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Strongly multiplicative labeling of some standard graphs

K. K. Kanani[1], T. M. Chhaya[2]

[1]Government Engineering College
Rajkot, Gujarat, India.

[2]L. E. College
Morbi, Gujarat, India.

Abstract

In this paper we investigate strongly multiplicative labeling of some standard graphs. We prove that helm, flower graph, fan, friendship graph and bistar are strongly multiplicative. We also prove that gear graph, double triangular snake, double fan and double wheel are strongly multiplicative.

Keywords: Strongly multiplicative labeling, helm, flower graph, fan graph, friendship graph, bistar, gear graph, double triangular snake, double fan, double wheel.
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1 Introduction

Graph Theory is one of the expanding branches of Discrete Mathematics. It has applications in almost all the disciplines of Science and Engineering. There are many potential fields of research in Graph Theory. Some of them are algebraic graph theory, domination in graphs, algorithmic graph theory, topological graph theory, fuzzy graph theory and graph labeling.

Graph labeling is an integral part of graph theory which assigns numeral values to vertices or edges or both, subject to certain conditions. A latest survey of all the graph labeling techniques can be found in Gallian Survey[2].

In this paper, by a graph we mean finite, connected, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)| = p$ and size $|E(G)| = q$.

Definition 1.1. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s).

Definition 1.2. A graph $G = (V(G), E(G))$ with p vertices is said to be *multiplicative* if the vertices of G can be labeled with p distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Multiplicative labeling was introduced by Beineke and Hegde[1] and they proved that every graph G admits multiplicative labeling and defined strongly multiplicative labeling as follows.

Definition 1.3. A graph $G = (V(G), E(G))$ with p vertices is said to be *strongly multiplicative* if the vertices of G can be labeled with p consecutive positive integers $1, 2, 3, ..., p$ such that the label induced on the edges by the product of labels of end vertices are all distinct.

Beineke and Hegde[1] proved that every cycle C_n, wheel W_n are strongly multiplicative. They also proved that the complete graph K_n is strongly multiplicative $\Leftrightarrow n \leq 5$ and the complete bipartite graph $K_{n,n}$ is strongly multiplicative $\Leftrightarrow n \leq 4$. Further they established that every spanning subgraph of a strongly multiplicative graph is strongly multiplicative and every graph is an induced subgraph of a strongly multiplicative graph. Vaidya et al.[4] proved arbitrary supersubdivisions of the graphs namely the tree T_n, complete bipartite graphs $K_{m,n}$, grid graph $P_m \times P_n$, $C_n \odot P_m$ and $C_n^{(m)}$ are strongly multiplicative.

Kanani and Chhaya[3] discussed strongly multiplicative labeling of some path related graphs and proved that the total graph $T(P_n)$ of the path P_n, the splitting Graph $S'(P_n)$ the path P_n, the shadow graph $D_2(P_n)$ of the path P_n, and the triangular snake TS_n are strongly multiplicative.

In this paper, we prove that helm, flower graph, fan, friendship graph and bistar are strongly multiplicative. We also prove that gear graph, double triangular snake, double fan and double wheel are strongly multiplicative.

Here we consider the following definitions of standard graphs. For any undefined term in this paper, we rely upon West[6].

Definition 1.4. The *helm* $(H_n, n \geq 3)$ is the graph obtained from a wheel W_n by attaching a pendant edge at each rim vertex.

Definition 1.5. The *flower graph* (Fl_n) is obtained from helm H_n by joining each pendant vertex to the central vertex of the helm.

Definition 1.6. The *fan graph*(f_n) is obtained by taking $n - 3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the *apex vertex.* In other words f_n is the join $P_n + K_1$.

Definition 1.7. The *friendship graph* is the one point union of n copies of cycle C_3. It is denoted by F_n.

Definition 1.8. The *bistar*$(B_{m,n})$ is the graph obtained by joining the apex vertices of two copies of star $K_{1,m}$ and $K_{1,n}$ by an edge.

Definition 1.9. The *double triangular snake* (DT_n) consists of two triangular snakes that have a common Path P_n.

Definition 1.10. The *gear graph* (G_n) is obtained from the wheel W_n by adding a vertex between every pair of vertices of the n-cycle.

Definition 1.11. The *double fan graph* (DF_n) is composed of $P_n + 2K_1$.

Definition 1.12. The *double wheel graph*(DW_n) of size n is composed of $2C_n + K_1$. It consists of two cycles of size n where vertices of two cycles are all connected to a central vertex.

2 Main Results

Theorem 2.1. The helm H_n is strongly multiplicative.

Proof: Let H_n be the helm. Let v_0 be the central vertex. Let $v_1, v_2, ..., v_n$ be rim vertices and $v_1', v_2', ..., v_n'$ be pendant vertices of H_n. We note that $|V(H_n)| = 2n + 1$ and $|E(H_n)| = 3n$.
We define vertex labeling $f : V(H_n) \to \{1, 2, ..., 2n + 1\}$ as follows:
$f(v_0) = p_1$; where p_1 is the highest prime number such that, $3 \leq p_1 \leq 2n + 1$;
$f(v_n) = p_2$; where p_2 is the second highest prime number such that, $3 \leq p_2 < p_1 \leq 2n + 1$;
$f(v_n') = p_2 + 1$.
Label the remaining vertices starting from $v_1, v_2, ..., v_{n-1}$ consecutively in clockwise direction from the set $\{1, 3, ..., 2n + 1\} \setminus \{p_1, p_2\}$ and label the remaining vertices starting from $v_1', v_2', ..., v_{n-1}'$ consecutively in clockwise direction from the set $\{2, 4, ..., 2n\} \setminus \{p_2 + 1\}$.
The labeling pattern defined above covers all the possibilities and in each case the graph H_n admits strongly multiplicative labeling. Therefore, the helm H_n is strongly multiplicative. ■

Illustration 2.2. The helm H_6 and its strongly multiplicative labeling is shown in *Figure 1.*

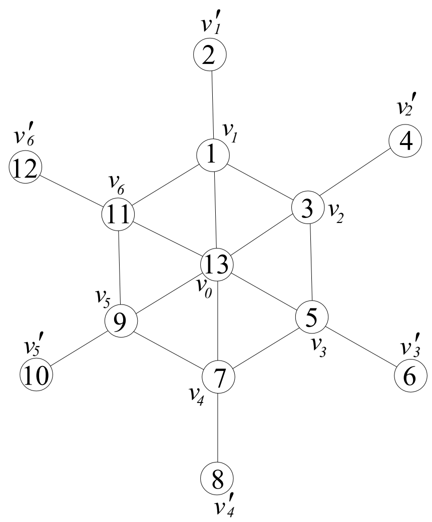

Figure 1: Strongly multiplicative labeling of helm H_6.

Theorem 2.3. The flower graph Fl_n is strongly multiplicative.

Proof: Let Fl_n be the flower graph. Let v_0 be the central vertex. Let $v_1, v_2, ..., v_n$ be the rim vertices and $v_1', v_2', ..., v_n'$ be the pendant vertices of Fl_n. We note that $|V(Fl_n)| = 2n + 1$ and $|E(Fl_n)| = 4n$.

We define vertex labeling $f : V(Fl_n) \rightarrow \{1, 2, ..., 2n+1\}$ as follows:

$f(v_0) = p_1$; where p_1 is the highest prime number such that, $3 \leq p_1 \leq 2n+1$;

$f(v_n) = p_2$; where p_2 is the second highest prime number such that, $3 \leq p_2 < p_1 \leq 2n+1$;

$f(v_n^{'}) = p_2 + 1$.

Now, label the remaining vertices starting from $v_1, v_2, ..., v_{n-1}$ consecutively in clockwise direction from the set $\{1, 3, ..., 2n+1\} \setminus \{p_1, p_2\}$ and label the remaining vertices starting from $v_1^{'}, v_2^{'}, ..., v_{n-1}^{'}$ consecutively in clockwise direction from the set $\{2, 4, ..., 2n\} \setminus \{p_2 + 1\}$.

The labeling pattern defined above satisfies the vertex conditions and edge conditions of strongly multiplicative labeling. Therefore, the flower graph Fl_n is strongly multiplicative. ■

Illustration 2.4. The flower graph Fl_5 and its strongly multiplicative labeling is shown in *Figure 2*.

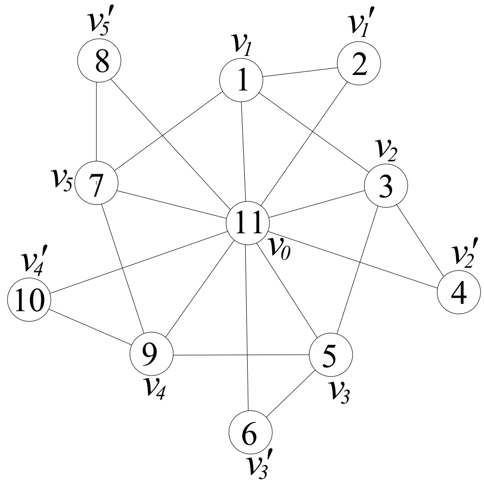

Figure 2: Strongly multiplicative labeling of the flower graph Fl_5.

Theorem 2.5. The fan graph f_n is strongly multiplicative.

Proof: Let f_n be the fan graph. Let v_0 be the apex vertex and $v_1, v_2, ..., v_n$ be path vertices of f_n. We note that $|V(f_n)| = n+1$ and $|E(f_n)| = 2n-1$.

We define vertex labeling $f : V(f_n) \rightarrow \{1, 2, ..., n+1\}$ as follows:

$f(v_0) = p$; where p is the highest prime number such that, $3 \leq p \leq n+1$;
Now, label the remaining vertices $v_1, v_2, ..., v_n$ consecutively from the set {1,2,...,n+1} \{p}.

The labeling pattern defined above covers all the possibilities and in each case the graph f_n admits strongly multiplicative labeling. Hence, the fan graph f_n is strongly multiplicative. ■

Illustration 2.6. The fan graph f_6 and its strongly multiplicative labeling is shown in *Figure 3*.

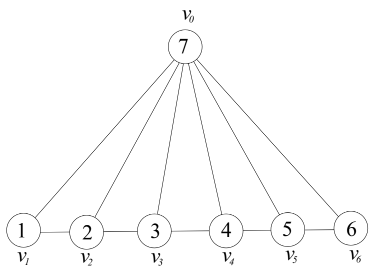

Figure 3: Strongly multiplicative labeling of fan graph f_6.

Theorem 2.7. The friendship graph F_n is strongly multiplicative.

Proof: Let F_n be the friendship graph. Let v_0 be the common vertex and $v_1, v_2, ..., v_{2n}$ be other vertices of F_n. We note that $|V(F_n)| = 2n + 1$ and $|E(F_n)| = 3n$.
We define vertex labeling $f : V(F_n) \to \{1, 2, ..., 2n + 1\}$ as follows:
$f(v_0) = p$; p is the highest prime number such that, $1 \leq p \leq 2n + 1$.
Now, label the remaining vertices $v_1, v_2, ..., v_{2n}$ consecutively from the set {1,2,...,2n+1} \{p}.
The labeling pattern defined above covers all the possibilities and in each case the graph F_n admits a strongly multiplicative labeling. Hence, the friendship graph F_n is strongly multiplicative. ■

Illustration 2.8. The friendship graph F_3 and its strongly multiplicative labeling is shown in *Figure 4*.

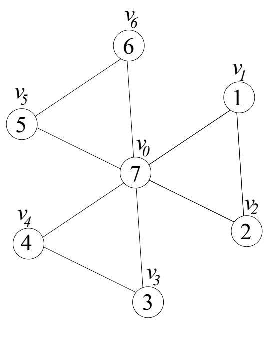

Figure 4: Strongly multiplicative labeling of friendship graph F_3.

Theorem 2.9. The bistar $B_{m,n}$ is strongly multiplicative.

Proof: Let $B_{m,n}$ be the bistar with vertex set $\{u, v, u_i, v_j : 1 \leq i \leq m\ , 1 \leq j \leq n\}$ where u_i, v_j are pendant vertices. We note that $|V(B_{m,n})| = m + n + 2$ and $|E(B_{m,n})| = m + n + 1$.
We define vertex labeling $f : V(B_{m,n}) \to \{1, 2, ..., m + n + 2\}$ as follows:
$f(v_0) = 1$;
$f(u_0) = p$; where p is the highest prime number such that, $3 \leq p \leq m + n + 2$.

Now, label the remaining vertices $u_1, u_2, ..., u_m$ and $v_1, v_2, ..., v_n$ consecutively from the set $\{2, 3, ..., m, m+1, ..., m+n+2\} \backslash \{\text{p}\}$.

The labeling pattern defined above satisfies the vertex conditions and edge conditions of strongly multiplicative labeling. Therefore, the bistar $B_{m,n}$ is strongly multiplicative. ■

Illustration 2.10. The bistar $B_{4,4}$ and its strongly multiplicative labeling is shown in Figure 5.

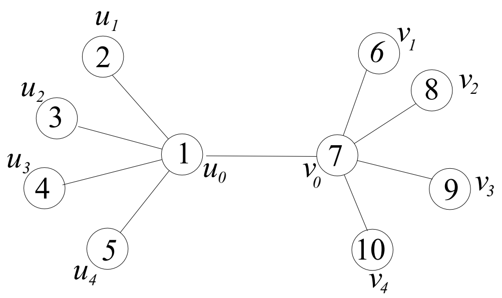

Figure 5: Strongly multiplicative labeling of the bistar $B_{4,4}$.

Theorem 2.11. The gear graph G_n is strongly multiplicative.

Proof: Let G_n be the gear graph. Let v_0 be apex vertex of a wheel. Let $v_1, v_2, ..., v_{2n}$ be the vertices of G_n. We note that $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n-2$.

We define vertex labeling $f : V(G_n) \to \{1, 2, ..., 2n+1\}$ as follows:

$f(v_0) = p_1$; where p_1 is the highest prime number such that,$3 \leq p_1 \leq 2n+1$;

$f(v_1) = 1$;

$f(v_{2n}) = p_2$; where p_2 is the second highest prime number such that, $3 \leq p_2 < p_1 \leq 2n+1$.

Now, label the remaining vertices $v_2, v_3, ..., v_{2n-1}$ consecutively from the set $\{2, 3, ..., 2n+1\} \setminus \{p_1, p_2\}$.

The labeling pattern defined above covers all the possibilities and in each case G_n admits strongly multiplicative labeling. Therefore, the gear graph G_n is strongly multiplicative. ■

Illustration 2.12. The gear graph G_6 and its strongly multiplicative labeling is shown in Figure 6.

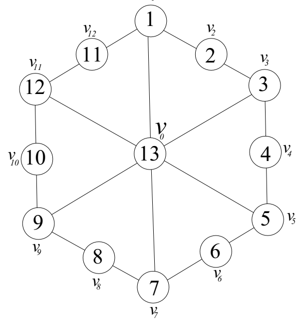

Figure 6: Strongly multiplicative labeling of gear graph G_6.

Theorem 2.13. The double triangular snake DT_n is strongly multiplicative.

Proof: Let DT_n be the double triangular snake obtained from path P_n. Let $v_1, v_2, ..., v_n$ be the vertices of the path P_n. Let $u_1, u_2, ..., u_{n-1}$ be the upper triangle vertices and $u_1^{'}, u_2^{'}, ..., u_{n-1}^{'}$ be the lower triangle vertices of DT_n consecutively. We note that $|V(DT_n)| = 3n - 2$ and $|E(DT_n)| = 5n - 5$.
We define vertex labeling $f : V(DT_n) \to \{1, 2, ..., 3n - 2\}$ as follows:
$f(u_i) = 3i - 2;\ 1 \leq i \leq n - 1;$
$f(v_i) = 3i - 1;\ 1 \leq i \leq n - 1;$
$f(v_n) = 3n - 2;$
$f(u_i^{'}) = 3i;\ 1 \leq i \leq n - 1;$
The labeling pattern defined above covers all the possibilities and in each case the graph DT_n admits strongly multiplicative labeling. Therefore, the double triangular snake DT_n is strongly multiplicative. ■

Illustration 2.14. The double triangular snake DT_6 and its strongly multiplicative labeling is shown in Figure 7.

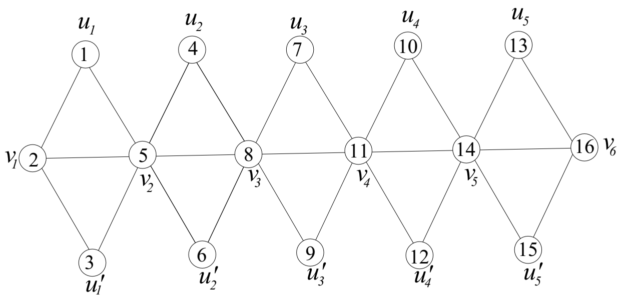

Figure 7: Strongly multiplicative labeling of double triangular snake DT_6.

Theorem 2.15. The double fan graph DF_n is strongly multiplicative.

Proof: Let DF_n be the double fan graph. Let v_0 and u_0 be two apex vertices. Let $v_1, v_2, ..., v_n$ be path vertices of P_n. We note that $|V(DF_n)| = n + 2$ and $|E(DF_n)| = 3n - 1$.
We define vertex labeling $f : V(DF_n) \to \{1, 2, ..., n + 2\}$ as follows:
$f(v_0) = p_1$; where p_1 is the highest prime number such that, $2 \leq p_1 \leq n + 2$;
$f(u_0) = p_2$; where p_2 is the second highest prime number such that, $2 \leq p_2 < p_1 \leq n + 2$;
Now, label the remaining vertices $v_1, v_2, ..., v_n$ consecutively from the set {1,2,...,n+2}$\setminus\{p_1, p_2\}$.
The labeling pattern defined above covers all the possibilities and in each case the graph DF_n admits strongly multiplicative labeling. Therefore, the double fan graph DF_n is strongly multiplicative. ■

Illustration 2.16. The double fan graph DF_6 and its strongly multiplicative labeling is shown in *Figure 8*.

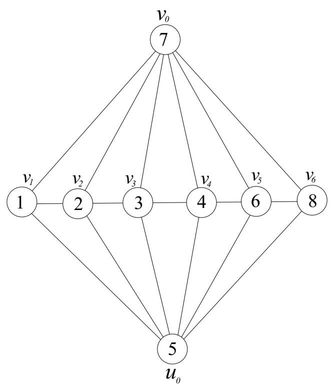

Figure 8: Strongly multiplicative labeling of double fan graph DF_6.

Theorem 2.17. The double wheel graph DW_n is strongly multiplicative.

Proof: Let DW_n be the double wheel graph.Let v_0 be the apex vertex. Let $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ be the rim vertices of DW_n. We note that $|V(G)| = 2n + 1$ and $|E(G)| = 2n$.
We define vertex labeling $f : V(DW_n) \rightarrow \{1, 2, ..., 2n + 1\}$ as follows:
$f(v_0) = p_1$; where p_1 is the highest prime number such that, $3 \leq p_1 \leq 2n + 1$;
$f(v_n) = p_2$; where p_2 is the second highest prime number such that, $5 \leq p_2 < p_1 \leq 2n + 1$;
Now, label the vertices $u_1, u_2, ..., u_n$ consecutively from the set $\{2, 4, ..., 2n\}$ and the vertices $v_1, v_2, ..., v_{n-1}$ consecutively from the set $\{1, 3, ..., 2n + 1\} \setminus \{p_1, p_2\}$.
The labeling pattern defined above covers all the possibilities and in each case the graph DW_n admits strongly multiplicative labeling. Therefore, the double wheel graph DW_n is strongly multiplicative. ∎

Illustration 2.18. The double wheel graph DW_4 and its strongly multiplicative labeling is shown in *Figure 9*.

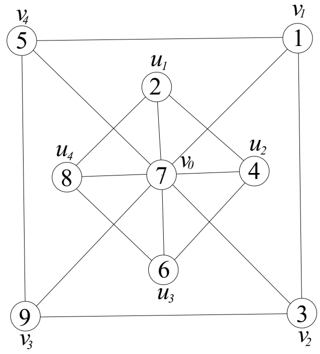

Figure 9: Strongly multiplicative labeling of double wheel graph DW_4.

Concluding Remark: We have derived nine results related to the strongly multiplicative labeling. To derive similar results for other graph families is an open problem.

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