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Relative pseudo-complementations on ADL'S

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Abstract

The notion of an Almost Distributive Lattice (abbreviated as ADL) is a common abstraction of several lattice theoretic generalization of Boolean algebras and Boolean rings. In this paper we introduce the notion of Relative pseudo-complementation on ADL's and discuss several properties of this.

Keywords: Almost Distributive Lattice (ADL), relative annulet, pseudo-complementation, relative pseudo-complementation.
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1 Introduction

The notion of an Almost Distributive Lattice (abbreviated as ADL) was introduced by U.M.Swamy and G.C.Rao [3] as a common abstraction of several lattice theoretic generalization of Boolean algebras and Boolean rings. Mark. Mandelker [1] has introduced the concepts of relative annhilators in lattices and relatively pseudo-complemented lattices and studied their properties. Further U.M.Swamy, G.C.Rao and G.N.Rao [4] have introduced the notion of pseudo-complementation on an Almost Distributive Lattice (ADL) and proved that the class of pseudo-complemented ADL's is equationally definable and discussed inter-relationship between the annhilator ideals and pseudo-complementations on an ADL. Also, they exhibited an one to one correspondence between the pseudo-complementations and the maximal elements of an ADL A, provided there is one pseudo-complementation on A. In this paper, we introduce the concepts of relative annulets and relative pseudo-complementations on an ADL A and discuss several properties of ADL's with a relative pseudo-complementation. In particular, we prove that an ADL with a relative pseudo-complementation is a pseudo-complemented ADL. It is observed that an ADL A can have more than one relative pseudo-complementation. In fact, there exists an induced surjective correspondence between the set of maximal elements and the set of relative pseudo-complementations on A, provided there is a relative pseudo-complementation on A.

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2 Preliminaries

In this section, we recall from [3] certain elementary concepts and results concerning ADL's.

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.

(1) $0 \wedge a = 0$

(2) $a \vee 0 = a$

(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(6) $(a \vee b) \wedge b = b$.

Example 2.2. Let X be a non-empty set and fix an arbitrarily chosen element $0 \in X$. For any a and $b \in X$, define

$$a \wedge b = \begin{cases} 0 & \text{if } a = 0 \\ b & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b & \text{if } a = 0 \\ a & \text{if } a \neq 0. \end{cases}$$

Then $(X, \wedge, \vee, 0)$ is an ADL and is called a discrete ADL.

Definition 2.3. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leq b \;\text{ if }\; a = a \wedge b \;(\Leftrightarrow a \vee b = b).$$

Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 2.4. The following hold for any elements a, b and c in an ADL $A = (A, \wedge, \vee, 0)$.

(1) $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$

(2) $a \wedge a = a = a \vee a$

(3) $a \wedge b \leq b \leq b \vee a$

(4) $a \wedge b = a \;\Leftrightarrow\; a \vee b = b$; and $a \wedge b = b \;\Leftrightarrow\; a \vee b = a$

(5) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(6) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(7) $If\ a \leq b,\ then\ a \wedge b = a = b \wedge a\ and\ a \vee b = b = b \vee a$

(8) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$ and $(a \vee b) \wedge c = (b \vee a) \wedge c$

(9) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

Definition 2.5. Let I be a non empty subset of an ADL A. Then I is called

(1) an ideal of A if $a \vee b \in I$ and $a \wedge x \in I$ for all a and $b \in I, x \in A$.

(2) a filter of A if $a \wedge b \in I$ and $x \vee a \in I$ for all a and $b \in I, x \in A$.

As consequence, for any ideal I of A, $x \wedge a \in I$ for all $x \in A$ and $a \in I$ and for any filter F of A, $a \vee x \in F$ for all $x \in A$ and $a \in F$.
The set $I(A)$ of all ideals of A is a complete distributive lattice under set inclusion in which, for any $I, J \in I(A), I \cap J$ is the infimum of I and J and the supremum is given by $I \vee J = \{i \vee j \mid i \in I, j \in J\}$. For any $X \subseteq A$, $(X] = \left\{\left(\bigvee_{i=1}^{n} a_i\right) \wedge x \mid a_i \in X, x \in A, n \in N\right\}$ is the smallest ideal of A containing X and is called the ideal generated by X. If $X = \{a\}$, then we write $(a]$ for $(X]$. Therefore, for any $a \in A$, $(a] = \{a \wedge x \mid x \in A\}$ is called the principal ideal generated by a and $[a) = \{x \vee a \mid x \in A\}$ is called the principal filter generated by a.

Theorem 2.6. Let A be an ADL. For any $a, b \in A$, we have the following:

(1) $(a] \vee (b] = (a \vee b] = (b \vee a]$

(2) $(a] \wedge (b] = (a \wedge b] = (b \wedge a]$

(3) $[a) \vee [b) = [a \wedge b) = [b \wedge a)$

(4) $[a) \wedge [b) = [a \vee b) = [b \vee a)$.

3 Relative Annulets

In this section, we introduce the concept of relative annulets in an ADL and study some basic properties of these annulets.
First, let us recall that, for any elements a and b in an ADL A, $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$ (since $a \wedge b = b \wedge a$). Let us recall from [2], for any subset S of A, the annihilator of S to be the set

$$Ann\ S = \{a \in A \mid a \wedge s = 0 \text{ for all } s \in S\}.$$

Then $Ann\ S$ is always an ideal of A for all $S \subseteq A$. It can be easily proved that $Ann\ S = Ann(S]$. For any $a \in A$, we have

$$Ann(a] = Ann\{a\} = \{x \in A \mid a \wedge x = 0\} = \{x \in A \mid x \wedge a = 0\}.$$

Definition 3.1. Let A be an ADL and $a \in A$. Then, for any $x \in A$, we define the annulet relative to a (or simply called relative annulet) as follows:

$$\langle x, a\rangle = \{y \in A : \ x \wedge y \in (a]\}$$

Lemma 3.2. For any x and a in an ADL A, the relative annulet $\langle x, a\rangle$ is an ideal of A.

Proof: $\langle x, a\rangle$ is a non empty subset of A since $x \wedge a \in (a]$ and hence $a \in \langle x, a\rangle$. Let $y, z \in \langle x, a\rangle$. Then $x \wedge y$ and $x \wedge z \in (a]$. Therefore $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \in (a]$ and hence $y \vee z \in \langle x, a\rangle$. Again let $y \in \langle x, a\rangle$ and $t \in A$. Then $x \wedge y \in (a]$ and hence $x \wedge y \wedge t \in (a]$. Therefore $y \wedge t \in \langle x, a\rangle$. Thus $\langle x, a\rangle$ is an ideal of A. ■

We observe that $\langle x, 0\rangle = \{y \in A/x \wedge y = 0\} = Ann\{x\}$.
In the following we give some elementary properties of relative annulets which can be proved directly.

Theorem 3.3. Let a, b be two elements in an ADL A. Then we have the following for any $x, y \in A$.

(1) $a \leq b \Rightarrow \langle x, a\rangle \subseteq \langle x, b\rangle$

(2) $\langle x, a \wedge b\rangle = \langle x, a\rangle \cap \langle x, b\rangle = \langle x, b \wedge a\rangle$

(3) $\langle x, a\rangle \vee \langle x, b\rangle \subseteq \langle x, a \vee b\rangle = \langle x, b \vee a\rangle$

(4) $x \leq y \Rightarrow \langle y, a\rangle \subseteq \langle x, a\rangle$

(5) $\langle x \wedge y, a\rangle = \langle y \wedge x, a\rangle$

(6) $\langle x \vee y, a\rangle = \langle x, a\rangle \cap \langle y, a\rangle$

(7) $\langle x \vee y, a\rangle = \langle y \vee x, a\rangle$

(8) $\langle x, a\rangle \vee \langle y, a\rangle \subseteq \langle x \wedge y, a\rangle$

(9) $x \in (a] \Leftrightarrow \langle x, a\rangle = A$

(10) $\langle a, a\rangle = A = \langle 0, a\rangle$

(11) a is maximal $\Rightarrow \langle x, a\rangle = A$, the converse is not true.

Proof:
(1) This follows from the fact that $a \leq b \Rightarrow (a] \subseteq (b]$.
(2) $y \in \langle x, a \wedge b\rangle \ \Leftrightarrow x \wedge y \in (a \wedge b] \Leftrightarrow x \wedge y \in (a] \cap (b] \Leftrightarrow x \in \langle x, a\rangle \cap \langle x, b\rangle$.
Therefore, $\langle x, a \wedge b\rangle = \langle x, a\rangle \cap \langle x, b\rangle$.
(3) Let $y \in \langle x, a\rangle \vee \langle x, b\rangle$. Then $y = y_1 \vee y_2$ for some $y_1 \in \langle x, a\rangle$ and $y_2 \in \langle x, b\rangle$. Hence, we get

$x \wedge y_1 \in (a]$ and $x \wedge y_2 \in (b]$.
Now, $x \wedge y = x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2) \in (a] \vee (b] = (a \vee b]$ and hence $y \in \langle x, a \vee b \rangle$.
Thus, $\langle x, a \rangle \vee \langle x, b \rangle \subseteq \langle x, a \vee b \rangle$.
(4) Suppose $x \leq y$. Then $x \wedge y = x$ and $x \vee y = y$. Now,

$$\begin{aligned} z \in \langle y, a \rangle &\Rightarrow y \wedge z \in (a] \\ &\Rightarrow x \wedge y \wedge z \in (a] \\ &\Rightarrow x \wedge z \in (a] \\ &\Rightarrow z \in \langle x, a \rangle. \end{aligned}$$

Thus, $\langle y, a \rangle \subseteq \langle x, a \rangle$.
(5) This follows from the Theorem 2.3(9)
(6)
$$\begin{aligned} z \in \langle x \vee y, a \rangle &\Leftrightarrow (x \vee y) \wedge z \in (a] \\ &\Leftrightarrow (x \wedge z) \vee (y \wedge z) \in (a] \\ &\Leftrightarrow x \wedge z \text{ and } y \wedge z \in (a] \\ &\Leftrightarrow z \in \langle x, a \rangle \cap \langle y, a \rangle. \end{aligned}$$
Therefore, $\langle x \vee y, a \rangle = \langle x, a \rangle \cap \langle y, a \rangle$.
(7) It is clear from (6).
(8) Let $z \in \langle x,\ a \rangle \vee \langle y,\ a \rangle$. Then $z = z_1 \vee z_2$ for some $z_1 \in \langle x,\ a \rangle$ and $z_2 \in \langle y,\ a \rangle$. We get $x \wedge z_1$, and $y \wedge z_2 \in (a]$ and hence $y \wedge x \wedge z_1$ and $x \wedge y \wedge z_2 \in (a]$. Now,
$x \wedge y \wedge z = (x \wedge y) \wedge (z_1 \vee z_2) = (x \wedge y \wedge z_1) \vee (x \wedge y \wedge z_2) \in (a]$.
Therefore $x \wedge y \wedge z \in (a]$ and hence $z \in \langle x \wedge y,\ a \rangle$.
Thus, $\langle x,\ a \rangle \vee \langle y,\ a \rangle \subseteq \langle x \wedge y, a \rangle$.
(9) $x \in (a] \Rightarrow a \wedge x = x$. Let $y \in A$. Then $a \wedge x \wedge y = x \wedge y$. Therefore $x \wedge y \in (a]$ and hence $y \in \langle x, a \rangle$. Thus $\langle x, a \rangle$=A. Converse is trivial.
(10) It is clear.
(11) If a is maximal in A, then $(a] = A$ and hence $\langle x, a \rangle = A$ for all $x \in A$. Converse is not true. For, in the distributive lattice L given in the figure-1, $\langle x, x \rangle = L$ but x is not maximal. ■
Note: Equality may not be true in (3) and (8) of above theorem. For example in the distributive lattice $L = \{0, a, b, x, 1\}$ whose Hasse diagram is given below

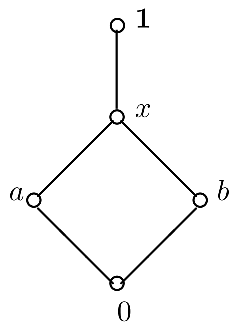


Figure 1

Here, $\langle x, a \vee b\rangle = \langle x, x\rangle = L$, and $\langle x, a\rangle = \{0, a\}$, $\langle x, b\rangle = \{0, b\}$.
Then, $\langle x, a\rangle \vee \langle x, b\rangle = (a \vee b] = (x] \neq L$.
Since $\langle a \wedge b, 0\rangle = \langle 0, 0\rangle = L$ and $\langle a, 0\rangle = \{0, b\}$, $\langle b, 0\rangle = \{0, a\}$, we have $\langle a, 0\rangle \vee \langle b, 0\rangle = \{0, a, b, x\} \neq L$.

4 Relative Pseudo-complementation on ADL's

The concept of pseudo-complementation on an ADL was first introduced by U.M. Swamy, G.C.Rao and G.N.Rao [4]. In this section, as a natural generalization of a pseudo-complementation on an ADL A, we introduce the concept of relative pseudo-complementations on an ADL A.

Definition 4.1. Let A be an ADL. Then a binary operation $*$ on A is called a relative pseudo-complementation on A if

$$\langle a, b\rangle = (a * b] \text{ for all } a, b \in A.$$

Example 4.2. Let A be a discrete ADL and with at least two elements. Then define for any $a, b \in A$,

$$a * b = \begin{cases} 0 & \text{if } a \neq 0, b = 0 \\ b & \text{otherwise} \end{cases}$$

Then $*$ is a relative pseudo-complementation on A.

Definition 4.3. An ADL A is said to be relatively pseudo-complemented if there exists a relative pseudo-complementation on A.

Example 4.4. Let A be a discrete ADL and with at least two elements. Fix $a_0 \in A$ and define for any $a, b \in A$,

$$a + b = \begin{cases} 0 & \text{if } a \neq 0, b = 0 \\ a_0 & \text{otherwise} \end{cases}$$

Then $+$ is a relative pseudo-complementation on A and hence A is relatively pseudo-complemented ADL which is not lattice.

Note that in Definition 4.1, if $a * b$ is unique such that $\langle a, b\rangle = (a * b]$, then A is a lattice and hence A is relatively pseudo-complemented lattice.
Let us recall that an element m in an ADL A is maximal in $(A, \leq)$ if and only if $m \wedge a = a (\Leftrightarrow m = m \vee a)$ for all $a \in A$, which is equivalent to saying that $(m] = A$.

Theorem 4.5. Let $*$ be a relative pseudo-complementation on an ADL A. Then for any $a, b \in A$ we have the following:

(1) $a * a$ is a maximal element in A.

(2) $a \wedge (a * 0) = 0 = (a * 0) \wedge a$.

(3) $(a \wedge b) * b$ is a maximal element in A.

(4) $a \leq b \Rightarrow a * b$ is a maximal element in A.

(5) $(a * b) \wedge b = b$.

(6) $b \in (a * 0] \Rightarrow a \wedge b = 0$.

Proof:
(1) Clearly $(a * a] = \langle a, a \rangle = A$ and hence $a * a$ is a maximal element in A.
(2) Since $(a * 0] = \langle a, 0 \rangle = Ann\{a\}$, we get that $a \wedge (a * 0) = 0 = (a * 0) \wedge a$.
(3) Since $a \wedge b \leq b$, $a \wedge b \in (b]$. By Theorem 3.3(9), $\langle a \wedge b, b \rangle = A$. This implies $((a \wedge b) * b] = A$ and hence $(a \wedge b) * b$ is maximal.
(4) It follows from (3).
(5) Since $b \in \langle a, b \rangle = (a * b]$, we get that $(a * b) \wedge b = b$.
(6) $b \in (a * 0] \Rightarrow b \in \langle a, 0 \rangle = Ann\{a\} \Rightarrow a \wedge b = 0$. ■

Note that a principal ideal in an ADL may have more than one generators. For example, in a discrete ADL X (given in the Example 2.2) for any $x, y \in X - \{0\}$ and $x \neq y$, we get $(x] = (y]$. But in the case of a lattice any principal ideal has a unique generator. However, for any a and b in an ADL, we have

$$\begin{aligned}(a] = (b] &\Leftrightarrow a \wedge b = b \text{ and } b \wedge a = a \\ &\Leftrightarrow a \vee b = a \text{ and } b \vee a = b\end{aligned}$$

and we denote this situation by writing $a \sim b$ and calling a and b as associates to each other. Note that $\sim$ is an equivalence relation on any ADL. In this context, we have the following.

Theorem 4.6. Let A be an ADL and let $*$ and $\perp$ be two relative pseudo-complementations on A. Then, for any $a, b \in A$, we have the following:

(1) $a * b \sim a \perp b$.

(2) $(a * b) \wedge (a \perp b) = a \perp b$ and $(a * b) \vee (a \perp b) = a * b$.

(3) $a * b \sim c * d \Leftrightarrow a \perp b \sim c \perp d$.

(4) $a * b = 0 \Leftrightarrow a \perp b = 0 \Leftrightarrow \langle a, b \rangle = \{0\}$.

(5) $(a * b) \wedge (0 \perp 0) \sim a \perp b$.

Proof:
(1) Since $*$ and $\perp$ are relative pseudo-complementations on A, we have
$(a * b] = \langle a, b\rangle = (a \perp b]$. Hence, $a * b \sim a \perp b$.
(2),(3),(4) follow from (1).
(5) Follows from (1) and Theorem 4.5(1). ■

Theorem 4.7. Let $*$ be a relative pseudo-complementation on an ADL A. Then for any $a, b, c \in A$ we have the following:

(1) If m is a maximal element in A, then $a * m$ is maximal in A and $a * m \sim m$.

(2) $a * (b \wedge c) \sim (a * b) \wedge (a * c)$.

(3) $(a \vee b) * c \sim (a * c) \wedge (b * c)$.

Proof: (1) Suppose m is a maximal element in A. Then $(a * m] = \langle a, m\rangle = A$ and hence, $a * m$ is a maximal element in A. Further, $(m] = A = (a * m]$.
Therefore $a * m \sim m$.
(2) By the Theorem 3.3 (2), $(a * (b \wedge c)] = (a * b] \cap (a * c] = ((a * b) \wedge (a * c)]$ and hence $a * (b \wedge c) \sim (a * b) \wedge (a * c)$.
(3) By the Theorem 3.3 (6), we have
$((a \vee b) * c] = (a * c] \cap (b * c] = ((a * c) \wedge (b * c)]$ and hence
$(a \vee b) * c \sim (a * c) \wedge (b * c)$. ■

Definition 4.8. Let A be an ADL and $*$ be a relative pseudo-complementation on A. Define, for any $a \in A$, $a^+ = a * 0$.

Theorem 4.9. Let $*$ be a relative pseudo-complementation on an ADL A. Then the following hold for any a and $b \in A$:

(1) 0^+ is a maximal element in A.

(2) m is maximal in $A \Rightarrow m^+ = 0$.

(3) $0^{++} = 0$.

(4) $a^+ \wedge a = 0 = a \wedge a^+$.

(5) $a^{++} \wedge a = a$.

(6) $a \wedge b = 0 \Leftrightarrow a^+ \wedge b = b \Leftrightarrow a^{++} \wedge b = 0 \Leftrightarrow a \wedge b^{++} = 0 \Leftrightarrow a^{++} \wedge b^{++} = 0$.

(7) $Ann\{a\} = Ann\{a^{++}\}$.

(8) $a^+ \sim a^{+++}$.

(9) $a^+ = 0 \Leftrightarrow a^{++}$ is maximal.

(10) $a = 0 \Leftrightarrow a^{++} = 0$.

Proof:
(1) $(0^+] = Ann\{0\} = A$ and hence 0^+ is maximal.
(2) m is maximal in $A \Rightarrow Ann\{m\} = \{0\}$
$\Rightarrow (m^+] = (m * 0] = \langle m, 0\rangle = Ann\{m\} = \{0\}$
$\Rightarrow m^+ = 0$.
(3) This follows from (1) and (2).
(4) From the Theorem 4.5(2), $(a * 0) \wedge a = 0$ and hence $a^+ \wedge a = 0 = a \wedge a^+$.
(5) Since $a^+ \wedge a = 0$, we have $a \in Ann\{a^+\} = \langle a^+, 0\rangle = (a^+ * 0] = (a^{++}]$. and hence $a^{++} \wedge a = a$.
(6)
$$
\begin{aligned}
a \wedge b = 0 &\Rightarrow b \in Ann\{a\} = \langle a, 0\rangle = (a * 0] = (a^+] \\
&\Rightarrow a^+ \wedge b = b \\
&\Rightarrow a^{++} \wedge b = a^{++} \wedge (a^+ \wedge b) = 0 \wedge b = 0 \\
&\Rightarrow a \wedge b = (a^{++} \wedge a) \wedge b = a \wedge a^{++} \wedge b = a \wedge 0 = 0 \\
&\Rightarrow b \wedge a = 0 \\
&\Rightarrow b^{++} \wedge a = 0 \\
&\Rightarrow a \wedge b^{++} = 0 \\
&\Rightarrow a^{++} \wedge b^{++} = 0 \\
&\Rightarrow a \wedge b = a^{++} \wedge a \wedge b^{++} \wedge b = a^{++} \wedge b^{++} \wedge a \wedge b = 0 \wedge a \wedge b = 0.
\end{aligned}
$$

(7) This follows from (6).
(8) By (7), we have $Ann\{a\} = Ann\{a^{++}\}$ and we get $(a * 0] = \langle a, 0\rangle = \langle a^{++}, 0\rangle = (a^{++} * 0]$. Therefore $(a^+] = (a^{+++}]$ implies that $a^+ \sim a^{+++}$.
(9) This follows from (1),(2) and (8) (Note that $x \sim 0 \Rightarrow x = 0$).
(10) This follows from (3) and (5). ■

Theorem 4.10. Let $*$ be a relative pseudo-complementation on an ADL A. Then the following hold for any a and $b \in A$,

(1) $(a \vee b)^+ \sim a^+ \wedge b^+$

(2) $a \sim b \Rightarrow a^+ \sim b^+$

(3) $(a \wedge b)^+ \sim (b \wedge a)^+$

(4) $(a \vee b)^+ \sim (b \vee a)^+$

(5) $(a \wedge b)^+ \wedge a^+ = a^+$

(6) $(a \wedge b)^+ \wedge b^+ = b^+$

(7) $(a \wedge b)^{++} \sim a^{++} \wedge b^{++}$.

Proof:

(1) We have
$$\begin{aligned}(a^+ \wedge b^+] &= (a^+] \cap (b^+] \\ &= (a * 0] \cap (b * 0] \\ &= \langle a, 0\rangle \cap \langle b, 0\rangle \\ &= \langle (a \vee b), 0\rangle \quad \text{(by the Theorem 3.3(6))} \\ &= ((a \vee b) * 0] = ((a \vee b)^+].\end{aligned}$$
Therefore, $(a \vee b)^+ \sim a^+ \wedge b^+$.

(2)
$$\begin{aligned}a \sim b \Rightarrow (a] = (b] &\Rightarrow Ann(a] = Ann(b] \\ &\Rightarrow \langle a, 0\rangle = \langle b, 0\rangle \\ &\Rightarrow (a * 0] = (b * 0] \\ &\Rightarrow (a^+] = (b^+] \\ &\Rightarrow a^+ \sim b^+.\end{aligned}$$

(3)$((a \wedge b)^+] = ((a \wedge b) * 0] = \langle a \wedge b, 0\rangle = \langle b \wedge a, 0\rangle = ((b \wedge a) * 0] = ((b \wedge a)^+]$. This implies that $(a \wedge b)^+ \sim (b \wedge a)^+$.

(4) This is similar to (3), since $\langle a \vee b, c\rangle = \langle b \vee a, c\rangle$

(5) Since $(a \wedge b) \wedge a^+ = b \wedge a \wedge a^+ = b \wedge 0 = 0$, by Theorem 4.9(6), we get that $(a \wedge b)^+ \wedge a^+ = a^+$.

(6) Since $(a \wedge b) \wedge b^+ = 0$, by Theorem 4.9(6), we get that $(a \wedge b)^+ \wedge b^+ = b^+$.

(7) We have $a \wedge b \wedge (a \wedge b)^+ = 0 = b \wedge a \wedge (a \wedge b)^+$. By repeated use of Theorem 4.9(6), we get that $a^{++} \wedge b^{++} \wedge (a \wedge b)^+ = 0$.

Therefore, $(a \wedge b)^+ \wedge a^{++} \wedge b^{++} = 0$ and hence $(a \wedge b)^{++} \wedge a^{++} \wedge b^{++} = a^{++} \wedge b^{++}$.
On the other hand, we have $(a \wedge b) \wedge b^+ = 0$ and hence by Theorem 4.9(6), $(a \wedge b)^{++} \wedge b^+ = 0$.
Therefore, $b^+ \wedge (a \wedge b)^{++} = 0$ and $b^{++} \wedge (a \wedge b)^{++} = (a \wedge b)^{++}$.
Similarly, $a^{++} \wedge (a \wedge b)^{++} = (a \wedge b)^{++}$ and therefore $a^{++} \wedge b^{++} \wedge (a \wedge b)^{++} = (a \wedge b)^{++}$.
Thus $(a \wedge b)^{++} \sim a^{++} \wedge b^{++}$. ■

From [4] recall that an unary operation $a \mapsto a^\perp$ on A is called a pseudo-complementation on A if, for any $a, b \in A$, the following independent axioms are satisfied

(1) $a \wedge b = 0 \;\Rightarrow\; a^\perp \wedge b = b$.

(2) $a \wedge a^\perp = 0$.

(3) $(a \vee b)^\perp = a^\perp \wedge b^\perp$.

Theorem 4.11. Every relatively pseudo-complemented ADL is a pseudo-complemented ADL.

Proof: Let A be an ADL and $*$ be a relative pseudo-complementation on A. Choose a maximal element m in A (A has one such; for example, $0 * 0$ is maximal in A). For any $a \in A$, define $a^{\perp} = a^{+} \wedge m$ where $a^{+} = a * 0$. Then, by Theorem 4.9(4) and (6) we have $a \wedge a^{\perp} = 0$ and for any $b \in A$, $a \wedge b = 0$ implies that $a^{\perp} \wedge b = b$. Also for any a and $b \in A$, we have $(a \vee b)^{+} \sim a^{+} \wedge b^{+}$ (by Theorem 4.10(1)). Now,

$$\begin{aligned} a^{\perp} \wedge b^{\perp} &= a^{+} \wedge m \wedge b^{+} \wedge m \\ &= a^{+} \wedge b^{+} \wedge m \\ &= (a \vee b)^{+} \wedge a^{+} \wedge b^{+} \wedge m \\ &= (a \vee b)^{+} \wedge m \wedge a^{+} \wedge m \wedge b^{+} \wedge m \\ &= (a \vee b)^{\perp} \wedge (a^{\perp} \wedge b^{\perp}). \end{aligned}$$

Similarly, $(a^{\perp} \wedge b^{\perp}) \wedge (a \vee b)^{\perp} = (a \vee b)^{\perp}$ and hence $(a \vee b)^{\perp} \sim a^{\perp} \wedge b^{\perp}$. Since $x^{\perp} \leq m$ for all $x \in A$, we have that m is an upper bound of $(a \vee b)^{\perp}$ and $a^{\perp} \wedge b^{\perp}$ and hence these two are commute to each other. This implies that

$$(a \vee b)^{\perp} = (a^{\perp} \wedge b^{\perp}) \wedge (a \vee b)^{\perp} = (a \vee b)^{\perp} \wedge (a^{\perp} \wedge b^{\perp}) = a^{\perp} \wedge b^{\perp}.$$

Thus $a \mapsto a^{\perp}$ is a pseudo-complementation on A and hence A is a pseudo-complemented ADL. ■

Theorem 4.12. Let A be an ADL with a maximal element. Then the following are equivalent to each other.

(1) A is relatively pseudo-complemented

(2) $[0, a]$ is relatively pseudo-complemented for all $a \in A$

(3) $[a, b]$ is relatively pseudo-complemented for all $a \leq b$ in A

(4) $[a, b]$ is pseudo-complemented for all $a \leq b$ in A.

Proof:
(1) $\Rightarrow$ (2) : Let $*$ be a relative pseudo-complementation on A and $a \in A$. Let $x, y \in [0, a]$, we have

$$\langle x, y \rangle = \{z \in A : x \wedge z \in (y]\} = (x * y].$$

Now, put $x \perp y = (x * y) \wedge a$. Then $x \perp y \in [0, a]$ and

$$x \wedge (x \perp y) = x \wedge (x * y) \wedge a \in (y] \cap (a]$$

which is the ideal generated by y in $[0, a]$. On the other hand, suppose $z \in [0, a]$ such that $x \wedge z \in (y]$. Then $z \in (x * y]$ and hence $(x * y) \wedge z = z$.

$$\text{Now,}\quad (x \perp y) \wedge z = (x * y) \wedge a \wedge z$$
$$= a \wedge (x * y) \wedge z = a \wedge z = z$$

so that $z \in (x \perp y]$. Therefore

$$\{z \in [0, a] : x \wedge z \in (y]\} = (x \perp y].$$

Thus, $\perp$ is a relative pseudo-complementation on $[0, a]$.
(2) $\Rightarrow$ (3) : Let $a, b \in A$ with $a \leq b$ and $*$ be a relative pseudo-complementation on $[0, b]$. Let $x, y \in [a, b]$. Put $x + y = (x * y) \vee a$. Since $x * y$ and $a \in [0, b]$, which is a lattice, we have

$$x + y = (x * y) \vee a = a \vee (x * y).$$

Consider $x \wedge (x + y) = x \wedge ((x * y) \vee a)$
$$= (x \wedge (x * y)) \vee (x \wedge a) \in (y]$$
since $x \wedge (x * y) \in (y]$ and $a \leq y$. On the other hand, suppose that $z \in [a, b]$ such that $x \wedge z \in (y]$. Then $z \in [0, b]$ and $z \in (x * y]$ and hence $(x * y) \wedge z = z$.

$$\text{Now,}\quad (x + y) \wedge z = ((x * y) \vee a) \wedge z$$
$$= ((x * y) \wedge z) \vee (a \wedge z)$$
$$= z \vee a = z.$$

Therefore $\{z \in [a, b] : x \wedge z \in (y]\} = (x + y]$. Thus $+$ is a relative pseudo-complementation on $[a, b]$.
(3) $\Rightarrow$ (4) follows from the Theorem 4.11.
(4) $\Rightarrow$ (1) : Assume the condition (4). Let $a, b \in A$ and x be the pseudo-complement of a in $[b \wedge a, a \vee m]$. Define

$a * b = x$. Now, we prove that $\langle a, b\rangle = (x]$.

Since $b \wedge a \leq x \leq a \vee m$ and $a \wedge x = b \wedge a \in (b]$, we get that $x \in \langle a, b\rangle$.
On the other hand let $y \in \langle a, b\rangle$. Then $a \wedge y \in (b]$ and hence $b \wedge a \wedge y = a \wedge y$.
Put $z = (y \vee (b \wedge a)) \wedge (a \vee m)$. Then $z \in [b \wedge a, a \vee m]$.

$$\text{Now,}\quad a \wedge z = a \wedge (y \vee (b \wedge a)) \wedge (a \vee m)$$
$$= ((a \wedge y) \vee (a \wedge b \wedge a)) \wedge (a \vee m)$$

$$= ((a \wedge y) \vee (b \wedge a)) \wedge (a \vee m)$$
$$= (b \wedge a) \wedge (a \vee m)$$
$$= b \wedge a.$$

This implies $z \leq x$. Now $y \wedge z = y \wedge (y \vee (b \wedge a)) \wedge (a \vee m) = y \wedge (a \vee m)$.
Therefore $y \wedge z \wedge y = y \wedge (a \vee m) \wedge y = y$ (since $a \vee m$ is maximal).
This implies $z \wedge y = y$ and hence $y = z \wedge y = z \wedge x \wedge y = x \wedge z \wedge y = x \wedge y$.
Hence $y \in (x]$. Thus $\langle a, b \rangle = (x]$ and hence A is relatively pseudo-complemented ADL. ■

Remark 4.13.

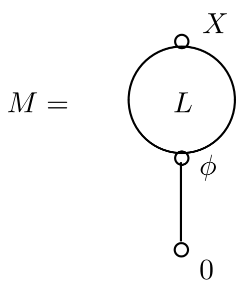
(1) If A is a bounded distributive lattice, there can be at most one relative pseudo-complementation on A and at most one pseudo-complementaion on A

(2) The converse of Theorem 4.11 is not true, even in the case of distributive lattices; for consider the following example.

Example 4.14. Let X be an infinite set and

$$L = \{A \subseteq X : A \text{ is finite or } A = X\}.$$

Then, L is a bounded distributive lattice under the usual set theoretic operations but not pseudo-complemented. For otherwise suppose $*$ is the pseudo-complementation on L. Then for each $x \in X$, we have $\{x\} \cap \{y\} = \phi$ for all $y \neq x$. This implies $\{y\} \subseteq \{x\}^*$ and hence $\{x\}^* = X$. Thus $\{x\} \cap \{x\}^* \neq \phi$, which is a contradiction.
Consider $M = L \cup \{0\}$ whose Hasse diagram is given below.



$$\text{For any } A \in M, \text{ define } A^* = \begin{cases} 0 & \text{if } A \neq 0 \\ X & \text{if } A = 0 \end{cases}$$

Then A^* is the pseudo-complement of A in M and hence M is pseudo-complemented distributive lattice, but $[\phi, X] = L$, which is not pseudo-complemented. Therefore by the Theorem 4.12, the condition (4) fails in M and hence (1) fails. Thus M is pseudo-complemented but not relatively pseudo-complemented.

Definition 4.15. Two relative pseudo-complementations $*$ and $+$ on an ADL A are said to be equivalent (and denote this by $* \approx +$) if $0*0 = 0+0$. Then clearly $\approx$ is an equivalence relation on the set $\mathcal{RPC}(A)$, of all relative pseudo-complementations on A.

Theorem 4.16. Let A be an ADL and $*$ a relative pseudo-complementation on A. Let $M(A)$ be the set of all maximal elements in A. For any $m \in M(A)$, define $*_m : A \times A \to A$ by $a *_m b = (a * b) \wedge m$ for all $a, b \in A$. Then the correspondence $m \mapsto *_m$ induces a bijection of M onto $\mathcal{RPC}(A)/\approx$.

Proof: Let $a, b \in A$ and $m \in M(A)$. Then $\langle a, b\rangle = (a*b] = (m \wedge (a*b)] = ((a*b) \wedge m] = (a *_m b]$ and hence $*_m$ is a relative pseudo-complementation on A. Let $m, n \in M(A)$ such that $*_m \approx *_n$. Then $0 *_m 0 = 0 *_n 0$ which implies that $(0 * 0) \wedge m = (0 * 0) \wedge n$ and hence $m = n$ since $0 * 0$ is maximal in A. Also, for any $+ \in \mathcal{RPC}(A)$, if $m = 0 + 0$, then $m \in M(A)$ and
$0 + 0 = (0 * 0) \wedge (0 + 0) = (0 * 0) \wedge m = 0 *_m 0$ and hence $*_m \approx +$.
Thus, $m \mapsto *_m$ is a bijection of $M(A)$ onto $\mathcal{RPC}(A)/\approx$. ■

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