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Polynomial Inequalities in Regions Bounded by Piecewise Asymptotically Conformal Curve with Nonzero Angles in the Bergman Space

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Abstract We continue the study of estimates of algebraic polynomials in regions bounded by a piecewise asymptotically conformal curve with interior non-zero angles in the weighted Bergman space.

Keywords: Algebraic polynomials, Conformal mapping, Asymptotically conformal curve.

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1 Introduction and Main Results

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Omega := extL := \overline{\mathbb{C}} \backslash \overline{G}$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\Delta := \{w : |w| > 1\}$ and let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$, and $\Psi := \Phi^{-1}$. For $t \geq 1$, $z \in \mathbb{C}$, we set:

$$L_t := \{z : \ |\Phi(z)| = t\} \ (L_1 \equiv L), \ G_t := intL_t, \ \Omega_t := extL_t.$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of distinct points on curve L, located in the positive direction. For some fixed R_0, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \quad z \in G_{R_0}, \tag{1.1}$$

where $\gamma_j > -2$, for all $j = 1, 2, ..., m$, and the function h_0 is uniformly separated from zero in G_{R_0}, i.e. there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that, for all $z \in G_{R_0}$

$$h_0(z) \geq c_0 > 0.$$

For any $p > 0$ and for Jordan region G, let's define:

$$\begin{aligned} \|P_n\|_p &:= \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \, |P_n(z)|^p \, d\sigma_z \right)^{1/p} < \infty, \ 0 < p < \infty; \\ \|P_n\|_\infty &:= \|P_n\|_{A_\infty(1,G)} := \|P_n\|_{C(\overline{G})}, \ p = \infty, \end{aligned} \tag{1.2}$$

where σ_z is the two-dimensional Lebesgue measure.

In this work, we continue the study of the following Nikolskii-type inequality:

$$\|P_n\|_\infty \leq c_1 \lambda_n(G, h, p) \, \|P_n\|_p, \tag{1.3}$$

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where $c_1 = c_1(G, h, p) > 0$ is a constant independent of n and P_n, and $\lambda_n(G, h, p) \to \infty$, $n \to \infty$, depending on the geometrical properties of region G, weight function h and of p. The estimate of (1.3)-type for some $(G,\ p,\ h)$ was investigated in [27, pp.122-133], [17], [26, Sect.5.3], [32], [15], [2]-[8] (see, also, references therein) and others. Further, analogous of (1.3) for some regions and the weight function $h(z)$ were obtained: in [8] for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary without cusps; in [11] for $p > 0$ and for regions bounded by quasiconformal curve; in [7] for $p > 1$ and for regions bounded by piecewise smooth curve without cusps; in [10] for $p > 0$ and for regions bounded by asymptotically conformal curve; in [16] for $p > 0$ and for regions bounded by piesewise smooth curves with interior (zero or nonzero) angles, in [12] for $p > 0$ and for regions bounded by piecewise asymptotically conformal curve having cusps and others.

In this work, we investigate similar problems for $z \in \overline{G}$ in regions bounded by piecewise asymptotically conformal curves having interior nonzero zero angles and for weight function $h(z)$, defined in (1.1) and for $p > 0$.

Now, we begin to give some definitions and notations.

Following [24, p.97], [28], the Jordan curve (or arc) L is called $K-$quasiconformal $(K \geq 1)$, if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let S be a Jordan curve and $z = z(s)$, $s \in [0,\ |S|]$, $|S| := mes\ S$, denote the natural representation of S. *Let z_1, $z_2 \in S$ be an arbitrary points and $S(z_1, z_2) \subset S$ denotes the subarc of S of shorter diameter with endpoints z_1 and z_2. The curve S is a quasicircle if and only if the quantity*

$$\sup_{z_1,z_2\in l;\ z\in l(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \tag{1.4}$$

is bounded. Following to Lesley [25], the curve S to be said "*$c-$quasiconformal*", if the quantity (1.4) bounded by positive constant c, independent from points z_1, z_2 and z. At the literature it is possible to find various functional definitions of the quasiconformal curves (see, for example, [29, pp.286-294], [24, p.105], [13, p.81], [30, p.107]).

The Jordan curve S is called asymptotically conformal [19], [30], if

$$\sup_{z_1,z_2\in S;\ z\in S(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1, \qquad |z_1 - z_2| \to 0. \tag{1.5}$$

We will denote this class as AC, and will write $G \in AC$, if $L := \partial G \in AC$.

The asymptotically conformal curves occupy a special place in the problems of the geometric theory of functions of a complex variable. These curves in various problems have been studied by Anderson, Becker and Lesley [14], Dyn'kin [20], Pommerenke, Warschawski [31], Gutlyanskii, Ryazanov [21], [22], [23] and others. According to the geometric criteria of quasiconformality of the curves ([13, p.81], [30, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [18], [24, p.104]). The same is true for asymptotically conformal curves.

A Jordan arc ℓ is called asymptotically conformal arc, when ℓ is a part of some asymptotically conformal curve.

Now, we define a new class of regions bounded by piecewise asymptotically conformal curve having exterior nonzero "angles" at the connecting points of boundary arcs.

Throughout this work, we will assume that $p > 0$ and the constants $c, c_0, c_1, c_2, ...$ are positive and constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive (generally, are different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. Also note that, for any $k \geq 0$ and $m > k$, notation $j = \overline{k, m}$ denotes $j = k, k+1, ..., m$.

Now, let's introduce "special angles" on L.

Definition 1.1. *We say that a Jordan region $G \in PAC(\nu_1, ..., \nu_m)$, $0 < \nu_j < 2$, $j = \overline{1, m}$, if $L = \partial G$ consists of the union of finite asymptotically conformal arcs $\{L_j\}_{j=1}^{m}$, connected at the points $\{z_j\}_{j=0}^{m} \in L$ such that in z_0- L locally asymptotically conformal and for any $z_j \in L$, $j = \overline{1, m}$, where two arcs L_{j-1} and L_j meet, there exist $r_j := r_j(L, z_j) > 0$ and $\nu_j := \nu_j(L, z_j)$, $0 < \nu_j < 2$, such that for some $0 \leq \theta_0 < 2$ a closed maximal circular sector $S(z_j; r_j, \nu_j) := \left\{\zeta : \zeta = z_j + r_j e^{i\theta\pi},\ \theta_0 < \theta < \theta_0 + \nu_j\right\}$ of radius r_j and opening $\nu_j\pi$ lies in $\overline{G} = \overline{intL}$ with vetrex at z_j.*

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Clearly , that $PAC(\nu_1) \subset PAC(\nu_2)$, if $\nu_2 \geq \nu_1$.

Definition 1.2. *We say that a Jordan region* $G \in PAC(\nu)$, *if* $G \in PAC(\nu_1, ..., \nu_m)$, $0 < \nu_j < 2,\ j = \overline{1,m}$, *where* $\nu = \min\{\nu_j : 0 < \nu_j < 2,\ j = \overline{1,m}\}$.

It is clear from Definition 1.1 (1.2), that each region $G \in PAC(\nu_1, ..., \nu_m)$, $0 < \nu_1, ..., \nu_m < 2$, ($G \in PAC(\nu)$) may have "singularity" at the boundary points $\{z_i\}_{i=1}^m \in L$. If it does not have such "singularity" (in this case we put $\nu_i = 1$, $i = \overline{1,m}$), then it is written as $G \in AC$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^m \in L$ defined in (1.1) and $\{\zeta_i\}_{i=1}^m \in L$ defined in Definition 1.1 (1.2) coincide. Without the loss of generality, we also will assume that the points $\{z_i\}_{i=1}^m$ are ordered in the positive direction on the curve L.

We state our new results. Assume that the curve L have "singularity" on the boundary points $\{z_i\}_{i=1}^m$, i.e., $\nu_i < 1$, for all $i = \overline{1,m}$, and the weight function h have "singularity" at the same points, i.e., $\gamma_i \neq 0$ for some $i = \overline{1,m}$. In this case, we have the following:

Theorem 1.1. *Let* $p > 0$. *Suppose that* $G \in PAC(\nu_1, ..., \nu_m)$ *for some* $0 < \nu_1, ..., \nu_m < 1$; $h(z)$ *defined as in (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and arbitrarily small* $\varepsilon > 0$, *there exists* $c_1 = c_1(G, p, \gamma_j) > 0$ *such that*

$$\|P_n\|_\infty \leq c_1 (n+1)^{\frac{2+\widetilde{\gamma}}{p}(2-\widetilde{\nu})+\varepsilon} \|P_n\|_p \,, \tag{1.6}$$

where $\widetilde{\gamma} := \max\{0, \gamma_i\}$ *and* $\widetilde{\nu} := \min\{\nu_i\}$, $i = \overline{1,m}$.

Theorem 1.2. *Let* $p > 0$. *Suppose that* $G \in PAC(\nu_1, ..., \nu_m)$ *for some* $0 < \nu_1, ..., \nu_m < 1$; $h(z)$ *defined as in (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and arbitrarily small* $\varepsilon > 0$, *there exists* $c_2 = c_2(G, p, \gamma_j) > 0$ *such that*

$$|P_n(z_j)| \leq c_2 \mu_n \|P_n\|_p \,,$$

where

$$\mu_n := \begin{cases} n^{\frac{(2+\gamma_j)(2-\nu_j)}{p}+\varepsilon}, & \text{if } \gamma_j > \frac{1}{2-\nu_j} - 2 - \varepsilon, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \gamma_j = \frac{1}{2-\nu_j} - 2 - \varepsilon, \\ n^{\frac{1}{p}}, & \text{if } -2 < \gamma_j < \frac{1}{2-\nu_j} - 2 - \varepsilon. \end{cases} \tag{1.7}$$

The sharpness of the estimations (1.6) and (1.7) can be discussed by comparing them with the following result:

Remark 1.1. *([9, Theorem 1.15], [2]) For any* $n \in \mathbb{N}$ *there exists a polynomials* $Q_n^*, T_n^* \in \wp_n$ *such that for unit disk* B *and weight function* $h^*(z) = |z - z_1|^2$ *the following is true:*

$$|Q_n^*(z)| \geq c_6 n \|Q_n^*\|_{A_2(B)} \,, \quad \text{for all } z \in \overline{B};$$
$$|T_n^*(z_1)| \geq c_7 n^2 \|T_n^*\|_{A_2(h^*,B)} \,;$$

2 Some Auxiliary Results

Throughout this work, for the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1. *[1] Let* L *be a* $K-$*quasiconformal curve,* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. *Then*

a) *The statements* $|z_1 - z_2| \preceq |z_1 - z_3|$ *and* $|w_1 - w_2| \preceq |w_1 - w_3|$ *are equivalent. So are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$;

b) *If* $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2},$$

where $0 < r_0 < 1$, $R_0 := r_0^{-1}$ *are constants, depending on* G.

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Lemma 2.2. *[25, p.342] Let L be an asymptotically conformal curve. Then, Φ and Ψ are Lipα for all $\alpha < 1$ in $\overline{\Omega}$ and $\overline{\Delta}$, correspondingly.*

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (1.1).

Lemma 2.3. *[5] Let L be a $K-$quasiconformal curve; $h(z)$ is defined in (1.1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n = 1, 2, ...,$ we have*

$$\|P_n\|_{A_p(h,G_R)} \preceq \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \ p > 0, \tag{2.1}$$

where $\widetilde{R} = 1 + c(R-1)$ and c is independent from n and R.

3 Proof of Theorems

3.1 Proof of Theorem 1.1

Proof. Suppose that $G \in PAC(\nu_1, \nu_2)$ for some $0 < \nu_1, \nu_2 < 1$ and $h(z)$ is defined as in (1.1). Let $\{\xi_j\}$, $1 \le j \le m \le n$, be the zeros (if any exist) of $P_n(z)$ lying on Ω. Let's define the function Blaschke with respect to the zeros $\{\xi_j\}$ of the polynomial $P_n(z)$:

$$\widetilde{B}_j(z) := \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)}, \ z \in \Omega, \tag{3.1}$$

and let

$$B_m(z) := \prod_{j=1}^{m} \widetilde{B}_j(z), \ z \in \Omega. \tag{3.2}$$

It is easy that the

$$B_m(\xi_j) = 0, \ |B_m(z)| \equiv 1, \ z \in L; \ |B_m(z)| < 1, \ z \in \Omega. \tag{3.3}$$

Then, for each ε_1, $0 < \varepsilon_1 < 1$, there exists a circle $\left\{w : |w| = R_1 := 1 + \varepsilon_2, 0 < \varepsilon_2 < \frac{\varepsilon_1}{n}\right\}$ such that for any $j = 1, 2,$ the following holds:

$$\left|\widetilde{B}_j(\zeta)\right| > 1 - \varepsilon_2, \ \zeta \in L_{R_1}.$$

So, from (3.2), we get:

$$|B_m(\zeta)| > (1 - \varepsilon_2)^m \succeq 1, \ \zeta \in L_{R_1}. \tag{3.4}$$

For any $p > 0$ and $z \in \Omega$ let us set:

$$Q_{n,p}(z) := \left[\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right]^{p/2}. \tag{3.5}$$

The function $Q_{n,p}(z)$ is analytic in Ω, continuous on $\overline{\Omega}$, $Q_{n,p}(\infty) = 0$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $Q_{n,p}(z)$ and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region Ω, we have:

$$Q_{n,p}(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} Q_{n,p}(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in \Omega_{R_1}. \tag{3.6}$$

According to (3.1) - (3.5), we have:

$$\begin{aligned} |P_n(z)|^{p/2} &= \frac{\left|B_m(z)\Phi^{n+1}(z)\right|^{\frac{p}{2}}}{2\pi d(z, L_{R_1})} \int_{L_{R_1}} \left|\frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)}\right|^{p/2} |d\zeta| \\ &\preceq \left|\Phi^{n+1}(z)\right|^{\frac{p}{2}} \int_{L_{R_1}} |P_n(\zeta)|^{p/2} \frac{|d\zeta|}{|\zeta - z|}. \end{aligned} \tag{3.7}$$

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Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, we obtain:

$$\left(\int\limits_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} |d\zeta|\right)^2 \leq \int\limits_{|t|=R_1} h(\Psi(t)) |P_n(\Psi(t))|^p |\Psi'(t)|^2 |dt| \cdot \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \tag{3.8}$$

$$\leq \int\limits_{|t|=R_1} h(\Psi(t)) |P_n(\Psi(t))|^p |\Psi'(t)|^2 |dt| \cdot \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2}$$

$$= \int\limits_{|t|=R_1} |f_{n,p}(t)|^p |dt| \cdot \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} =: A_n \cdot D_n(w),$$

where $f_{n,p}(t) := h^{\frac{1}{p}}(\Psi(t)) P_n(\Psi(t)) (\Psi'(t))^{\frac{2}{p}}$, $|t| = R_1$.

For the estimate integral A_n, we divide the circle $|t| = R_1$ into n equal parts δ_n with $mes\delta_n = \frac{2\pi R_1}{n}$ and by applying the mean value theorem, we get:

$$A_n := \int\limits_{|t|=R_1} |f_{n,p}(t)|^p |dt|$$

$$= \sum_{k=1}^{n} \int\limits_{\delta_k} |f_{n,p}(t)|^p |dt| = \sum_{k=1}^{n} \left|f_{n,p}\left(t_k'\right)\right|^p mes\delta_k, \quad t_k' \in \delta_k.$$

On the other hand, by applying mean value estimation

$$\left|f_{n,p}\left(t_k'\right)\right|^p \leq \frac{1}{\pi\left(|t_k'| - 1\right)^2} \iint\limits_{|\xi - t_k'| < |t_k'| - 1} |f_{n,p}(\xi)|^p d\sigma_\xi,$$

we obtain:

$$(A_n)^2 \preceq \sum_{k=1}^{n} \frac{mes\ \delta_k}{\pi\left(|t_k'| - 1\right)^2} \iint\limits_{|\xi - t_k'| < |t_k'| - 1} |f_{n,p}(\xi)|^p d\sigma_\xi, \ t_k' \in \delta_k.$$

By taking into account, at most two of the discs with center t_k' are intersecting, we have:

$$A_n \preceq \frac{mes\delta_1}{\left(|t_1'| - 1\right)^2} \iint\limits_{1<|\xi|<R} |f_{n,p}(\xi)|^p d\sigma_\xi \preceq n \cdot \iint\limits_{1<|\xi|<R} |f_{n,p}(\xi)|^p d\sigma_\xi.$$

According to Lemma 2.3, for A_n we get:

$$A_n \preceq n \iint\limits_{G_R \setminus G} h(\zeta) |P_n(\zeta)|^p d\sigma_\zeta \preceq n \cdot \|P_n\|_p^p. \tag{3.9}$$

To estimate the integral $D_n(w)$, denoted by $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, for any fixed $\rho > 1$,we introduce:

$$\Delta_1(\rho) := \left\{t = re^{i\theta} : r > \rho, \ \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\}, \tag{3.10}$$

$$\Delta_2(\rho) := \left\{t = re^{i\theta} : r > \rho, \ \frac{\varphi_1 + \varphi_2}{2} \leq \theta < \frac{\varphi_1 + \varphi_0}{2}\right\};$$

$$\Delta_j := \Delta_j(1), \ \Omega^j := \Psi(\Delta_j), \ \Omega_\rho^j := \Psi(\Delta_j(\rho));$$
$$L^j := L \cap \overline{\Omega}^j, \ L_\rho^j := L_\rho \cap \overline{\Omega}_\rho^j, \ j = 1, 2; \ L = L^1 \cup L^1, \ L_\rho = L_\rho^1 \cup L_\rho^2.$$

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Under these notations, from (3.8) for the $D_n(w)$, we get:

$$D_n(w) = \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2} \tag{3.11}$$

$$\preceq \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\prod\limits_{j=1}^{2} |\Psi(t)-\Psi(w_j)|^{\gamma_j}\,|\Psi(t)-\Psi(w)|^2}$$

$$\asymp \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_j)|^{\gamma_j}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{j=1}^{2} D_{n,j}(w),$$

since the points $\{z_j\}_{j=1}^2 \in L$ are distinct. So, we need to evaluate the $D_{n,j}(w)$. For this, we take $z \in L_R$ and introduce the notations:

$$\Phi(L_{R_1}) = \Phi(\bigcup_{j=1}^{2} L^j_{R_1}) = \bigcup_{j=1}^{2} \Phi(L^j_{R_1}) = \bigcup_{j=1}^{2}\bigcup_{i=1}^{2} K^j_i(R_1), \tag{3.12}$$

where

$$K^j_1(R_1) := \left\{t \in \Phi(L^j_{R_1}) : \quad |t-w_j| < c_1\right\}$$
$$K^j_2(R_1) := \Phi(L^j_{R_1}) \backslash K^j_1(R_1),\ j=1,2.$$

Analogously,

$$\Phi(L_R) = \Phi(\bigcup_{j=1}^{2} L^j_R) = \bigcup_{j=1}^{2} \Phi(L^j_R) = \bigcup_{j=1}^{2}\bigcup_{i=1}^{2} K^j_i(R),$$

where

$$K^j_1(R) := \left\{t \in \Phi(L^j_R) : |\tau - w_j| < 2c_1\right\}$$
$$K^j_2(R) := \Phi(L^j_R) \backslash K^j_1(R),\ j=1,2.$$

Then, after these definitions, taking arbitrary fixed $w = \Phi(z) \in \Phi(L_R)$, the quantity $D_{n,j}(w)$ can be written as follows:

$$D_{n,j}(w) = \sum_{i=1}^{2} \int\limits_{K^j_i(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_j)|^{\gamma_j}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{i=1}^{2} D^i_{n,j}(w) \tag{3.13}$$

The quantity $D^i_{n,j}(w)$ we shall estimate for each $i=1,2$ and $j=1,2$ in cases separately, depending of location of the $w \in \Phi(L_R)$. Let $\varepsilon > 0$ arbitrary small fixed number.

Case 1. Let $w \in \Phi(L^1_R)$.

According to the above notations, we will make evaluations for case $w \in K^1_i(R)$ for each $i = 1,2,3$.

1.1) Let $w \in K^1_1(R)$. In this case, we will estimate the quantity

$$D_{n,1}(w) = \sum_{i=1}^{2} \int\limits_{K^1_i(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{i=1}^{2} D^i_{n,1}(w) \tag{3.14}$$

for $\gamma_1 \geq 0$ and $\gamma_1 < 0$ separately.

For each $i=1,2$ and $j=1,2$ we put: $K^j_{i,1}(R_1) := \left\{t \in \Phi(L^j_{R_1}) : |t-w_j| \geq |t-w|\right\}$, $K^j_{i,2}(R_1) := K^j_i(R_1) \setminus K^j_{i,1}(R_1)$.

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1.1.1) If $\gamma_1 \geq 0$, then

$$D^1_{n,1}(w) = \int\limits_{K^1_1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} \tag{3.15}$$

$$= \int\limits_{K^1_{1,1}(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K^1_{1,2}(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}}$$

$$=: D^{1,1}_{n,1}(w) + D^{1,2}_{n,1}(w).$$

Since $G \in PAC(\nu_1,\nu_2)$ for some $0<\nu_1,\nu_2<1$, according to [25], $\psi \in Lip\nu_i$ and $\Phi \in Lip\frac{1}{2-\nu_i}$, $i=1,2$, in a some fixed neighborhood of point z_j. Therefore, we get:

$$D^{1,1}_{n,1}(w) \preceq \int\limits_{K^1_{1,1}(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(2-\nu_1)}} \preceq n^{(2+\gamma_1)(2-\nu_1)-1}, \tag{3.16}$$

and

$$D^{1,2}_{n,1}(w) \preceq \int\limits_{K^1_{1,2}(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)(2-\nu_1)}} \preceq n^{(2+\gamma_1)(2-\nu_1)-1}, \tag{3.17}$$

If $\gamma_1 < 0$, then

$$D^1_{n,1}(w) = \int\limits_{K^1_1(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{(-\gamma_1)}\,|dt|}{|\Psi(t)-\Psi(w)|^2} \tag{3.18}$$

$$\preceq \int\limits_{K^1_1(R_1)} \frac{|dt|}{|t-w|^{2(2-\nu_1)}} \preceq \int\limits_{K^1_1(R_1)} \frac{|dt|}{|t-w|^{2(2-\nu_1)}}$$

$$\preceq n^{2(2-\nu_1)-1}.$$

1.1.2) If $\gamma_1 \geq 0$, then

$$D^2_{n,1}(w) = \int\limits_{K^1_2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} \tag{3.19}$$

$$= \int\limits_{K^1_{2,1}(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K^1_{2,2}(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}}$$

$$=: D^{2,1}_{n,1}(w) + D^{2,2}_{n,1}(w).$$

and, so from Lemma 2.1 and 2.2, we get:

$$D^{2,1}_{n,1}(w) \preceq \int\limits_{K^1_{2,1}(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(2-\nu_1)}} \preceq n^{(2+\gamma_1)(2-\nu_1)-1}, \tag{3.20}$$

and

$$D^{2,2}_{n,1}(w) \preceq 1. \tag{3.21}$$

Therefore, from (3.19)-(3.21) for $\gamma_1 \geq 0$, we have:

$$D^2_{n,1}(w) \preceq n^{(2+\gamma_1)(2-\nu_1)-1}. \tag{3.22}$$

For $\gamma_1 < 0$ from (3.14), we have:

$$D^2_{n,1}(w) = \int\limits_{K^1_2(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{(-\gamma_1)}\,|dt|}{|\Psi(t)-\Psi(w)|^2} \tag{3.23}$$

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$$\preceq \int\limits_{K_2^1(R_1)} \frac{|dt|}{|t-w|^{2(1+\varepsilon)}} \preceq n^{1+\varepsilon},\ \forall \varepsilon > 0.$$

1.2) Let $w \in K_2^1(R)$.
1.2.1) For any $\gamma_1 > -2$

$$D_{n,1}^1(w) = \int\limits_{K_{1,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K_{1,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} =: D_{n,1}^{1,1}(w) + D_{n,1}^{1,2}(w), \tag{3.24}$$

and so, according to Lemmas 2.1 and 2.2, we obtain:

$$D_{n,1}^{1,1}(w) \preceq \int\limits_{K_{1,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} \preceq 1,$$

and

$$D_{n,1}^{1,2}(w) \preceq \int\limits_{K_{1,2}^1(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)(2-\nu_1)}} \preceq n^{(2+\gamma_1)(2-\nu_1)-1}. \tag{3.25}$$

1.2.2) For any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we have:

$$D_{n,1}^2(w) \preceq \int\limits_{K_{2,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K_{2,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}}. \tag{3.26}$$

$$\preceq \int\limits_{K_{2,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)1+\varepsilon}} + 1 \preceq n^{(2+\gamma_1)(1+\varepsilon)-1},\ \forall \varepsilon > 0.$$

Combining estimates (3.14)-(3.26), for $w \in \Phi(L_R)$, we have:

$$D_{n,1} \preceq n^{\left(2+\widetilde{\gamma}_1\right)(2-\nu_1)-1+\varepsilon},\ \widetilde{\gamma}_1 := \max\{0;\gamma_1\}. \tag{3.27}$$

Case 2. Let $w \in \Phi(L_R^2)$. Analogously to the Case 1, we will obtain estimates for $w \in K_1^2(R)$ and $w \in K_2^2(R)$

$$D_{n,2}(w) \preceq n^{(2+\gamma_2)(2-\nu_2)-1+\varepsilon},\ \widetilde{\gamma}_2 := \max\{0;\gamma_2\} \tag{3.28}$$

Therefore, comparing relations (3.11), (3.13), (3.27) and (3.28), we have:

$$D_n(w) \preceq n^{\left(2+\widetilde{\gamma}_1\right)(2-\nu_1)-1} + n^{\left(2+\widetilde{\gamma}_2\right)(2-\nu_2)-1}, \tag{3.29}$$

where $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ defined as in (3.27) and (3.28).

Now, from (3.7), (3.8), (3.9) and (3.29), for any $z \in L_R$, we get:

$$|P_n(z)| \preceq [n^{\left(2+\widetilde{\gamma}_1\right)(2-\nu_1)} + n^{\left(2+\widetilde{\gamma}_2\right)(2-\nu_2)}] \, \|P_n\|_p$$

Since this estimate holds for any $z \in L_R$, then it is also true for $z \in \overline{G}$. Therefore, we complete the proof of theorem. □

3.2 Proof of Theorem 1.2

Proof. Suppose that $G \in PAC(\nu_1, \nu_2)$ for some $0 < \nu_1, \nu_2 < 1$ and $h(z)$ is defined as in (1.1). For each $R > 1$, let $w = \varphi_R(z)$ denote a univalent conformal mapping G_R onto the B, normalized by $\varphi_R(0) = 0,\ \varphi_R'(0) > 0$, and let $\{\zeta_j\},\ 1 \le j \le m \le n$, be a zeros of $P_n(z)$ (if any exist) lying on G_R. Let

$$b_{m,R}(z) := \prod_{j=1}^{m} \widetilde{b}_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)}, \tag{3.30}$$

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denote a Blaschke function with respect to zeros $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$ ([33]). Clearly,

$$|b_{m,R}(z)| \equiv 1, \ z \in L_R, \ \text{and} \ |b_{m,R}(z)| < 1, \ z \in G_R. \tag{3.31}$$

For any $p > 0$ and $z \in G_R$, let us set

$$T_{n.p}(z) := \left[\frac{P_n(z)}{b_{m,R}(z)}\right]^{p/2} \quad . \tag{3.32}$$

The function $T_{n,p}(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_{n,p}(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_{n,p}(z)$ at the $z = z_1$ gives:

$$T_{n,p}(z_1) = \frac{1}{2\pi i}\int\limits_{L_R} T_{n,p}(\zeta)\frac{d\zeta}{\zeta - z_1}.$$

Then, according to (3.31), we obtain:

$$\begin{aligned} |P_n(z_1)|^{p/2} &\leq \frac{|b_{m,R}(z_1)|^{p/2}}{2\pi}\int\limits_{L_R}\left|\frac{P_n(\zeta)}{b_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta - z_1|} \\ &\preceq \int\limits_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta - z_1|} \ . \end{aligned} \tag{3.33}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, we obtain:

$$\begin{aligned} &\left(\int\limits_{L_R}|P_n(\zeta)|^{\frac{p}{2}}\frac{|d\zeta|}{|\zeta - z_1|}\right)^2 \\ &\leq \int\limits_{|t|=R} h(\Psi(t))\,|P_n(\Psi(t))|^p\,|\Psi'(t)|^2\,|dt| \cdot \int\limits_{|t|=R}\frac{|dt|}{h(\Psi(t))\,|\Psi(t) - \Psi(w_1)|^2} \\ &= \int\limits_{|t|=R}|f_{n,p}(t)|^p\,|dt| \cdot \int\limits_{|t|=R}\frac{|dt|}{h(\Psi(t))\,|\Psi(t) - \Psi(w_1)|^2}, \end{aligned} \tag{3.34}$$

where $f_{n,p}(t)$ has been defined as in (3.8). Since $R > 1$ is arbitrary, then (3.34) holds also for $R = R_1 := 1 + \frac{\varepsilon_1}{n}$, $0 < \varepsilon_1 < 1$. So, we have:

$$\begin{aligned} &\left(\int\limits_{L_{R_1}}|P_n(\zeta)|^{\frac{p}{2}}\frac{|d\zeta|}{|\zeta - z_1|}\right)^2 \\ &\leq \left(\int\limits_{|t|=R_1}|f_{n,p}(t)|^p\,|dt|\right)\cdot\left(\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t) - \Psi(w_1)|^2}\right) \\ &=: A_n \cdot D_n(w_1), \end{aligned} \tag{3.35}$$

and, A_n and $D_n(w_j)$ have been defined as in (3.8) for $R = R_1$. Therefore, from (3.33) and (3.35), we have:

$$|P_n(z_1)| \preceq A_n \cdot D_n(w_1), \tag{3.36}$$

where, according to (3.9), the estimate

$$A_n \preceq n \cdot \|P_n\|_p^p$$

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is satisfied. For the estimate of the quantity $D_n(w_1)$ we use the notations at the estimation of the $D_n(w)$ as in (3.11)-(3.13). Therefore, under these notations, for the $D_n(w_1)$, we get:

$$D_n(w_1) \preceq \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \tag{3.37}$$

$$\preceq \sum_{i=1}^{2} \int\limits_{K^1_i(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} =: \sum_{i=1}^{2} D^i_{n,1}(w_1).$$

So, we need to evaluate the $D^i_{n,1}(w_1)$ for each $i = 1, 2$. We have:

$$D^1_{n,1}(w_1) = \int\limits_{K^1_1(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \tag{3.38}$$

$$\preceq \int\limits_{K^1_1(L_{R_1})} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)(2-\nu_1)}} \preceq \begin{cases} n^{(2+\gamma_1)(2-\nu_1)-1}, & \text{if } (2+\gamma_1)(2-\nu_1) > 1, \\ \ln n, & \text{if } (2+\gamma_1)(2-\nu_1) = 1, \\ 1, & \text{if } (2+\gamma_1)(2-\nu_1) < 1, \end{cases}$$

and

$$D^2_{n,1}(w_1) = \int\limits_{K^2_1(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \preceq \int\limits_{K^2_1(L_{R_1})} \frac{|dt|}{|t-w_1|^{2+\gamma_1+\varepsilon}} \preceq n^{(2+\gamma_1)(1+\varepsilon)-1}. \tag{3.39}$$

Combining relations (3.37) - (3.39), we have:

$$D_n(w_1) \preceq \begin{cases} n^{(2+\gamma_1)(2-\nu_1)-1+\varepsilon}, & \text{if } (2+\gamma_1)(2-\nu_1) > 1-\varepsilon, \\ \ln n, & \text{if } (2+\gamma_1)(2-\nu_1) = 1-\varepsilon, \\ 1, & \text{if } (2+\gamma_1)(2-\nu_1) < 1-\varepsilon, \end{cases} \tag{3.40}$$

From the estimations (3.36) and (3.40), we obtain:

$$|P_n(z_1)| \preceq \begin{cases} n^{\frac{(2+\gamma_1)(2-\nu_1)}{p}+\varepsilon}, & \text{if } (2+\gamma_1)(2-\nu_1) > 1-\varepsilon, \\ (n\ln n)^{\frac{1}{p}}, & \text{if } (2+\gamma_1)(2-\nu_1) = 1-\varepsilon, \\ n^{\frac{1}{p}}, & \text{if } (2+\gamma_1)(2-\nu_1) < 1-\varepsilon, \end{cases} \|P_n\|_p,$$

and we complete the proof of theorem. □

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