

Infinite multi-series arising from generalized q -alpha difference equation

G. Britto Antony Xavier¹, T.G. Gerly²

Department of Mathematics
Sacred Heart College,
Tirupattur - 635 601
Vellore District, Tamil Nadu, India.
¹E-mail : brittoshc@gmail.com

Abstract

In this paper, we extend certain results of finite q -alpha multi-series to infinite q -alpha multi-series and obtain the sum of infinite q -alpha multi-series of polynomials, polynomial factorials and logarithmic functions.

Keywords: Generalized polynomial factorial, infinite multi-series, q -alpha difference operator, summation solution.

AMS Subject Classification(2010): 39A10, 39A11, 39A13.

1 Introduction

The theory of q -derivative equations of q -calculus or quantum calculus is based on the definition of the q -derivative operator, which was introduced by Jackson [3, 4]. Several groups have intensified their research on the amazing mathematics world featuring q -calculus. However, between 1930 and 1980 the theory of linear q -difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. Since 1980s, an extensive and somewhat surprising interest in the subject reappeared in many areas of mathematics, physics and applications including new difference calculus and orthogonal polynomials, q -combinatorics, q -arithmetics, integrable systems and variational q -calculus.

In 1984, Jerzy Popenda [5] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Ross [7] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. Recently, Britto Antony Xavier et al. [1] introduced a q -difference operator Δ_q defined as $\Delta_q u(k) = u(qk) - u(k)$ and obtained a summation solution of the generalized q -difference equation $\Delta_q^t v(k) = u(k)$, $k \in (-\infty, \infty)$ and $q \neq 1$, in the form

$$\Delta_q^{-t} u(k) \Big|_{\frac{k}{q^m}}^k = \sum_{(r)_{1 \rightarrow t}}^m u\left(k \prod_{i=1}^t q^{-r_i}\right).$$

¹ Research supported by University Grants Commission, SERO, Hyderabad, India.

Then we extended this q -difference equation to generalized higher order q -alpha difference equation

$$\Delta_{(q_1)\alpha_1} \left(\Delta_{(q_2)\alpha_2} \left(\cdots \Delta_{(q_t)\alpha_t} (v(k)) \cdots \right) \right) = u(k), k \in (-\infty, \infty), \quad (1)$$

and obtained many results. Also we derived finite q -alpha multi-series formula and finite higher order q -alpha series formula [2]. In this paper, we derive infinite q -alpha multi-series for polynomials, polynomial factorials and logarithmic functions by equating summation and closed form solutions of the equation (1).

2 Preliminaries

We begin with some notations, basic definitions and preliminary results which are used in the subsequent sections. Let $u(k)$ be a real valued function on $(-\infty, \infty)$, α and q be non-zero reals and m be a positive integer. Throughout this paper, we use following notations:

$$(i) \quad \sum_{(r_1)_{1 \rightarrow i}}^m = \sum_{r_1=0}^m \sum_{r_2=0}^m \cdots \sum_{r_i=0}^m; \quad (ii) \quad \Delta_{q_1 \rightarrow t}^{-1} = \Delta_{q_1}^{-1} \Delta_{q_2}^{-1} \Delta_{q_3}^{-1} \cdots \Delta_{q_t}^{-1} \quad \text{and}$$

$$(iii) \quad \Delta_{(q,\alpha)_{1 \rightarrow t}}^{-1} = \Delta_{(q_1)\alpha_1}^{-1} \Delta_{(q_2)\alpha_2}^{-1} \cdots \Delta_{(q_t)\alpha_t}^{-1}.$$

Definition 2.1. [2] Let $u(k)$ be a real valued function on $(-\infty, \infty)$ and q be a fixed real. Then the q -alpha difference operator, denoted by $\Delta_{(q)\alpha}$, on $u(k)$ is defined as

$$\Delta_{(q)\alpha} u(k) = u(qk) - \alpha u(k), \quad (2)$$

and the inverse of the q -alpha difference operator, denoted by $\Delta_{(q)\alpha}^{-1}$, is defined as below:

$$\text{if } \Delta_{(q)\alpha} v(k) = u(k), \text{ then we write } v(k) = \Delta_{(q)\alpha}^{-1} u(k). \quad (3)$$

Lemma 2.2. [1, 2] Let $k \in (0, \infty)$ and $\alpha \neq 1$. Then we have

$$\Delta_{(q)\alpha}^{-1}(1) = \frac{1}{1-\alpha}, \quad \Delta_{(q)1}^{-1}(1) = \frac{\log(k)}{\log(q)}, \quad q \neq 1 \quad (4)$$

and

$$\Delta_{(q)\alpha}^{-1} \log(k) = \frac{\log(k)}{1-\alpha} - \frac{\log(q)}{(1-\alpha)^2}. \quad (5)$$

Lemma 2.3. [6] Let s_r^n be the Stirling numbers of first kind, $n \in N(1)$. If

$k_q^{(n)} = \prod_{i=0}^{n-1} (k - iq)$ and $\left(\frac{1}{k}\right)_q^{(n)} = \prod_{i=0}^{n-1} \left(\frac{1}{k} - iq\right)$ for $k, q \neq 0$, then

$$k_q^{(n)} = \sum_{r=1}^n s_r^n q^{n-r} k^r \quad \text{and} \quad \left(\frac{1}{k}\right)_q^{(n)} = \sum_{r=1}^n s_r^n q^{n-r} \left(\frac{1}{k}\right)^r. \quad (6)$$

3 Main Results

The purpose of this section is to obtain the sum of infinite q -alpha multi-series by equating summation and closed form solutions of the generalized higher order q -alpha difference equation (1).

Theorem 3.1. (Infinite q -alpha Series Formula) Let $q, \alpha \neq 0$, $u(k)$ be a real valued function defined on $(-\infty, \infty)$ and if $\lim_{h \rightarrow \infty} \frac{1}{\alpha^h} \Delta_{(q)\alpha}^{-1} u(q^h k) = 0$, then

$$\Delta_{(q)\alpha}^{-1} u(k) = \frac{-1}{\alpha} \sum_{h=0}^{\infty} \frac{1}{\alpha^h} u(q^h k) \quad (7)$$

is an infinite series solution of the q -alpha difference equation (1) for $t = 1$.

Proof: Taking $\Delta_{(q)\alpha}^{-1} u(k) = v(k)$ and by Definition 2.1, we have

$$v(k) = \frac{-1}{\alpha} u(k) + \frac{1}{\alpha} v(qk). \quad (8)$$

Replacing k by qk in (8), we get $v(qk) = \frac{-1}{\alpha} u(qk) + \frac{1}{\alpha} v(q^2 k)$. Therefore (8) becomes

$$v(k) = \frac{-1}{\alpha} \left\{ u(k) + \frac{1}{\alpha} u(qk) \right\} + \frac{1}{\alpha^2} v(q^2 k). \quad (9)$$

Again replacing k by $q^2 k, q^3 k, \dots$ in (8) repeatedly and putting the resultant expressions in (9), we get $v(k) = \frac{-1}{\alpha} \sum_{h=0}^{\infty} \frac{1}{\alpha^h} u(q^h k)$, which completes the proof. ■

Theorem 3.2. (Infinite q -alpha Multi-Series Formula) Let $k \in (-\infty, \infty)$ and $\alpha_i, q_i \neq 0$. If $\lim_{h_i \rightarrow \infty} \frac{1}{\alpha_i^{h_i}} \Delta_{(q_i)\alpha_i}^{-1} u(q_i^{h_i} k) = 0$ for $i = 1, 2, \dots, t$, then we have

$$\sum_{(h)_1 \rightarrow t}^{\infty} \prod_{p=1}^t \alpha_p^{-h_p} u\left(\prod_{p=1}^t q_p^{h_p} k\right) = (-1)^t \prod_{p=1}^t \alpha_p \Delta_{(q,\alpha)_{1 \rightarrow t}}^{-1} u(k), \quad (10)$$

which is a solution of the equation (1).

Proof: Replacing q, α, h by q_2, α_2, h_2 in (7), we get

$$u(k) + \frac{1}{\alpha_2} u(q_2 k) + \frac{1}{\alpha_2^2} u(q_2^2 k) + \cdots + \infty = -\alpha_2 \Delta_{(q_2)\alpha_2}^{-1} u(k). \quad (11)$$

Replacing k by $q_1^{h_1} k$ and dividing by $\alpha_1^{h_1}$ for $h_1 = 1, 2, 3, \dots, \infty$ in (11), we obtain

$$\frac{1}{\alpha_1^{h_1}} \left\{ u(q_1^{h_1} k) + \frac{1}{\alpha_2} u(q_1^{h_1} q_2 k) + \frac{1}{\alpha_2^2} u(q_1^{h_1} q_2^2 k) + \cdots + \infty \right\} = \frac{-\alpha_2}{\alpha_1^{h_1}} \Delta_{(q_2)\alpha_2}^{-1} u(q_1^{h_1} k)$$

for $h_1 = 1, 2, 3, \dots, \infty$.

Summing the above equations with (11), we have

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{u(q_1^{h_1} q_2^{h_2} k)}{\alpha_1^{h_1} \alpha_2^{h_2}} = (-\alpha_2) \sum_{h_1=0}^{\infty} \Delta_{(q_2)\alpha_2}^{-1} \frac{u(q_1^{h_1} k)}{\alpha_1^{h_1}}. \quad (12)$$

Applying (7) in (12), we obtain

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{u(q_1^{h_1} q_2^{h_2} k)}{\alpha_1^{h_1} \alpha_2^{h_2}} = (-\alpha_1)(-\alpha_2) \Delta_{(q_1)\alpha_1}^{-1} \Delta_{(q_2)\alpha_2}^{-1} u(k). \quad (13)$$

Replacing $q_1, q_2, \alpha_1, \alpha_2, h_1, h_2$ by $q_2, q_3, \alpha_2, \alpha_3, h_2, h_3$ in (13), we get

$$\sum_{h_2=0}^{\infty} \sum_{h_3=0}^{\infty} \frac{u(q_2^{h_2} q_3^{h_3} k)}{\alpha_2^{h_2} \alpha_3^{h_3}} = (-\alpha_2)(-\alpha_3) \Delta_{(q_2)\alpha_2}^{-1} \Delta_{(q_3)\alpha_3}^{-1} u(k). \quad (14)$$

Replacing k by $q_1^{h_1} k$ and dividing by $\alpha_1^{h_1}$ for $h_1 = 1, 2, 3, \dots, \infty$ in (14) and then summing all the corresponding expressions with (14) yields

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \sum_{h_3=0}^{\infty} \frac{u(q_1^{h_1} q_2^{h_2} q_3^{h_3} k)}{\alpha_1^{h_1} \alpha_2^{h_2} \alpha_3^{h_3}} = (-\alpha_2)(-\alpha_3) \sum_{h_1=0}^{\infty} \Delta_{(q_2)\alpha_2}^{-1} \Delta_{(q_3)\alpha_3}^{-1} \frac{u(q_1^{h_1} k)}{\alpha_1^{h_1}}. \quad (15)$$

Applying (7) in (15) gives

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \sum_{h_3=0}^{\infty} \frac{u(q_1^{h_1} q_2^{h_2} q_3^{h_3} k)}{\alpha_1^{h_1} \alpha_2^{h_2} \alpha_3^{h_3}} = (-\alpha_1)(-\alpha_2)(-\alpha_3) \Delta_{(q_1)\alpha_1}^{-1} \Delta_{(q_2)\alpha_2}^{-1} \Delta_{(q_3)\alpha_3}^{-1} u(k).$$

Proceeding like this, we derive

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \cdots \sum_{h_t=0}^{\infty} \prod_{p=1}^t \alpha_p^{-h_p} u \left(\prod_{p=1}^t q_p^{h_p} k \right) = (-1)^t \prod_{p=1}^t \alpha_p \Delta_{(q_1)\alpha_1}^{-1} \Delta_{(q_2)\alpha_2}^{-1} \cdots \Delta_{(q_t)\alpha_t}^{-1} u(k),$$

which yields (10). ■

Corollary 3.3 gives a formula for infinite series involving logarithmic function.

Corollary 3.3. Let $\alpha_1, \alpha_2 \neq 0, 1, q_i \neq 0$ and $k > 0$. If $\lim_{h_i \rightarrow \infty} \frac{1}{\alpha_i^{h_i}} \Delta_{(q, \alpha)_{1 \rightarrow i}}^{-1} \log(q_i^{h_i} k) = 0$ for $i = 1, 2$, then we have

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{\alpha_1^{h_1} \alpha_2^{h_2}} \log\left(\frac{1}{q_1^{h_1} q_2^{h_2} k}\right) = \frac{\alpha_1 \alpha_2}{(1 - \alpha_1)(1 - \alpha_2)} \left\{ \log\left(\frac{1}{k}\right) - \frac{\log(1/q_1)}{1 - \alpha_1} - \frac{\log(1/q_2)}{1 - \alpha_2} \right\} \quad (16)$$

Proof: Consider $t = 2$ and $u(k) = \log(1/k)$ in (10). Then we have

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{\alpha_1^{h_1} \alpha_2^{h_2}} \log\left(\frac{1}{q_1^{h_1} q_2^{h_2} k}\right) = \alpha_1 \alpha_2 \Delta_{(q_1)\alpha_1}^{-1} \Delta_{(q_2)\alpha_2}^{-1} \log\left(\frac{1}{k}\right). \quad (17)$$

From Definition (2.1), we have $\Delta_{(q_2)\alpha_2} \log\left(\frac{1}{k}\right) = \log\left(\frac{1}{q_2}\right) + (1 - \alpha_2) \log\left(\frac{1}{k}\right)$, which gives

$$\Delta_{(q_2)\alpha_2}^{-1} \log\left(\frac{1}{k}\right) = \frac{1}{1 - \alpha_2} \left\{ \log\left(\frac{1}{k}\right) - \frac{1}{1 - \alpha_2} \log\left(\frac{1}{q_2}\right) \right\}.$$

Operating $\Delta_{(q_1)\alpha_1}^{-1}$ on both sides, we obtain

$$\Delta_{(q_1)\alpha_1}^{-1} \Delta_{(q_2)\alpha_2}^{-1} \log\left(\frac{1}{k}\right) = \frac{1}{(1 - \alpha_1)(1 - \alpha_2)} \left\{ \log\left(\frac{1}{k}\right) - \frac{1}{1 - \alpha_1} \log\left(\frac{1}{q_1}\right) - \frac{1}{1 - \alpha_2} \log\left(\frac{1}{q_2}\right) \right\}. \quad (18)$$

We complete the proof by substituting (18) in (17). ■

The following example is a verification of Corollary 3.3.

Example 3.4. Putting $k = 20, q_1 = 3, q_2 = 4, \alpha_1 = 2$ and $\alpha_2 = 3$ in (16), we get

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{2^{h_1} 3^{h_2}} \log\left(\frac{1}{3^{h_1} 4^{h_2} 20}\right) = \frac{6}{(-1)(-2)} \left\{ \log\left(\frac{1}{20}\right) + \log\left(\frac{1}{3}\right) + \frac{1}{2} \log\left(\frac{1}{4}\right) \right\},$$

which gives $\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{2^{h_1} 3^{h_2}} \log(3^{h_1} 4^{h_2} 20) = 3 \log(120)$.

Corollary 3.5. For any real valued function $u(k)$ on $(-\infty, \infty)$, if $\alpha \neq 0$ and $\lim_{h_i \rightarrow \infty} \frac{1}{\alpha^{h_i}} \Delta_{(q)_{1 \rightarrow i} \alpha}^{-1} u(q_i^{h_i} k) = 0$ for $i = 1, 2, \dots, t$, then we have

$$\sum_{(h)_{1 \rightarrow t}}^{\infty} \prod_{p=1}^t \alpha^{-h_p} u\left(\prod_{p=1}^t q_p^{h_p} k\right) = (-\alpha)^t \Delta_{(q)_{1 \rightarrow t} \alpha}^{-1} u(k). \quad (19)$$

Proof: The proof is obvious by putting $\alpha_1 = \alpha_2 = \dots = \alpha_t = \alpha$ in (10). ■

Corollary 3.6. (Infinite q Multi-Series Formula) For $0 \neq q_i \in (-\infty, \infty)$ and if $\lim_{h_i \rightarrow \infty} \Delta_{(q)_{1 \rightarrow i}}^{-1} u(q_i^{h_i} k) = 0$, $i = 1, 2, \dots, t$, then

$$\sum_{(h)_{1 \rightarrow t}}^{\infty} u \left(\prod_{p=1}^t q_p^{h_p} k \right) = (-1)^t \Delta_{(q)_{1 \rightarrow t}}^{-1} u(k). \quad (20)$$

Proof: The proof completes by taking $\alpha = 1$ in Corollary 3.5. ■

Corollary 3.7. Let $k \in (0, \infty)$ and $q_i \neq 0, 1$. If $\lim_{h_i \rightarrow \infty} \Delta_{(q)_{1 \rightarrow i}}^{-1} \frac{1}{q_i^{h_i} k} = 0$ for $i = 1, 2$, then we have

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{q_1^{h_1} q_2^{h_2} k} = \frac{q_1 q_2}{(1 - q_1)(1 - q_2)k} \quad (21)$$

Proof: Consider $t = 2$ and $u(k) = \frac{1}{k}$ in Corollary 3.6. Then we get

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{q_1^{h_1} q_2^{h_2} k} = \Delta_{(q_1)}^{-1} \Delta_{(q_2)}^{-1} \frac{1}{k}. \quad (22)$$

From Definition 2.1, we write, $\Delta_{(q_1)}^{-1} \Delta_{(q_2)}^{-1} \frac{1}{k} = \frac{q_1 q_2}{(1 - q_1)(1 - q_2)k}$. Hence the proof follows by applying the above value in (22). ■

The following example illustrates Corollary 3.7.

Example 3.8. Taking $q_1 = 3$, $q_2 = 4$ and $k = 19$ in (21), we get $\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{1}{3^{h_1} 4^{h_2} 19} = \frac{12}{6 \times 19}$.

Corollary 3.9. (Infinite q -series) Let $k \in (-\infty, \infty)$. If $\lim_{h_i \rightarrow \infty} \Delta_{(q)}^{-t} u(q^{h_i} k) = 0$ for $i = 1, 2, \dots, t$, then we have

$$\sum_{(h)_{1 \rightarrow t}}^{\infty} u \left(\prod_{p=1}^t q^{h_p} k \right) = (-1)^t \Delta_{(q)}^{-t} u(k). \quad (23)$$

Proof: We can easily prove this corollary by putting $q_i = q$ in 20. ■

Theorem 3.10. (Higher order Infinite q -alpha Series Formula) Let $\alpha, q \neq 0$ and $k \in (-\infty, \infty)$.

If $\lim_{h \rightarrow \infty} \binom{h+t-1}{t-1} \frac{1}{\alpha^h} u(q^h k) = 0$, then we have

$$\sum_{h=0}^{\infty} \binom{h+t-1}{t-1} \frac{1}{\alpha^h} u(q^h k) = (-\alpha)^t \Delta_{(q)\alpha}^{-t} u(k), \quad (24)$$

which is a solution of the equation (1) for each $\alpha_i = \alpha$.

Proof: From Theorem 3.1, we have

$$u(k) + \frac{1}{\alpha}u(qk) + \frac{1}{\alpha^2}u(q^2k) + \cdots + u(\infty) = -\alpha \Delta_{(q)\alpha}^{-1} u(k). \quad (25)$$

Replacing k by qk and dividing by α , we get

$$\frac{1}{\alpha} \left\{ u(qk) + \frac{1}{\alpha}u(q^2k) + \frac{1}{\alpha^2}u(q^3k) + \cdots + u(\infty) \right\} = -\Delta_{(q)\alpha}^{-1} u(qk). \quad (26)$$

Again replacing k by q^2k, q^3k, \dots and dividing the corresponding expressions by $\alpha^2, \alpha^3, \dots$ in (25) and then summing up all the resultant expressions, we arrive

$$\sum_{h=0}^{\infty} \frac{(h+1)}{\alpha^h} u(q^h k) = -\alpha \sum_{h=0}^{\infty} \frac{1}{\alpha^h} \Delta_{(q)\alpha}^{-1} u(q^h k). \quad (27)$$

Applying Theorem 3.1 in (27), we find that

$$\sum_{h=0}^{\infty} \binom{h+1}{1} \frac{1}{\alpha^h} u(q^h k) = (-\alpha)^2 \Delta_{(q)\alpha}^{-2} u(k).$$

Repeating the above procedure on (27) and using Theorem 3.1, we obtain

$$\sum_{h=0}^{\infty} \binom{h+2}{2} \frac{1}{\alpha^h} u(q^h k) = (-\alpha)^3 \Delta_{(q)\alpha}^{-3} u(k).$$

Continuing like this, we get proof of this theorem. ■

Corollary 3.11. (Higher order Infinite q -Series Formula) Let $k \in (-\infty, \infty)$ and $t \in N(1)$.

If $\lim_{h \rightarrow \infty} \binom{h+t-1}{t-1} u(q^h k) = 0$, then we have

$$\sum_{h=0}^{\infty} \binom{h+t-1}{t-1} u(q^h k) = (-1)^t \Delta_{(q)}^{-t} u(k). \quad (28)$$

Proof: Taking $\alpha = 1$ in (24) completes the proof of this corollary. ■

Corollary 3.12. Let $k \in (0, \infty)$ and $\alpha, q, 1 - \alpha q^2 \neq 0$. If $\lim_{h \rightarrow \infty} \binom{h+2}{2} \frac{1}{\alpha^h (q^h k)^2} = 0$, then we have

$$\sum_{h=0}^{\infty} \binom{h+2}{2} \frac{1}{\alpha^h (q^h k)^2} = (-\alpha)^3 \left(\frac{q^2}{1 - \alpha q^2} \right)^3 \frac{1}{k^2}. \quad (29)$$

Proof: Consider $t = 3$ and $u(k) = \frac{1}{k^2}$ in (24). Then we get

$$\sum_{h=0}^{\infty} \binom{h+2}{2} \frac{1}{\alpha^h (q^h k)^2} = (-\alpha)^3 \Delta_{(q)\alpha}^{-3} \left(\frac{1}{k^2} \right) \quad (30)$$

From (2), $\Delta_{(q)\alpha} \left(\frac{1}{k^2} \right) = \left(\frac{1}{q^2} - \alpha \right) \frac{1}{k^2}$, which gives $\Delta_{(q)\alpha}^{-3} \left(\frac{1}{k^2} \right) = \left(\frac{q^2}{1 - \alpha q^2} \right)^3 \frac{1}{k^2}$.

Substituting this value in (30) yields (29). ■

Example 3.13. Taking $k = 18, q = 4$ and $\alpha = 3$ in (29), we get

$$\sum_{h=0}^{\infty} \binom{h+2}{2} \frac{1}{3^h} \frac{1}{(4^h 18)^2} = (-3)^3 \frac{4^6}{(1 - 3 \times 16)^3 18^2}.$$

Corollary 3.14. Let $q \neq 0, 1, k \in (0, \infty)$ and $1 - q^2 \neq 0$. Then we have

$$\sum_{h=0}^{\infty} \binom{h+1}{1} \left(\frac{1}{(q^h k)^2} - \frac{q}{q^h k} \right) = \left(\frac{q^2}{1 - q^2} \right)^2 \frac{1}{k^2} - q \left(\frac{q}{1 - q} \right)^2 \frac{1}{k}. \quad (31)$$

Proof: Taking $t = 2$ and $u(k) = \left(\frac{1}{k} \right)_q^{(2)}$ in (28), we get

$$\sum_{h=0}^{\infty} \binom{h+1}{1} \left(\frac{1}{q^h k} \right)_q^{(2)} = \Delta_{(q)}^{-2} \left(\frac{1}{k} \right)_q^{(2)}. \quad (32)$$

From Lemma 2.3, we have

$$\Delta_{(q)}^{-1} \left(\frac{1}{k} \right)_q^{(2)} = \left(\frac{q^2}{1 - q^2} \right) \frac{1}{k^2} - q \left(\frac{q}{1 - q} \right) \frac{1}{k}. \quad (33)$$

So the proof follows by substituting (33) in (32). ■

The following example illustrates Corollary 3.14.

Example 3.15. Putting $q = 4$ and $k = 26$ in (31), we get

$$\sum_{h=0}^{\infty} \binom{h+1}{1} \left(\frac{1}{(4^h \times 26)^2} - \frac{4}{4^h 26} \right) = \left(\frac{16}{1 - 16} \right)^2 \frac{1}{26^2} - 4 \left(\frac{4}{1 - 4} \right)^2 \frac{1}{26}.$$

4 Conclusion

In this paper, multi-series solution and closed form solutions of the higher order infinite q -alpha difference equation have been obtained to get the sum of infinite q -alpha multi-series.

Acknowledgement: One of the authors (Dr.G.Britto Antony Xavier) acknowledges University Grants Commission, SERO, Hyderabad, India for financial assistance (MRP-4021/11, SERO,UGC).

References

- [1] Britto G. Antony Xavier, T.G. Gerly and H. Nasira Begum, *Finite series of polynomials and polynomial factorials arising from generalised q -difference operator*, Far East Journal of Mathematical Sciences, 94(1) (2014), 47-63.
- [2] Britto G. Antony Xavier, T.G. Gerly and S.U. Vasantha Kumar, *Multi-Series Solution of Generalized q -alpha Difference Equation*, International Journal of Applied Engineering Research, 10(72) (2015), 97-101.
- [3] F.H. Jackson, *On q -functions and a Certain Difference Operator*, Trans. Roy.Soc.Edin, 46 (1908), 64-72.
- [4] F.H. Jackson, *On q -definite integrals* Qust.J. Pure Appl. Math., 41 (1910), 193-203.
- [5] Jerzy Popenda and Blazej Szmanda, *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, 17(1) (1984), 153-164.
- [6] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, *Theory of Generalized Difference Operator and Its Applications*, Far East Journal of Mathematical Sciences, 20(2) (2006), 163-171.
- [7] K.S. Miller and B. Ross, *Fractional difference calculus, in "Univalent functions, fractional calculus and the applications(Koriyama, 1988)"*, Horwood, Chichester, UK, 139-152, 1989.