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Edge domination in various snake graphs

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Abstract

A set $F \subseteq E(G)$ is an edge dominating set if each edge in $E(G)$ is either in F or is adjacent to an edge in F. An edge dominating set F is called a minimal edge dominating set if no proper subset F' of F is an edge dominating set. The edge domination number $\gamma'(G)$ is the minimum cardinality among all minimal edge dominating sets. We investigate the edge domination number of some graphs called snakes which are obtained from path P_n by replacing its edges by cycles C_3 and C_4.

Keywords: Edge dominating set, minimal edge dominating set, edge domination number.
AMS Subject Classification(2010): 05C38, 05C69.

1 Introduction

The concept of domination in graphs has received considerable attention due to its diversified applications ranging from design analysis of network to military surveillance and linear algebra to social sciences. The comprehensive bibliography on the concept of domination and its variants can be found in Hedetniemi and Laskar [5]. This paper is focused on edge domination in graphs.

Throughout the paper, by a graph G we mean a simple, finite, connected and undirected graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V(G)$, the open neighborhood $N(v)$ of v is defined as $N(v) = \{u \in V(G) : uv \in E(G)\}$ while $N[v] = N(v) \bigcup \{v\}$ is called the closed neighborhood of v. For a set $D \subseteq V(G)$, the open neighborhood $N(D)$ is defined to be $\bigcup\limits_{v \in D} N(v)$, and the closed neighborhood of D is $N[D] = N(D) \bigcup D$. A set $D \subseteq V(G)$ in a graph G is called a dominating set if $N[D] = V(G)$. The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$.

An edge e of a graph G is said to be incident with the vertex v if v is an end vertex of e. Two edges are adjacent if they have an end vertex in common.

A set $F \subseteq E(G)$ is an edge dominating set if each edge in $E(G)$ is either in F or is adjacent to an edge in F. An edge dominating set F is called a minimal edge dominating set (MEDS) if no proper subset F' of F is an edge dominating set. The edge domination number $\gamma'(G)$ is the minimum cardinality among all minimal edge dominating sets. The concept of edge domination was introduced by Mitchell and Hedetniemi [8] and further explored by Arumugam and Velammal [2].

Moreover, Yannakakis and Gavril [13], Dutton and Klostermeyer [3], Kulli and Soner [7], Jayaram [6], Mojdeh and Sadeghi [9], Arumugam and Jerry [1], Vaidya and Pandit [10, 11] and Zelinka [14] studied the concept of edge domination in various contexts.

We provide a brief summary of definitions which are useful for the present investigation.

Definition 1.1. For each $e \in E(G)$, $N(e)$ denotes the open neighborhood of e in G. That is, the set of all edges which are adjacent to e in G.

Definition 1.2. The degree of an edge $e = uv$ of G is defined by $deg(e) = deg(u) + deg(v) - 2$, that is, the number of edges adjacent to it. The maximum degree of an edge in G is denoted by $\triangle'(G)$.

For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n. For the various graph theoretic notations and terminology, we follow West [12] while the terms related to the concept of domination are used in the sense of Haynes *et al.* [4].

The present work is to investigate some new results on edge domination in graphs.

2 Main Results

Proposition 2.1. [6] An edge dominating set S is minimal if and only if for each $e \in S$, one of the following two conditions holds:

(a) $N(e) \cap S = \emptyset$.

(b) there exists an edge $f \in E(G) - S$, such that $N(f) \cap S = \{e\}$.

Definition 2.2. The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3.

Theorem 2.3. For the triangular snake T_n, $\gamma'(T_n) = \lceil \frac{n}{2} \rceil$.

Proof: Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Let the triangular snake T_n be obtained by replacing every edge of P_n by a triangle C_3. Then, $|V(T_n)| = 2n - 1$ and $|E(T_n)| = 3(n - 1)$.

Now, in order to dominate the cycles C_3 obtained by replacing the pendant edges of P_n in T_n, an edge dominating set of T_n must have an edge from each C_3. Moreover, $deg(e_i) = \triangle'(T_n)$ for $1 < i < n - 1$. Hence, to attain the minimum cardinality of an edge set of T_n, we construct an edge set $F \subset E(T_n)$ as follows:

$$F = \begin{cases} \{e_1, e_3, \ldots, e_{n-1}\} & \text{for even } n \\ \\ \{e_1, e_3, \ldots, e_{n-2}, e_{n-1}\} & \text{for odd } n. \end{cases}$$

Then $|F| = \lceil \frac{n}{2} \rceil$.

Since each edge in $E(T_n)$ is either in F or is adjacent to an edge in F, it follows that the set F is an edge dominating set of T_n.

Moreover, for each edge $e \in F$, there exists an edge $f \in E(T_n) - F$ for which $N(f) \cap F = \{e\}$. Therefore, by Proposition 2.1, the set F is a minimal edge dominating set of T_n. Now, we claim that F is an edge dominating set with minimum cardinality. If possible, let F_1 be an edge dominating set such that $|F_1| < |F|$. F_1 cannot contain all the edges $e_i \in F_1$ such that $deg(e_i) = 6 = \triangle'(T_n)$. Furthermore, $(\lceil \frac{n}{2} \rceil - 1)(\triangle'(T_n) - 1) + \lceil \frac{n}{2} \rceil - 1 < |E(T_n)|$ for even n while $(\lceil \frac{n}{2} \rceil - 1)(\triangle'(T_n) - 1) + \lceil \frac{n}{2} \rceil - 1 = |E(T_n)|$ for odd n. But for odd n, it is not possible as there are at most $\lceil \frac{n}{2} \rceil - 2$ edges having degree $\triangle'(T_n) - 1$ for attaining the minimum cardinality of F_1. Therefore, $|F_1| > \lceil \frac{n}{2} \rceil - 1$ which is a contradiction. Hence, the set F is of minimum cardinality.

Thus, the set F is an MEDS of T_n with minimum cardinality implying that $\gamma'(T_n) = \lceil \frac{n}{2} \rceil$. ∎

Definition 2.4. The double triangular snake DT_n consists of two triangular snakes that have a common path.

The following Theorem 2.5 can be proved by the arguments analogous to the proof of Theorem 2.3.

Theorem 2.5. For the double triangular snake DT_n, $\gamma'(DT_n) = \lceil \frac{n}{2} \rceil$.

Definition 2.6. The alternate triangular snake AT_n is obtained from a path P_n by replacing every alternate edge of a path P_n by a cycle C_3.

Theorem 2.7. For the alternate triangular snake AT_n, $\gamma'(AT_n) = \lfloor \frac{n}{2} \rfloor$.

Proof: Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Let the alternate triangular snake AT_n be obtained by replacing every alternate edges of P_n by a triangle C_3.

Note that $|E(AT_n)| = \begin{cases} 2n-1 & \text{for even } n \\ 2n-2 & \text{for odd } n. \end{cases}$

For $n = 2, 3$, the set $F = \{e_1\}$ is obviously an MEDS with minimum cardinality. Hence, $\gamma'(AT_n) = 1 = \lfloor \frac{n}{2} \rfloor$.

In order to dominate the pendant edges as well as the cycles C_3 obtained by replacing the pendant edges of P_n in AT_n, an edge dominating set of AT_n must have an edge from C_3 for even n while for odd n, it should contain the edges e_1 and e_{n-2} for attaining its minimum cardinality. Also, $deg(e_i) = \triangle'(AT_n)$ for $1 < i < n-1$. Hence, in order to attain the minimum cardinality of an edge set of AT_n, we can construct an edge set F of AT_n as follows:

$$F = \begin{cases} \{e_1, e_3, \ldots, e_{n-2}\} & \text{for odd } n \\ \\ \{e_1, e_3, \ldots, e_{n-1}\} & \text{for even } n \end{cases}$$

Then $|F| = \lfloor \frac{n}{2} \rfloor$.

Since each edge in $E(AT_n)$ is either in F or is adjacent to an edge in F, it follows that the set F is an edge dominating set of AT_n. Moreover, for each edge $e \in F$, $N(e) \cap F = \emptyset$. Therefore, by Proposition 2.1, the set F is an MEDS of AT_n.

For $n \geq 4$, as $\triangle'(AT_n) = 4$, an edge of AT_n can dominate five distinct edges including itself. For $4 \leq n \leq 7$, $(\lfloor \frac{n}{2} \rfloor - 1) \triangle' (AT_n) + \lfloor \frac{n}{2} \rfloor - 1 < |E(AT_n)|$. But, $|F| = \lfloor \frac{n}{2} \rfloor$. This implies that $\gamma'(AT_n) = \lfloor \frac{n}{2} \rfloor$ where $4 \leq n \leq 7$. For $n \geq 8$, we claim that F is an edge dominating set with minimum cardinality $\lfloor \frac{n}{2} \rfloor$. Suppose, if possible, an edge set $F_1 \subseteq E(AT_n)$, $F_1 \neq F$ is an edge dominating set of AT_n with $|F_1| = \lfloor \frac{n}{2} \rfloor - 1 < |F|$. In order to attain the minimum cardinality of F_1, we cannot take all the edges $e_i \in F_1$ where $deg(e_i) = \triangle'(AT_n)$. Moreover, $(\lfloor \frac{n}{2} \rfloor - 1)(\triangle'(AT_n) - 1) + \lfloor \frac{n}{2} \rfloor - 1 < |E(AT_n)|$. Therefore, F_1 is not an edge dominating set of AT_n, which is a contradiction. Hence, the set F is of minimum cardinality.

Thus, the set F is an MEDS of AT_n with minimum cardinality which implies that $\gamma'(AT_n) = \lfloor \frac{n}{2} \rfloor$. ■

Remark 2.8. The double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes which have a common path. By the arguments analogous to the proof of Theorem 2.7, we can prove that $\gamma'(DA(T_n)) = \lfloor \frac{n}{2} \rfloor$.

In Figure 1, the double alternate triangular snake $DA(T_7)$ is shown in which the set of dotted edges is its edge dominating set with minimum cardinality.

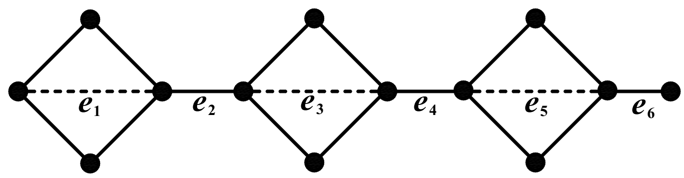

Figure 1

Definition 2.9. The quadrilateral snake Q_n is obtained from a path P_n by replacing every edge of a path P_n by a cycle C_4.

Theorem 2.10. For the quadrilateral snake Q_n, $\gamma'(Q_n) = n$.

Proof: Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. In order to obtain Q_n, replace every edge of P_n by a cycle C_4. Let $C_4^{(1)}, C_4^{(2)}, \ldots, C_4^{(n-1)}$ denote the cycles C_4 obtained by replacing the edges $e_1, e_2, \ldots, e_{n-1}$ of P_n in Q_n and let e_i, x_i, y_i, z_i be the edges of the cycle $C_4^{(i)}$ for $1 \leq i \leq n-1$. Then, $|V(Q_n)| = 3n - 2$ and $|E(Q_n)| = 4(n-1)$. Now, none of the edges $y_1, y_2, \ldots, y_{n-1}$ of Q_n are adjacent to each other and there is no edge which is adjacent to any two of the edges $y_1, y_2, \ldots, y_{n-1}$ of Q_n. Therefore, at least $n-1$ edges are required to dominate the edges $y_1, y_2, \ldots, y_{n-1}$. Hence, we construct an edge set $F \subseteq E(Q_n)$ as follows:
$F = \{x_1, x_2, x_3, \ldots, x_{n-1}, z_{n-1}\}$.
Then $|F| = n$.

The set F is an edge dominating set of Q_n because each edge in $E(Q_n)$ is either in F or adjacent to an edge in F.

Moreover, for each edge $e \in F$, there exists an edge $f \in E(Q_n) - F$ for which $N(f) \cap F = \{e\}$. Therefore, by Proposition 2.1, the set F is a minimal edge dominating set of Q_n. Furthermore, $\triangle'(Q_n) = 6$ and from the adjacency nature of the edges in Q_n, it can be seen that only $n-1$ edges are not enough to dominate all the edges of Q_n. Therefore, for every edge dominating set F_1, $|F_1| > n-1$. As $|F| = n$, it follows that F is an MEDS with minimum cardinality. Thus, $\gamma'(Q_n) = n$. ■

Illustration 2.11. In Figure 2, the quadrilateral snake Q_5 is shown in which the set of dotted edges is its edge dominating set with minimum cardinality.

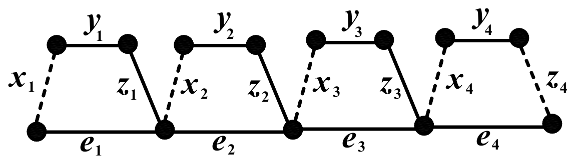

Figure 2

Definition 2.12. The double quadrilateral snake DQ_n consists of two quadrilateral snakes that have a common path.

Theorem 2.13. For the double quadrilateral snake DQ_n, $\gamma'(DQ_n) = 2(n-1)$.

Proof: Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of the common path P_n of DQ_n. In order to obtain DQ_n, every edge e_i of P_n is replaced by two distinct cycles namely C_4 and C_4' having the common edge e_i. Let e_i, x_i, y_i, z_i be the edges of C_4 and let e_i, x_i', y_i', z_i' be the edges of C_4' obtained by replacing the edge e_i of P_n in DQ_n. Then, $|V(DQ_n)| = 5n-4$ and $|E(DQ_n)| = 7(n-1)$.

Now, from the adjacency nature of the edges of DQ_n, it can be seen that at least $2(n-1)$ distinct edges are required to dominate the edges $y_1, y_2, \ldots, y_{n-1}$ and $y_1', y_2', \ldots, y_{n-1}'$ of DQ_n because

(i) none of the edges $y_1, y_2, \ldots, y_{n-1}$ are adjacent to each other.
(ii) none of the edges $y_1', y_2', \ldots, y_{n-1}'$ are adjacent to each other.
(iii) there is no edge which is adjacent to any two of the edges $y_1, y_2, \ldots, y_{n-1}$.
(iv) there is no edge which is adjacent to any two of the edges $y_1', y_2', \ldots, y_{n-1}'$.
(v) there is no edge which is adjacent to any two of the edges y_i, y_i' where $1 \le i \le n-1$.

Since $2(n-1)$ edges can also dominate the remaining edges of DQ_n, it follows that every edge dominating set F of DQ_n must have at least $2(n-1)$ edges of DQ_n. Thus, $|F| \ge 2(n-1)$ which implies that $\gamma'(DQ_n) = 2(n-1)$ as required. ■

Definition 2.14. The alternate quadrilateral snake $A(QS_n)$ is obtained from a path P_n by replacing every alternate edge of a path P_n by a cycle C_4.

Theorem 2.15. For the alternate quadrilateral snake $A(QS_n)$,

$$\gamma'(A(QS_n)) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil & \text{if } n \equiv 0(\text{mod } 2) \\ \\ \left\lfloor \frac{3n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof: Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. By replacing the alternate edges of P_n by a cycle C_4, we obtain $A(QS_n)$. Let e_i, x_i, y_i, z_i be the edges of the cycle C_4 obtained by replacing the edge e_i of P_n by C_4 in $A(QS_n)$.

Then, $|E(A(QS_n))| = \begin{cases} \frac{5n-2}{2} & \text{if } n \equiv 0(\text{mod } 2) \\ \\ \frac{5n-5}{2} & \text{if } n \equiv 1(\text{mod } 2). \end{cases}$

For $n = 2, 3$, since the set $F_1 = \{x_1, z_1\}$ is clearly an MEDS with minimum cardinality, it follows that $\gamma'(A(QS_n)) = 2$.

Now, none of the edges y_i of $A(QS_n)$ are adjacent to each other and there is no edge which is adjacent to any two of the edges y_i. Therefore, at least either an edge adjacent to y_i or the edge y_i itself must belong to an edge dominating set of $A(QS_n)$. Also, for $n \geq 4$, $deg(e_i) = \triangle'(A(QS_n))$ for $1 < i < n-1$. Hence, for attaining the minimum cardinality of an edge set of $A(QS_n)$ $(n \geq 4)$, we can construct an edge set $F \subseteq E(A(QS_n))$ as follows:

$$F = \begin{cases} \{x_{4i+1}, e_{4j+2}, z_{4k+3}\} & \text{for } n \equiv 0, 1, 3(\text{mod } 4) \\ \\ \{x_{4i+1}, e_{4j+2}, z_{4k+3}\} \cup \{e_{n-1}\} & \text{otherwise.} \end{cases}$$

where $0 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor, 0 \leq j < \left\lceil \frac{n-2}{4} \right\rceil, 0 \leq k < \left\lceil \frac{n-3}{4} \right\rceil$.

Then $|F| = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil & \text{if } n \equiv 0(\text{mod } 2) \\ \\ \left\lfloor \frac{3n}{4} \right\rfloor & \text{otherwise.} \end{cases}$

Since each edge in $E(A(QS_n))$ is either in F or is adjacent to an edge in F, it follows that the set F is an edge dominating set of $A(QS_n)$.

Moreover, for each edge $e \in F$, $N(e) \cap F = \emptyset$. Hence, by Proposition 2.1, the set F is an MEDS of $A(QS_n)$. Now, none of the edges y_i of $A(QS_n)$ are adjacent to each other and there is no edge which is adjacent to any two of the edges y_i. Therefore, at least $\left\lceil \frac{n}{2} \right\rceil$ edges are required to dominate the edges y_i of $A(QS_n)$. Furthermore, $\triangle'(A(QS_n)) = 4$ and from the adjacency nature of the edges in $A(QS_n)$, it follows that the set F is of minimum cardinality. That is, the set F is an MEDS with minimum cardinality.

Thus,

$$\gamma'(A(QS_n)) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil & \text{if } n \equiv 0 (\bmod 2) \\ \\ \left\lfloor \frac{3n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

■

Illustration 2.16. In Figure 3, the alternate quadrilateral snake $A(QS_6)$ is shown in which the set of dotted edges is its edge dominating set with minimum cardinality.

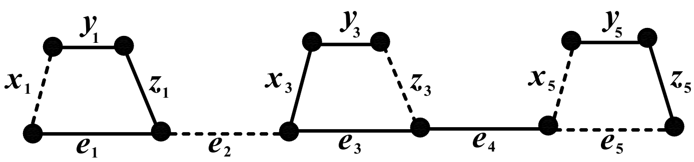

Figure 3

Definition 2.17. The double alternate quadrilateral snake $DA(QS_n)$ consists of two alternate quadrilateral snakes that have a common path.

The following Theorem 2.18 can be proved by the arguments analogous to the proof of Theorem 2.15.

Theorem 2.18. For the double alternate quadrilateral snake $DA(QS_n)$,

$$\gamma'(DA(QS_n)) = \begin{cases} n & \text{if } n \equiv 0 (\bmod 2) \\ \\ n-1 & \text{otherwise.} \end{cases}$$

3 Concluding Remarks

To investigate the edge domination number of graphs is always interesting and challenging as well. We have investigated the edge domination number of various snakes.

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