, it is natural to expect that Eqs. (1) and (2) will lead to an analytic expression for $d_l^{(1)}(k)$ for all SIPs. But this is true only for a special class of SIPs. Among all known SIPs, for which $f(a_1)$ is a translation, only the Coulomb potential satisfies this criterion [6]. The difference between the phase shifts produced by $V^{(1)}$ and $V^{(2)}$ for both SIP and non-SIP do not obey the SSQM predicted relation, if $V^{(1)}$ and its supersymmetric partner $V^{(2)}$ are taken with l and (l+1) respectively, whereas it is obeyed when both of $V^{(1)}$ and $V^{(2)}$ are taken with the same l. This is to say that if $V^{(1)}$ is the effective potential including the centrifugal repulsion term corresponding to the l-th partial wave, then $V^{(2)}$ is just the SUSY partner of $V^{(1)}$, *without any explicit centrifugal repulsion term corresponding to the (l+1)-th partial wave*. Thus, unless the parameter involved in the shape invariance relation, Eq. (1), changes automatically from l to (l+1), one must not change l for the partner potential. This has been established for the square well and generalized Pöschl-Teller potential [7]. In this communication, I present the calculation for the Eckart potential which is shape invariant (SIP).

This paper is organized as follows. In section 2, the necessary calculation for obtaining the phase shift has been discussed. In section 3, numerical results are discussed and finally the conclusion is drawn in section 4.

2. Calculation of the phase shift for Eckart Potential

Let us consider a spherically symmetric potential $V^{(1)}(r)$ having a continuum part in its energy spectrum. The energy scale is so chosen that the ground state of orbital angular momentum l in this potential has zero energy. Assuming that $V^{(1)}(r)$ goes slower than r^{-2} for r→0 and approaches a finite constant value V∞ for r→∞, the Schrödinger equation for the ground state of orbital angular momentum l in this potential is

$$H^{(1)}y_0^{(1)}(r)=[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+Veff^{(1)}(r)]y_0^{(1)}(r)=0. \qquad (3)$$

where, $Veff^{(1)}(r)=\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}+V^{(1)}(r)$. Then $Veff^{(1)}(r)$ can be expressed as

$$Veff^{(1)}(r)=\frac{\hbar^2}{2m}\frac{y_0^{(1)''}(r)}{y_0^{(1)}(r)}. \qquad (4)$$

The 'superpotential' $W(r)$ is defined as [4]

$$W(r)=-\frac{\hbar}{\sqrt{2m}}\frac{y_0^{(1)'}(r)}{y_0^{(1)}(r)}, \qquad (5)$$

$$\text{so that} \quad V_{eff}^{(1)}(r) = W^2(r) - \frac{\hbar}{\sqrt{2m}}W'(r) \qquad (6)$$

The partner potential $Veff^{(2)}(r)$ is given by

$$V_{eff}^{(2)}(r) = W^2(r) + \frac{\hbar}{\sqrt{2m}} W'(r) \quad (7)$$

From Eqs. (5)–(7), $Veff^{(2)}(r)$ can be expressed as [5]

$$Veff^{(2)}(r) = \frac{\hbar^2}{2m}\frac{(l+1)(l+2)}{r^2} + V^{(1)}(r) - \frac{\hbar^2}{m}\frac{d^2}{dr^2}[\ln\{\frac{y_0^{(1)}(r)}{r^{l+1}}\}]. \quad (8)$$

From Eq. (8), it can be observed that the r→0 behaviour of $Veff^{(2)}(r)$ corresponds to angular momentum l+1, since for r→0, $y_0^{(1)}(r)$ goes as r^{l+1}. Both the potentials $Veff^{(1)}(r)$ and $Veff^{(2)}(r)$ approach the *same value* V∞ in the limit r→∞. This justifies the subscript (l+1) of $d^{(2)}$. $d_{l+1}^{(2)}(k)$ is the phase shift produced by $Veff^{(2)}(r)$, which is the supersymmetric partner of $Veff^{(1)}(r)$, and *not of the potential* $[V^{(2)}(r) + \frac{\hbar^2}{2m}\frac{(l+1)(l+2)}{r^2}]$ (where $V^{(2)}(r)$ is the supersymmetric partner of $V^{(1)}(r)$), even though $d_{l+1}^{(2)}(k)$ corresponds to the phase shift of the (l+1)-th partial wave.

For the Eckart potential [4] (plotted in Fig. 1), the corresponding superpotential $W(r)$ has the form

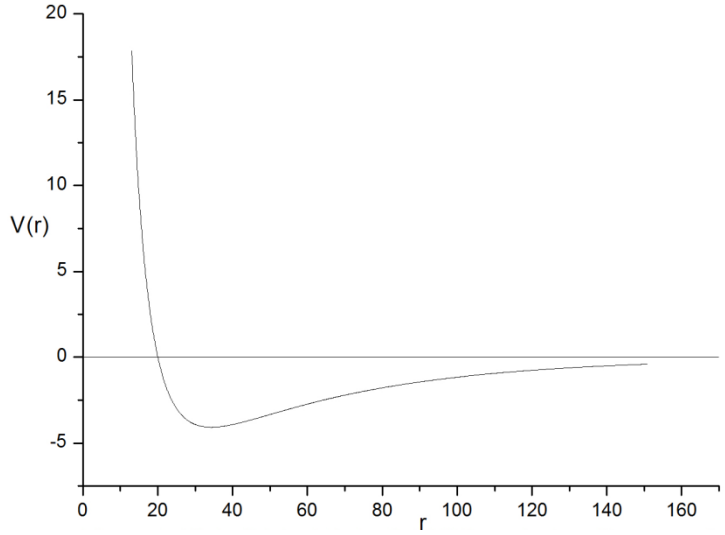

Figure 1: Plot of Eckart Potential (V(r)) against r with parameters A=1.0, B=3.0, α=0.01

$$W(r) = -A\coth\alpha r + \frac{B}{A} \quad (9)$$

where A, B and α are constants (A^2<B).
Substituting $W(r)$ in Eqs. (6) and (7) , we get (for l=0)

$$V_{eff}^{(1)}(r) = A^2 + \frac{B^2}{A^2} + A(A-\alpha)cosech^2\alpha r - 2Bcoth\alpha r$$

$$V_{eff}^{(2)}(r) = A^2 + \frac{B^2}{A^2} + A(A+\alpha)cosech^2\alpha r - 2Bcoth\alpha r \quad (10)$$

For the potential $V(1)eff(r)$ (vanishes asymptotically) the radial Schrödinger equation for the l-th partial wave takes the form

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+V(1)eff(r)-E]y_E^{(1)}(r)=0. \quad (11)$$

The asymptotic solution of Eq. (11) is given by $y_E^{(1)}(r) = \sin[kr - \frac{l\pi}{2} + d_l^{(1)}(k)]$, where $k=\sqrt{\frac{2mE}{\hbar^2}}$. Eq. (11) is numerically solved using the Runga-Kutta algorithm from r→0 (subject to $y_E^{(1)}(0)=0$) to two large values of r (say R_1 and R_2) to obtain $h_1=y_E^{(1)}(R_1)$ and $h_2=y_E^{(1)}(R_2)$. Then

$$d_l^{(1)}(k) = tan^{-1}[\frac{h_2 \sin\left(kR_1-\frac{l\pi}{2}\right)-h_1 \sin\left(kR_2-\frac{l\pi}{2}\right)}{h_1 \cos\left(kR_2-\frac{l\pi}{2}\right)-h_2 \cos\left(kR_1-\frac{l\pi}{2}\right)}] \quad (12)$$

For calculation of $d_{l+1}^{(2)}(k)$, the above procedure can be followed, replacing $V(1)eff(r)$ by its SUSY partner $V(2)eff(r)$ in Eq. (11) without inserting explicit l-dependent term in $V(2)eff(r)$.

3. Numerical Results and Discussion

To calculate the phase shift for the Eckart potential, the corresponding Schrödinger equation is solved numerically by Runga-Kutta algorithm with appropriate boundary conditions for r→0 and r→∞Phase shifts were calculated using Eq. (12). Calculated phase shifts satisfy the supersymmetric relation Eq. (1), within the estimated numerical errors, when $d_l^{(1)}(k)$ and $d_{l+1}^{(2)}(k)$ are obtained from the solutions of the Schrödinger equation with $Veff^{(1)}(r)$ and $Veff^{(2)}(r)$ respectively. It is observed that Eq. (1) is valid when $V^{(1)}$ and $V^{(2)}$ are taken with same l. But the phase shift relation is not valid if $V^{(1)}$ and $V^{(2)}$ are taken with l and l+1, respectively. The calculated results are presented in Table 1.

Table 1. Results of calculation for the Eckart potential. $d_l^{(1)}(k)$ and $d_{l+1}^{(2)}(k)$ stand for the phase shifts produced by the potentials $V(1)eff(r)$ and $V(2)eff(r)$, respectively.

Energy (E)	$d_l^{(1)}(k)$	$d_{l+1}^{(2)}(k)$	$d_l^{(1)}(k)-d_{l+1}^{(2)}(k)$	$\tan^{-1}(\frac{g_0}{k})$
0.310	-1.02774900	0.78960000	1.32424365	1.29928285
0.320	1.07363600	-0.24029300	1.31392900	1.29515353
0.330	0.06006700	-1.24590400	1.30597100	1.29109763
0.340	-0.92826700	0.91697900	1.29634665	1.28711196
0.350	1.24893600	-0.03845700	1.28739300	1.28319352
0.360	0.30534900	-0.97120900	1.27655800	1.27933953
0.370	-0.61461600	1.25799500	1.26898165	1.27554742
0.380	-1.51325300	0.36669900	1.26164065	1.27181477
0.390	0.74867000	-0.50358900	1.25225900	1.26813932
0.400	-0.11220600	-1.35727100	1.24506500	1.26451896

4. Summary and Conclusion

In this work, the relationship between phase shifts produced by two supersymmetric (SUSY) partner potentials has been examined critically for the Eckart potential, which is an SIP. From the numerical results, it is observed that the analytic relation (Eq. (1)) between the phase shifts produced by the l-th partial wave of $V^{(1)}(r)$; i.e, by

the potential $V_{eff}^{(1)}(r)=V^{(1)}(r)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$ and that by its SUSY partner $V_{eff}^{(2)}(r)$ is well obeyed without changing the '*l*' value in $V_{eff}^{(2)}(r)$, even though $d_{l+1}^{(2)}(k)$ corresponds to the $(l+1)$-th partial wave.

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