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Some notes on the solutions of BSM equation

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Abstract

The solution of Black-Scholes-Merton (BSM) Partial Differential Equation represents the model for pricing an option. It is a very useful application for trading terminal. The solution gives the theoretical value of an option (Call/Put). In the present paper we apply Fourier Transform Method to solve the equation for plain vanilla payoff function and Log payoff function, which are the boundary conditions for the BSM partial differential equation. Also, we observe and show that averages of these two payoff functions will give exactly the average of two solutions. And we also extend this result.

Keywords: Black-Scholes-Merton model, Partial Differential Equation, Call/Put option, Fourier Transform Method.
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1 Introduction

In Mathematical Finance, the Black-Scholes-Merton equation is a Partial Differential Equation to find the value of European Call/Put option. Suppose $C(S,t)$ is the value of European call option. The equation [1]

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} - rC = 0$$

is known as a Black-Scholes-Merton Partial Differential Equation, where,

S is Spot price of asset (i.e. the price of asset at time $t = 0$)
X is Exercise price or strike price
T is Total period of time
r is Risk free interest rate
σ is Volatility
$t \in [0,T]$ and $C(S,t) = 0$ for all t

$$C(S,t) \to S \text{ as } S \to \infty.$$

Consider the European call option whose final payoff at the expiry time T is given by a function f of the final spot price S (we note that in the literature it is often denoted by many as S_T) which is assumed to be a continuous function, that need not be differentiable. Also we demand,

$$\lim_{t \to T_-} C(S,t) = f(S)$$

We can convert the Black-Scholes-Merton Partial Differential Equation into heat equation using the following substitutions:

$$y = T - t$$
$$x = \ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)$$
$$D(x,y) = e^{r(T-t)} C(S,t)$$

These substitutions also convert the above mentioned boundary condition,

$$\lim_{t \to T_-} C(S,t) = f(S)$$

to the initial condition,

$$\lim_{y \to 0^+} D(x,y) = f(Xe^x).$$

Thus the Black-Scholes-Merton Partial Differential Equation gets converted into the following Heat equation with the stated initial condition:

$$\frac{\partial D}{\partial y} = \frac{\sigma^2}{2} \frac{\partial^2 D}{\partial x^2} \text{ with } \lim_{y \to 0^+} D(x,y) = f(Xe^x).$$

Several people have solved the same problem by using the Method of separation of variables and Laplace Transform Method. Here we use Fourier Transform Method to solve the problem. Applying Fourier Transform on Heat equation we get,

$$\frac{\partial}{\partial y} F(D) + \frac{\sigma^2 \lambda^2}{2} F(D) = 0$$

$$\text{Therefore,} \qquad F(D) = C_1 e^{-\frac{\sigma^2 \lambda^2}{2} y}.$$

Now we get, $F(D(x,0)) = G(\lambda)$ because $D(x,0) = f(Xe^x)$.
Here G is the Fourier Transform of f, so that C_1 is determined and now we have:

$$F(D) = G(\lambda) e^{-\frac{\sigma^2 \lambda^2}{2} y}.$$

Taking inverse Fourier Transform on both the sides we get,

$$D(x,y) = F^{-1}\left(G(\lambda)e^{-\frac{\sigma^2\lambda^2}{2}y}\right)$$

Now, using Convolution theorem and the facts, that

$$F^{-1}(G(\lambda)) = f(Xe^x)\,\text{and}\,F^{-1}\left(e^{-\frac{\sigma^2\lambda^2}{2}y}\right) = \frac{1}{\sigma\sqrt{y}}e^{\frac{-x^2}{2\sigma^2 y}}$$

we get,

$$D(x,y) = \tfrac{1}{\sqrt{2\pi}}\textstyle\int_{-\infty}^{\infty} f(\nu)\tfrac{1}{\sigma\sqrt{y}}e^{\frac{-(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu$$

Therefore, $\quad D(x,y) = \tfrac{1}{\sigma\sqrt{2\pi y}}\textstyle\int_{-\infty}^{\infty} f(\nu)e^{\frac{-(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu.$

2 Solution of the problem using different payoff functions

2.1 Plain Vanilla Payoff

This is a very basic and commonly used payoff function. Many financial organizations use this as a payoff. The very first BSM formula was derived using [4].
Now we consider Plain Vanilla payoff function which is as follows:

$$f(S) = \max\{S - X,\ 0\} = \begin{cases} S - X & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}$$

Therefore,

$$f(Xe^x) = \max\left\{X(e^x - 1),\ 0\right\} = \begin{cases} X(e^x - 1) & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\begin{aligned} D(x,y) &= \frac{1}{\sigma\sqrt{2\pi y}}\int_{-\infty}^{\infty} X(e^\nu - 1)e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu \\ &= \frac{X}{\sigma\sqrt{2\pi y}}\left[\int_0^{\infty} e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu - \int_0^{\infty} e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu\right] \end{aligned}$$

Substituting, $Z = \frac{\nu - x}{\sigma\sqrt{y}}$ we get,

$$D(x,y) = \frac{X}{\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{-\frac{(Z^2 - 2\sigma\sqrt{y}Z + \sigma^2 y)}{2}}\,\mathrm{d}Z - \frac{X}{\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{-\frac{Z^2}{2}}\,\mathrm{d}Z$$

$$
\begin{aligned}
&= \frac{X}{\sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^{\infty} e^{-\frac{t^2}{2}} \mathrm{d}t - \frac{X}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{-\frac{Z^2}{2}} \mathrm{d}Z \\
&= \frac{X}{\sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\frac{x+\sigma^2 y}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}} \mathrm{d}t - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}} \mathrm{d}t
\end{aligned}
$$

Therefore, $D(x,y) = Xe^{x+\frac{\sigma^2 y}{2}} N(d_1) - XN(d_2)$

where, $d_1 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}}$, $d_2 = \frac{x}{\sigma\sqrt{y}}$ and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t$.

Hence, $C(S,t) = SN(d_1) - Xe^{-r(T-t)} N(d_2)$

where
$$
d_1 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}} = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}
$$
$$
d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.
$$

2.2 Log Payoff

Paul Wilmott discussed BSM formula for Log payoff function in [5]. Other than this several types of option pricing formulas have been derived with different payoff functions [3].
Now we consider the payoff function which is known as Log payoff, which is as:

$$
f(S) = \max\left\{\ln\left(\frac{S}{X}\right),\ 0\right\} = \begin{cases} \ln\left(\frac{S}{X}\right) & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}
$$

Therefore,

$$
\begin{aligned}
f(Xe^x) &= \max\{x,\ 0\} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \\
D(x,y) &= \frac{1}{\sigma\sqrt{2\pi y}} \int_0^{\infty} \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \mathrm{d}\nu.
\end{aligned}
$$

Taking $Z = \frac{\nu - x}{\sigma\sqrt{y}}$, we get,

$$
\begin{aligned}
D(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} (x + \sigma\sqrt{y} Z) e^{-\frac{Z^2}{2}} \mathrm{d}Z \\
&= \frac{x}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{-\frac{Z^2}{2}} \mathrm{d}Z + \sigma\sqrt{\frac{y}{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} Z e^{-\frac{Z^2}{2}} \mathrm{d}Z \\
&= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}} \mathrm{d}t + \sigma\sqrt{\frac{y}{2\pi}} e^{-\frac{x^2}{2\sigma^2 y}}
\end{aligned}
$$

Hence, $D(x,y) = xN(d) + \sigma\sqrt{\frac{y}{2\pi}}e^{-\frac{x^2}{2\sigma^2 y}}$ where, $d = \frac{x}{\sigma\sqrt{y}}$.

Therefore, $C(S,t) = e^{-r(T-t)}\left[\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]N(d) + \frac{1}{\sqrt{2\pi}}e^{-r(T-t)}\sigma\sqrt{T-t}e^{-\frac{d^2}{2}}$

where, $d = \frac{\ln\left(\frac{S}{X}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.

Now, we consider average of two payoff functions Plain Vanilla and Log,

$$f(S) = \begin{cases} \frac{\ln\left(\frac{S}{X}\right)+S-X}{2} & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}$$

Therefore,

$$\begin{aligned} f(Xe^x) &= \begin{cases} \frac{X(e^x-1)+x}{2} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \qquad \text{and} \end{cases} \\ D(x,y) &= \frac{1}{2\sigma\sqrt{2\pi y}}\int_0^\infty (X(e^\nu - 1) + \nu)e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu \\ &= \frac{X}{2\sigma\sqrt{2\pi y}}\left[\int_0^\infty e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu - \int_0^\infty e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu\right] + \frac{1}{2\sigma\sqrt{2\pi y}}\int_0^\infty \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu \\ &= \frac{X}{2\sigma\sqrt{2\pi y}}\int_0^\infty e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu - \frac{X}{2\sigma\sqrt{2\pi y}}\int_0^\infty e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu + \frac{1}{2\sigma\sqrt{2\pi y}}\int_0^\infty \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu \end{aligned}$$

Substituting $Z = \frac{\nu - x}{\sigma\sqrt{y}}$ we get,

$$\begin{aligned} D(x,y) &= \frac{X}{2\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{Z^2-2\sigma\sqrt{y}Z+\sigma^2 y}{2}}\,\mathrm{d}Z - \frac{X}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\ &\quad + \frac{1}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty (x + \sigma\sqrt{y}Z)e^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\ &= \frac{X}{2\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{(Z-\sigma\sqrt{y})^2}{2}}\,\mathrm{d}Z - \frac{(X-x)}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\ &\quad + \frac{\sigma\sqrt{y}}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty Ze^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\ &= \frac{X}{2\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t - \frac{(X-x)}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t + \frac{\sigma}{2}\sqrt{\frac{y}{2\pi}}e^{-\frac{x^2}{2\sigma^2 y}} \\ &= \frac{X}{2\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\int_{-\infty}^{\frac{x+\sigma^2 y}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}}\,\mathrm{d}t - \frac{(X-x)}{2\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}}\,\mathrm{d}t + \frac{\sigma}{2}\sqrt{\frac{y}{2\pi}}e^{-\frac{d_2^2}{2}} \\ &= \frac{X}{2}e^{x+\frac{\sigma^2 y}{2}}N(d_1) - \frac{(X-x)}{2}N(d_2) + \frac{\sigma}{2}\sqrt{\frac{y}{2\pi}}e^{-\frac{d_2^2}{2}} \end{aligned}$$

where, $d_1 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}}$, $d_2 = \frac{x}{\sigma\sqrt{y}}$ and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{t^2}{2}}\,\mathrm{d}t$.

Hence we get,

$$C(S,t) = \frac{S}{2}N(d_1) - \frac{1}{2}e^{-r(T-t)}\left[X - \ln\left(\frac{S}{X}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]N(d_2) + e^{-r(T-t)}\frac{\sigma}{2}\sqrt{\frac{T-t}{2\pi}}e^{-\frac{d_2^2}{2}}$$

where, $d_1 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}} = \frac{\ln\left(\frac{S}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{X}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.

Theorem 2.1. Let C_f and C_g be the solutions of BSM equation with the boundary conditions $\lim_{t\to T_-} C_f(S,t) = f(S)$ and $\lim_{t\to T_-} C_g(S,t) = g(S)$. Then $\alpha C_f + \beta C_g$ is a solution of BSM with the boundary condition $\alpha f + \beta g$.

Proof: Let $C_{\alpha f+\beta g}$ be the solution of BSM with the boundary condition $\alpha f + \beta g$.
Define, $V(S,t) = \alpha C_f + \beta C_g - C_{\alpha f+\beta g}$. Here $V(S,t)$ is a solution of BSM with the boundary condition $\lim_{t\to T_-} V(S,t) = 0$. Hence we have,

$$V(S,t) = 0 \Rightarrow \alpha C_f + \beta C_g - C_{\alpha f+\beta g} = 0 \Rightarrow \alpha C_f + \beta C_g = C_{\alpha f+\beta g}. \quad \blacksquare$$

Conclusion: Theorem 2.1 opens up a new direction of considering averages of two known payoff functions. The theorem confirms that the averages of payoff functions behaves the way one expect it to behave. In this paper, we consider the averages of most well-known and useful payoff functions, namely Plain Vanilla and Log payoff functions. It is now open for researchers to consider the Modified Log Payoff function [2] along with any of the two payoff function considered in Theorem 2.1. While choosing appropriate payoff function for the known data, this result provides one more choice, the averages of the two approximate payoff functions. In other words, for curve fitting and surface fitting there is one more natural choice.

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