International Journal of Mathematics and Soft Computing
Vol.7, No.1 (2017), 89 - 94.

ISSN Print : 2249 - 3328
ISSN Online : 2319 - 5215

Complete Star of a graph and its Balanced cordial labeling

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Abstract

In this paper we introduce a complete star of a graph G. We prove that for a balanced cordial graph G, star of G, $P_n \times G$ and its complete star are balanced cordial graph. We also prove that the star of a vertex balanced graph is also a vertex balanced graph.

Keywords: Binary vertex labeling, balanced cordial graph, complete star of a graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

In this paper, by a graph G we mean a finite, undirected and simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. We follow Harary [4] for the basic notation and terminology of graph theory.

Star of a graph G is denoted by G^* and it is obtained by $p+1$ copies $G^{(0)}, G^{(1)}, G^{(2)}, \ldots, G^{(p)}$ of the graph G, where $V(G) = \{v_1, v_2, \ldots, v_p\}$. It is obtained by joining each vertex of $G^{(0)}$ with the corresponding vertex v_i of $G^{(i)}, \forall\ i = 1, 2, \ldots, p$. We call $G^{(0)}$ as central copy of G^* and here the graph G may or may not be a connected graph. It is obvious that $K_1^* = K_2, K_2^* = P_6$ and $(K_1 \cup K_2)^* = P_6 \cup 2K_2 \cup 2K_1$. $P_n \times G$ is the graph obtained by joining n copies of the graph G say $G^{(1)}, G^{(2)}, G^{(3)}, \ldots, G^{(n)}$ by joining all p edges of $G^{(i)}$ with $G^{(i+1)}$ to corresponding vertex v of $G^{(i)}$ with same vertex v of $G^{(i+1)}$, $\forall\ i = 1, 2, \ldots, n-1$. That is, $V(P_n \times G) = \bigcup_{i=1}^{n} V(G^{(i)})$ and $E(P_n \times G) = \bigcup_{i=1}^{n} E(G^{(i)}) \bigcup \{(v_j^{(i)}, v_j^{(i+1)}) /\ \forall\ v_j^{(i)} \in V(G^{(i)}),\ \forall\ j = 1, 2, \ldots, p,\ \forall\ i = 1, 2, \ldots, n-1\}$.

Complete star of a graph G is the graph obtained from $p+1$ copies $G^{(0)}, G^{(1)}, G^{(2)}, \ldots, G^{(p)}$ of the graph G and it is obtained by joining each vertex of $G^{(0)}$ with all corresponding vertex of all the copies $G^{(1)}, G^{(2)}, \ldots, G^{(p)}$. We denote such graph by $\overline{G}^*$ and we call $G^{(0)}$ as central copy of $\overline{G}^*$. It is obvious that $\overline{K_1}^* = K_1^* = K_2$ and $\overline{K_2}^* = P_2 \times P_3$.

Cahit [1] introduced cordial labeling of a graph G, a variation of graceful and harmonious labelings. In [2], Cahit proved that all trees are cordial. A function $f : V(G) \longrightarrow \{0, 1\}$ is called a binary vertex labeling of a graph G. $f(v)$ is called label of the vertex v of G under f. For any edge $e=(u,v)\in E(G)$, the edge induced function $f^*: E(G)\longrightarrow\{0,1\}$ is defined as $f^*(e)=|f(u)-f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having 0 and 1 vertex label respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges of G having 0 and 1 edge labels respectively under f^*. A binary vertex labeling function f of a graph G is called cordial labeling if $|e_f(0)-e_f(1)|\leq 1$ and $|v_f(0)-v_f(1)|\leq 1$. A graph G is called a cordial graph if it admits a cordial labeling.

A cordial graph G with a cordial labeling f is called a balanced cordial graph if $|e_f(0)-e_f(1)|=|v_f(0)-v_f(1)|=0$. It is said to be edge balanced cordial graph if $|e_f(0)-e_f(1)|=0$ and $|v_f(0)-v_f(1)|=1$. Similarly it is said to be vertex balanced cordial graph if $|e_f(0)-e_f(1)|=1$ and $|v_f(0)-v_f(1)|=0$. A cordial graph G is said to be unbalanced cordial graph if $|v_f(0)-v_f(1)|=|e_f(0)-e_f(1)|=1$.

For any cordial graph G, if f is a cordial labeling of G and it is of one of above four categories, then $1-f$ is also a cordial labeling for G and it is of the same category.

2 Main Results

Theorem 2.1. If G is a balanced cordial graph, then so is G^*.

Proof: Let $V(G)=\{v_1,v_2,\ldots,v_p\}$ and $q=|E(G)|$. Let $f: V(G)\longrightarrow\{0,1\}$ be a balanced cordial labeling for G. It is obvious that p and q are both even and $v_f(0)=v_f(1)=\frac{p}{2}$, $e_f(0)=e_f(1)=\frac{q}{2}$.

Let H be the star of the graph G and $V(H)=\bigcup_{i=0}^{p}(V(G_{(i)})=\{v_1^{(i)},v_2^{(i)},\ldots,v_p^{(i)}/\ \forall\ i=0,1,\ldots,p\}$. Here $|V(H)|=p(p+1)$ and $|E(H)|=(p+1)q+p$.

Define $g: V(H)\to\{0,1\}$ as follows:

For any $v_j^{(i)}\in V(H)$,

$$\begin{aligned} g(v_j^{(i)}) &= f(v_j) && \text{when } i \text{ is even} \\ &= 1-f(v_j) && \text{when } i \text{ is odd};\quad \forall\ i=0,1,2,\ldots,p,\ \forall\ j=1,2,\ldots,p. \end{aligned}$$

It is observed that,

$v_g(0)=v_f(0)+\frac{p}{2}v_f(1)+\frac{p}{2}v_f(0)=\frac{p}{2}+2(\frac{p}{2}.\frac{p}{2})=\frac{p}{2}(1+p)$,

$v_g(1)=v_f(1)+\frac{p}{2}v_f(0)+\frac{p}{2}v_f(1)=\frac{p}{2}(1+p)$,

$e_g(0)=(p+1)e_f(0)+\frac{p}{2}=\frac{1}{2}(pq+p+q)$ and

$e_g(1)=(pq+p+q)$.

Because for each $(v_j^{(0)},v_j^{(j)})\in E(H)$,

$$
\begin{aligned}
g^*((v_j^{(0)}, v_j^{(j)})) &= |g(v_j^{(0)}) - g(v_j^{(j)})| \\
&= \begin{cases} |f(v_j) - f(v_j)|, & \text{when } j \text{ is even} \\ |f(v_j) - 1 + f(v_j)|, & \text{when } j \text{ is odd} \end{cases} \\
&= \begin{cases} 0, & \text{when } j \text{ is even} \\ 1, & \text{when } j \text{ is odd}; \quad \forall\, j = 1, 2, \ldots, p. \end{cases}
\end{aligned}
$$

Thus, $H = G^*$ is a graph which satisfies $v_g(0) = v_g(1) = \frac{1}{2}p(p+1)$ and $e_g(0) = e_g(1) = \frac{1}{2}(pq + p + q)$. So, it is a balanced cordial graph. ■

Illustration 2.2. A balanced cordial graph G with its balanced cordial labeling and its star G^* with balanced cordial labeling are shown in Figures 1 and 2 respectively.

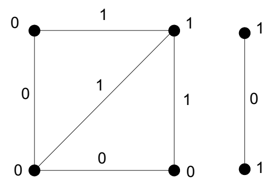

Figure 1: Graph G with its balanced cordial labeling f, which satisfies $v_f(1) = v_f(0) = 3$ and $e_f(0) = e_f(1) = 3$.

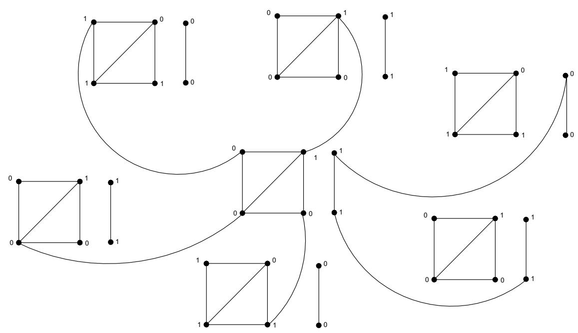

Figure 2: G^* and its balanced cordial labeling g with $v_g(0) = v_g(1) = 21$ and $e_g(0) = e_g(1) = 24$.

Theorem 2.3. Let n be an odd integer. If G is a balanced cordial graph, then so is $P_n \times G$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $q = |E(G)|$. Let $f : V(G) \longrightarrow \{0, 1\}$ be a balanced cordial labeling for G. It is obvious that p and q are both even and $v_f(0) = v_f(1) = \frac{p}{2}$, $e_f(0) = e_f(1) = \frac{q}{2}$ hold in G. Let $G^{(1)}, G^{(2)}, G^{(3)}, \ldots, G^{(n)}$ be n copies of the graph G. Join vertex $v_j^{(i)}$ of $G^{(i)}$ and vertex $v_j^{(i+1)}$ of $G^{(i+1)}$ by an edge, $\forall\, i = 1, 2, \ldots, n-1$, $\forall\, j = 1, 2, \ldots, p$, to form the graph $P_n\times$ G. Thus, $|V(P_n \times G)| = |\bigcup_{i=1}^{n} V(G^i)| = \sum_{i=1}^{n} |V(G)| = np$ and $|E(P_n \times G| = nq + (n-1)p$.

Define $g : V(P_n \times G) \longrightarrow \{0,1\}$ as follows:

For any $v_j^{(i)} \in V(P_n \times G)$,

$$\begin{aligned} g(v_j^{(i)}) &= f(v_j); \quad \text{if} \quad i \equiv 0,1 \pmod 4, \\ &= 1-f(v_j); \quad \text{if} \quad i \equiv 2,3 \pmod 4; \ \forall\, i=1,2,\dots,n,\ \forall\, j=1,2,\dots,p. \end{aligned}$$

It is obvious that $v_g(0) = \dfrac{np}{2} = v_g(1)$ and $e_g(0) = \dfrac{nq}{2} + \dfrac{(n-1)p}{2} = e_g(1)$. Because for each $(v_j^{(i)}, v_j^{(i+1)}) \in E(P_n \times G)$,

$$\begin{aligned} g^*((v_j^{(i)}, v_j^{(i+1)})) &= |g(v_j^{(i)}) - g(v_j^{(i+1)})| \\ &= \begin{cases} |2f(v_j)-1|, & \text{when } i \text{ is odd} \\ |f(v_j)-f(v_j)|, & \text{when } i \text{ is even} \end{cases} \\ &= \begin{cases} 0, & \text{when } i \text{ is even} \\ 1, & \text{when } i \text{ is odd}; \quad \forall\, i=1,2,\dots,n-1,\ \forall\, j=1,2,\dots,p. \end{cases} \end{aligned}$$

Thus, $P_n \times G$ (n is odd) is a graph which satisfies $v_g(0) = v_g(1) = \dfrac{np}{2}$ and $e_g(0) = e_g(1) = \dfrac{1}{2}(n(p+q)-p)$. So, it is a balanced cordial graph. ∎

Theorem 2.4. If G is a balanced cordial graph, then so is $\overline{G}^*$.

Proof: Let $G^{(0)}$ be the central copy of G in $\overline{G}^*$, $V(G^{(0)}) = \{v_1^{(0)}, v_2^{(0)}, \dots, v_p^{(0)}\}$ and $|E(G^{(0)}| = q$. Let $f : V(G^{(0)}) \longrightarrow \{0,1\}$ be a balanced cordial labeling for $G = G^{(0)}$.

Let H be the complete star of the graph G. That is, $H = \overline{G}^*$. It is obvious that $P = |V(H)| = p(p+1)$, $Q = |E(H)| = (p+1)q + p^2$. Here $V(H) = \bigcup_{i=0}^{p} V(G^{(i)}) = \{v_1^{(i)}, v_2^{(i)}, \dots, v_p^{(i)} / \ \forall\ i = 0,1,\dots,p\}$. Since G is a balanced cordial graph, p, q both are even and $v_f(0) = v_f(1) = \dfrac{p}{2}$, $e_f(0) = e_f(1) = \dfrac{q}{2}$ hold in G.

Define $g : V(H) \longrightarrow \{0,1\}$ as follows:

For any $v_j^{(i)} \in V(H)$,

$$\begin{aligned} g(v_j^{(i)}) &= f(v_j^{(0)}) \quad \text{when } i \text{ is even} \\ &= 1 - f(v_j^{(0)}) \quad \text{when } i \text{ is odd}, \quad \forall\ i = 0,1,2,\dots,p, \quad \forall\ j=1,2,\dots,p. \end{aligned}$$

It observed that,

$$v_g(0) = (p+1)v_f(0) = \frac{p(p+1)}{2} = \frac{P}{2} \text{ and } v_g(1) = (p+1)v_f(1) = \frac{p(p+1)}{2} = \frac{P}{2}.$$

That is, $v_g(0) = v_g(1) = \dfrac{P}{2}$. Moreover $e_g(1) = (p+1)e_f(1) + p(\dfrac{p}{2}) = (p+1)\dfrac{q}{2} + \dfrac{p^2}{2}$ and $e_g(0) = (p+1)e_f(0) + p(\dfrac{p}{2})$. Hence $e_f(0) = e_f(1) = \dfrac{Q}{2}$. Therefore, H is a balanced cordial graph. ∎

Theorem 2.5. If G is a vertex balanced graph, then so is G^*.

Proof: Let $G^{(0)}$ be the central copy of G in G^*. Take $V(G^{(0)}) = \{v_1^{(0)}, v_2^{(0)}, \ldots, v_p^{(0)}\}$ and $q = |E(G^{(0)}|$. Let $f : V(G^{(0)}) \longrightarrow \{0,1\}$ be a balanced cordial labeling for $G^{(0)} = G$. It is obvious that p is even and q is odd and $v_f(0) = v_f(1) = \frac{p}{2}$, $|e_f(0) - e_f(1)| = 1$ hold in $G^{(0)}$.

Let $H = G^*$ and $V(H) = \bigcup_{i=0}^{p} V(G^{(i)}) = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_p^{(i)}/i = 0, 1, \cdots, p\}$. Here $P = |V(H)| = (p+1)p$ and $Q = |E(H)| = pq + p + q$.

To define $g : V(H) \longrightarrow \{0,1\}$ we have to consider the following two cases.

Case 1: $e_f(0) - e_f(1) = 1$.

Define $g/V(G^{(0)}) = f/V(G^{(0)})$ and $g/V(G^{(i)}) = 1 - g/V(G^{(0)})$, $\forall\ i = 1, 2, \ldots, p$.

Case 2: $e_f(0) - e_f(1) = -1$.

Define $g/V(G^{(0)}) = f/V(G^{(0)})$ and $g/V(G^{(i)}) = g/V(G^{(0)})$, $\forall\ i = 1, 2, \ldots, p$.

Above defined labeling pattern gives rise to a cordial labeling to the graph $H = G^*$ with $v_g(0) = v_g(1) = \frac{p(p+1)}{2} = \frac{P}{2}$ and $|e_g(0) - e_g(1)| = |e_f(0) - e_f(1)| = 1$.

Because in Case 1, $e_g(0) = (p+1)e_f(0) = (p+1)(\frac{q+1}{2}) = \frac{1}{2}(pq + p + q + 1) = \frac{Q+1}{2}$ and $e_g(1) = (p+1)e_f(1) + p = (p+1)(\frac{q-1}{2}) + p = \frac{Q-1}{2}$.

That is, $e_g(0) - e_g(1) = \frac{1}{2}(Q+1) - \frac{1}{2}(Q-1) = 1$.

and in Case 2, $e_g(0) = (p+1)e_f(0) + p = (p+1)(\frac{q-1}{2}) + p = \frac{Q-1}{2}$ and $e_g(1) = (p+1)e_f(1) = (p+1)(\frac{q+1}{2}) = \frac{Q+1}{2}$.

That is, $e_g(0) - e_g(1) = \frac{Q-1}{2} - \frac{Q+1}{2} = -1$.

Hence, g is a vertex balanced cordial labeling for H and so, H is a vertex balanced cordial graph. ■

Illustration 2.6. A vertex balanced cordial graph G with its vertex balanced cordial labeling and its star G^* with vertex balanced cordial labeling are shown in figure-3 and 4 respectively.

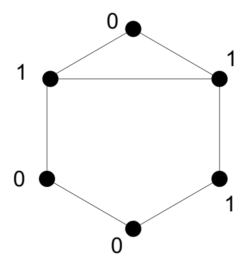

Figure 3: A graph G with its a vertex balanced cordial labeling f, which satisfies $v_f(1) = v_f(0) = 3$ and $e_f(1) - e_f(0) = 1$.

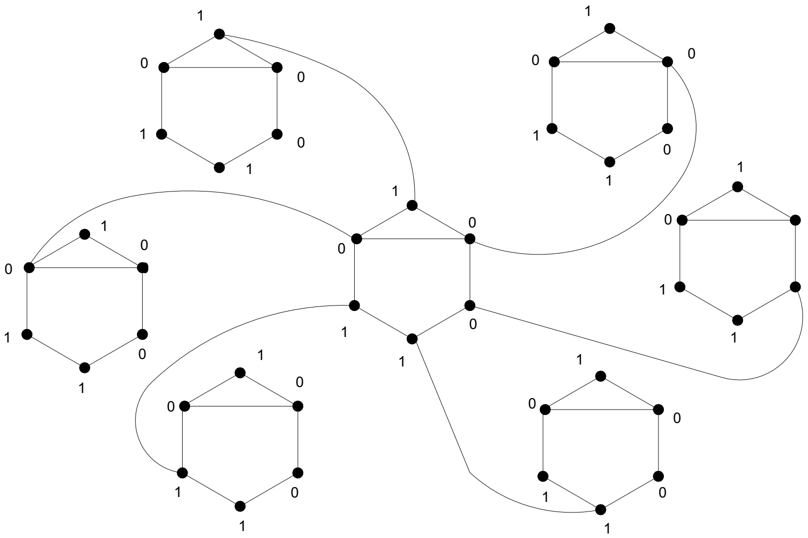

Figure 4: G^* and its a vertex balanced cordial labeling g with $v_g(0) = v_g(1) = 21$, $e_g(1) = 28$ and $e_g(0) = 27$.

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