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Edge regular property of cartesian product and composition of two fuzzy graphs

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Abstract

In this paper, a necessary and sufficient condition for the cartesian product and composition of two edge regular fuzzy graphs to be an edge regular fuzzy graph is determined.

Keywords: Cartesian product, composition, regular fuzzy graph, edge regular fuzzy graph. **Subject Classification (2010):** 03E72, 05C72, 05C76.

1 Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975 [11]. Mordeson and Peng introduced the concept of operations on fuzzy graphs [2]. The degrees of vertices in fuzzy graphs obtained from two given fuzzy graphs using these operations were discussed by Nagoor Gani and Radha [5]. Radha and Kumaravel introduced the concept of degree of an edge and total degree of an edge in fuzzy graphs [8]. We study the cartesian product and composition of two fuzzy graphs. In general, cartesian product and composition of two edge regular fuzzy graphs G_1 and G_2 need not be edge regular. In this paper, we find necessary and sufficient condition for cartesian product and composition of two fuzzy graphs to be edge regular fuzzy graph. First we present some basic concepts.

A fuzzy subset of a set V is a mapping σ from V to [0, 1]. A fuzzy graph G is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, (i.e.) $\mu(xy)\le\sigma(x)\wedge\sigma(y)$ for all $x, y \in V$. The underlying crisp graph of $G:(\sigma,\mu)$ is denoted by $G^*:(V,E)$ where $E \subseteq V \times V$. Throughout this paper, $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $|V_i| = p_i, i=1,2$. Also $d_{G_i^*}(u_i)$ denotes the degree of u_i in G_i^* and $d_{\overline{G_i^*}}(u_i)$ denotes the degree of u_i in $\overline{G_i}^*$, where $\overline{G_i}^*$ is the complement of G_i^*. Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The degree of a vertex u is $d_G(u) = \sum_{u \neq v} \mu(uv)$. The minimum degree of G is $\delta(G) = \wedge\{d_G(v), \forall\ v \in V\}$ and the maximum degree of G is $\Delta(G) = \vee\{d_G(v), \forall\ v \in V\}$. The total degree of a vertex $u \in V$ is defined by $td_G(u) = \sum_{u \neq v} \mu(uv) + \sigma(u)$. If each vertex in G has same degree k, then G is said to be a regular fuzzy graph or k – regular fuzzy graph. If each vertex in G has same total degree k, then G is said to be a totally regular fuzzy graph or k – totally regular fuzzy graph. The order and size of a fuzzy graph G are defined by $O(G) = \sum_{u \in V} \sigma(u)$ and $S(G) = \sum_{uv \in E} \mu(uv)$. Let $G^*:(V,E)$ be a graph and let $e = uv$ be an edge in G^*. Then the degree of an edge $e = uv \in E$ is defined by $d_{G^*}(uv) = d_{G^*}(u) + d_{G^*}(v) - 2$. If each and every pair of distinct vertices is joined by an edge, then $G^*:(V,E)$ is said to be complete graph. Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The degree of an edge uv is $d_G(uv) = d_G(u) + d_G(v) - 2\mu(uv)$. This is equivalent to $d_G(uv) = \sum_{\substack{uw \in E \\ w \neq v}} \mu(uw) + \sum_{\substack{wv \in E \\ w \neq u}} \mu(wv)$. The total degree of an edge $uv \in E$ is defined by $td_G(uv) = d_G(u) + d_G(v) - \mu(uv)$. This is equivalent to $td_G(uv) = \sum_{\substack{uw \in E \\ w \neq v}} \mu(uw) + \sum_{\substack{wv \in E \\ w \neq u}} \mu(wv) + \mu(uv) = d_G(uv) + \mu(uv)$. The minimum edge degree and maximum edge degree of G are $\delta_E(G) = \wedge\{d_G(uv), \forall\ uv \in E\}$ and $\Delta_E(G) = \vee\{d_G(uv), \forall\ uv \in E\}$. If each edge in G has same degree k, then G is said to be an edge regular fuzzy graph or k – edge regular fuzzy graph. If each edge in G has same total degree k, then G is said to be a totally edge regular fuzzy graph or k – totally edge regular fuzzy graph. A fuzzy Graph G is said to be strong, if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $xy \in E$. A fuzzy Graph G is said to be complete, if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.

Definition 1.1.[3] Let $G^* = G_1^* \times G_2^* = (V,E)$ be the cartesian product of two graphs G_1^* and G_2^*, where $V = V_1 \times V_2$ and $E = \{(u_1,u_2)(v_1,v_2) : u_1 = v_1, u_2v_2 \in E_2\ \ (or)\ \ u_1v_1 \in E_1, u_2 = v_2\}$. Then the cartesian product of two fuzzy graphs G_1 and G_2 is a fuzzy graph

$G = G_1 \times G_2 = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ defined by $(\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$, $\forall (u_1, u_2) \in V$ and

$$(\mu_1 \times \mu_2)((u_1,u_2)(v_1,v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2), & if\ u_1 = v_1, u_2v_2 \in E_2 \\ \mu_1(u_1v_1) \wedge \sigma_2(u_2), & if\ u_1v_1 \in E_1, u_2 = v_2 \end{cases}.$$

Definition 1.2.[4] Let $G^* = G_1^* \circ G_2^* = (V, E)$ be the composition of two graphs G_1^* and G_2^*, where $V = V_1 \times V_2$ and $E = \{(u_1,u_2)(v_1,v_2) : u_1 = v_1, u_2v_2 \in E_2$ or $u_1v_1 \in E_1, u_2 = v_2$ or $u_1v_1 \in E_1$, $u_2v_2 \notin E_2\}$. Then the composition of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 \circ G_2 = G_1 \circ G_2 : (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ defined by

$$(\sigma_1 \circ \sigma_2)(u_1,u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2), \forall (u_1,u_2) \in V \text{ and}$$

$$(\mu_1 \circ \mu_2)((u_1,u_2)(v_1,v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2), & \text{if } u_1 = v_1, u_2v_2 \in E_2 \\ \mu_1(u_1v_1) \wedge \sigma_2(u_2), & \text{if } u_1v_1 \in E_1, u_2 = v_2 \\ \mu_1(u_1v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), & \text{if } u_1v_1 \in E_1, u_2v_2 \notin E_2 \end{cases}$$

2 Edge regular properties of cartesian product of two fuzzy graphs

Remark 2.1. If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two edge regular fuzzy graphs, then $G_1 \times G_2$ need not be an edge regular fuzzy graph.

Example 2.2. Consider the following two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

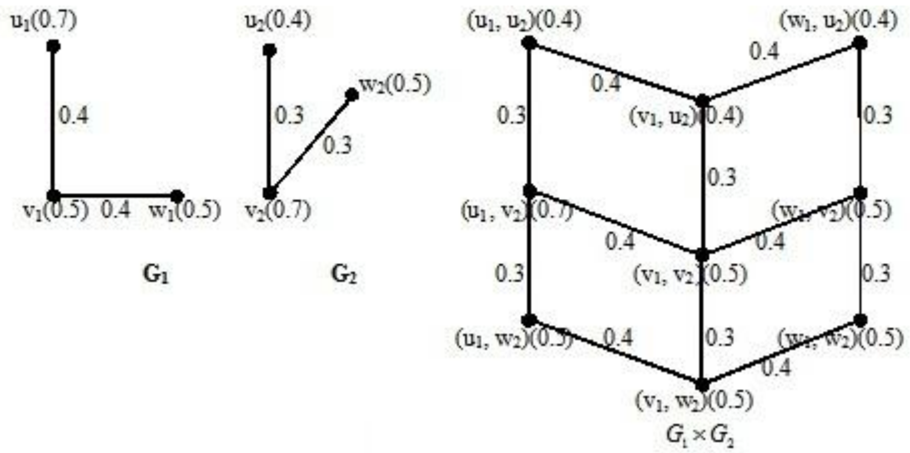

Figure 1

Here both G_1 and G_2 are edge regular fuzzy graphs of edge degree 0.4 and 0.3. In $G_1 \times G_2$, $d_{G_1 \times G_2}((u_1,u_2)(u_1,v_2))$ = 1.1 and $d_{G_1 \times G_2}((u_1,u_2)(v_1,u_2))$ = 1.0. Hence $G_1 \times G_2$ is not an edge regular fuzzy graph.

Remark 2.3. If $G_1 \times G_2$ is an edge regular fuzzy graph, then $G_1 : (\sigma_1, \mu_1)$ (or) $G_2 : (\sigma_2, \mu_2)$ need not be an edge regular fuzzy graph.

Example 2.4. Consider the following two fuzzy graphs $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$.

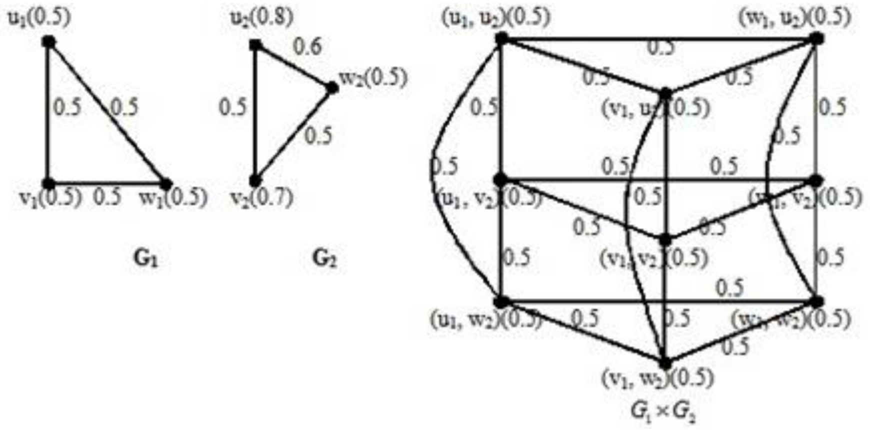

Figure 2

Here, $G_1 \times G_2$ is a 3 – edge regular fuzzy graph. But G_2 is not an edge regular fuzzy graph.

Theorem 2.5. [8] Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs. If $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$, then for any $(u_1,u_2)(v_1,v_2) \in E$,

(1) When $u_1 = v_1, u_2v_2 \in E_2$, $d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d_{G_1}(u_1) + d_{G_2}(u_2v_2)$,

(2) When $u_1v_1 \in E_1, u_2 = v_2$, $d_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2d_{G_2}(u_2)$.

Theorem 2.6. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two regular fuzzy graphs of same degree on $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Then G_1 and G_2 are edge regular fuzzy graphs of same degree if and only if $G_1 \times G_2$ is an edge regular fuzzy graph.

Proof: Let $d_{G_1}(u_1) = d_{G_2}(u_2) = m \ \forall u_1 \in V_1,\ u_2 \in V_2$, where m is a constant.

Assume that G_1 and G_2 are k – edge regular fuzzy graphs, where k is a constant.

Then $d_{G_1}(u_1v_1) = d_{G_2}(u_2v_2) = k \ \forall u_1v_1 \in E_1,\ u_2v_2 \in E_2$.

By Theorem 2.5, for any $(u_1,u_2)(v_1,v_2) \in E$, when $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d_{G_1}(u_1) + d_{G_2}(u_2v_2)$$
$$= 2m + k \tag{2.1}$$

Similarly, when $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = 2m + k \tag{2.2}$$

From (2.1) and (2.2), $G_1 \times G_2$ is an edge regular fuzzy graph.

Conversely, let $u_1v_1, w_1x_1 \in E_1$ be any two edges of G_1. Fix $u \in V_2$.

Then $(u_1,u)(v_1,u)$ and $(w_1,u)(x_1,u) \in E$, $d_{G_1\times G_2}((u_1,u)(v_1,u)) = d_{G_1\times G_2}((w_1,u)(x_1,u))$.

$$d_{G_1}(u_1v_1) + 2d_{G_2}(u) = d_{G_1}(w_1x_1) + 2d_{G_2}(u)$$

$$d_{G_1}(u_1v_1)+2m = d_{G_1}(w_1x_1)+2m$$

$$d_{G_1}(u_1v_1) = d_{G_1}(w_1x_1),\ \forall\ u_1v_1,\ w_1x_1 \in E_1.$$

Therefore, G_1 is an edge regular fuzzy graph. Similarly, G_2 is an edge regular fuzzy graph.

Suppose that G_1 is k_1 – edge regular fuzzy graph and G_2 is k_2 – edge regular fuzzy graph with $k_1 \neq k_2$.

Then, $d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d_{G_1}(u_1)+d_{G_2}(u_2v_2)$

$$= 2m + k_2 \tag{2.3}$$

Therefore, $d_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1)+2d_{G_2}(u_2)$

$$= k_1 + 2m \tag{2.4}$$

From (2.3) and (2.4), $d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) \neq d_{G_1\times G_2}((u_1,u_2)(v_1,u_2))$, since $k_1 \neq k_2$. This is a contradiction to our assumption that $G_1 \times G_2$ is an edge regular fuzzy graph. Hence, G_1 and G_2 are edge regular fuzzy graphs of same degree. ■

Theorem 2.7. [8] Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs.

1. If $\sigma_1 \leq \mu_2$ and σ_1 is a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$, then for any $(u_1,u_2)(v_1,v_2) \in E$,
   (i) When $u_1 = v_1, u_2v_2 \in E_2$, $d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d_{G_1}(u_1) + c_1(d_{G_2^*}(u_2)+d_{G_2^*}(v_2)-2)$.
   (ii) When $u_1v_1 \in E_1, u_2 = v_2$, $d_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2c_1 d_{G_2^*}(u_2)$.
2. If $\sigma_2 \leq \mu_1$ and σ_2 is a constant function with $\sigma_2(u) = c_2$ for all $u \in V_2$, then for any $(u_1,u_2)(v_1,v_2) \in E$,
   (i) When $u_1 = v_1, u_2v_2 \in E_2$, $d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2c_2 d_{G_1^*}(u_1) + d_{G_2}(u_2v_2)$.
   (ii) When $u_1v_1 \in E_1, u_2 = v_2$, $d_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = c_2(d_{G_1^*}(u_1)+d_{G_1^*}(v_1)-2) + 2d_{G_2}(u_2)$.

Theorem 2.8. [9] Let $G:(\sigma,\mu)$ be a fuzzy graph on k – regular crisp graph $G^*:(V,E)$. Then μ is constant if and only if G is both regular and edge regular.

Theorem 2.9. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs on regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_1 \leq \mu_2$ and μ_1 be a constant function with $\mu_1(e) = c_1$ for all $e \in E_1$. Then, $G_1 \times G_2$ is an edge regular fuzzy graph.

Proof: Given $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ are two fuzzy graphs on regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_1 \leq \mu_2$ and $\mu_1(e) = c_1$ for all $e \in E_1$ respectively.

Then $G_1:(\sigma_1,\mu_1)$ is both regular and edge regular. (Using Theorem 2.8)

Then $d_{G_1}(u_1v_1)=k, d_{G_1}(u_1)=m$, $d_{G_2^*}(u_2)=n,\quad \forall u_1,v_1\in V_1$ and $u_2\in V_2$, where k, m and n are constants. We prove the theorem in two cases using Theorem 2.7. Let $(u_1,u_2)(v_1,v_2)\in E$.

Case 1: When $u_1=v_1, u_2v_2\in E_2$,

$$d_{G_1\times G_2}((u_1,u_2)(u_1,v_2)) = 2d_{G_1}(u_1)+c_1(d_{G_2^*}(u_2)+d_{G_2^*}(v_2)-2).$$

$$=2m+c_1(n+n-2).$$

$$=2(m+c_1(n-1)) \tag{2.5}$$

Case 2: When $u_2=v_2, u_1v_1\in E_1$,

$$d_{G_1\times G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1)+2c_1d_{G_2^*}(u_2).$$

$$=k+2c_1n.$$

$=2m-2c_1+2c_1n$. (By definition of edge degree $k=2m-2c_1$)

$$=2(m+c_1(n-1)) \tag{2.6}$$

From (2.5) and (2.6), $G_1\times G_2$ is an edge regular fuzzy graph. ■

Corollary 2.10. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs with $\sigma_1\le\mu_2$ and σ_1 be a constant function with $\sigma_1(u)=c_1$ for all $u\in V_1$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be regular underlying crisp graphs. If G_1 is strong, then $G_1\times G_2$ is an edge regular fuzzy graph.

Proof: Given $G_1:(\sigma_1,\mu_1)$ is strong with $\sigma_1(u)=c_1$ for all $u\in V_1$. Then $\mu_1(e)=c_1$ for all $e\in E_1$. Therefore the result follows from Theorem 2.9.

Theorem 2.11. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs on regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_2\le\mu_1$ and μ_2 be a constant function with $\mu_2(e)=c_2$ for all $e\in E_2$. Then $G_1\times G_2$ is an edge regular fuzzy graph.

Proof: The proof is similar to the proof of Theorem 2.9. ■

Corollary 2.12. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs with $\sigma_2\le\mu_1$ and let σ_2 be a constant function with $\sigma_2(u)=c_2$ for all $u\in V_2$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be regular underlying crisp graphs. If G_2 is strong, then $G_1\times G_2$ is an edge regular fuzzy graph.

Proof: The proof is similar to the proof of Theorem 2.10. ■

3 Edge regular properties of composition of two fuzzy graphs

Remark 3.1. If $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two edge regular fuzzy graphs, then $G_1\circ G_2$ need not be an edge regular fuzzy graph.

Example 3.2. Consider the two fuzzy graphs $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$.

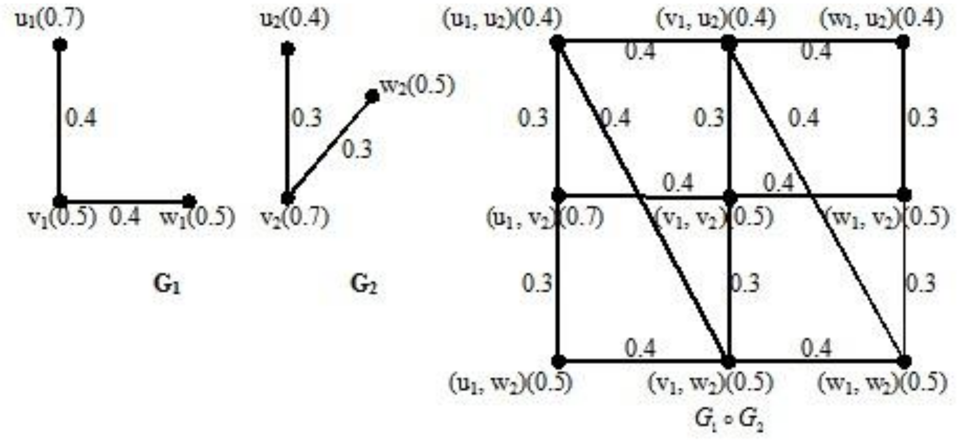

Figure 3

Here both G_1 and G_2 are edge regular fuzzy graphs of edge degree 0.4 and 0.3. In $G_1 \circ G_2$, $d_{G_1 \circ G_2}((u_1, u_2)(v_1, w_2))$ = 1.8 and $d_{G_1 \circ G_2}((v_1, v_2)(w_1, v_2))$ = 1.6. Hence $G_1 \circ G_2$ is not an edge regular fuzzy graph.

Remark 3.3. If $G_1 \circ G_2$ is an edge regular fuzzy graph, then $G_1 : (\sigma_1, \mu_1)$ (or) $G_2 : (\sigma_2, \mu_2)$ need not be edge regular fuzzy graph.

Example 3.4. Consider the two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

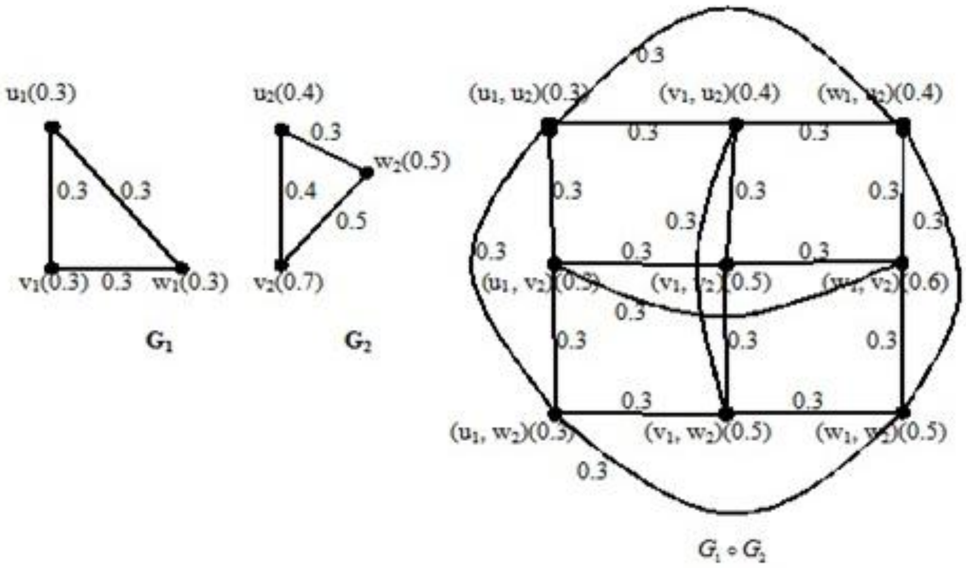

Figure 4

Here $G_1 \circ G_2$ is 1.8 – edge regular fuzzy graph, but G_1 is not an edge regular fuzzy graph.

Theorem 3.5. [8] Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs. If $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$, then for any $(u_1, u_2)(v_1, v_2) \in E$,

(1) When $u_1 = v_1, u_2v_2 \in E_2$, $d_{G_1[G_2]}((u_1,u_2)(u_1,v_2)) = 2p_2 d_{G_1}(u_1) + d_{G_2}(u_2v_2)$,

(2) When $u_1v_1 \in E_1, u_2 = v_2$, $d_{G_1[G_2]}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2d_{G_2}(u_2) + (p_2-1)(d_{G_1}(u_1) + d_{G_1}(v_1))$,

(3) When $u_1v_1 \in E_1, u_2v_2 \notin E_2$, $d_{G_1[G_2]}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + (p_2-1)(d_{G_1}(u_1) + d_{G_1}(v_1)) + d_{G_2}(u_2)$.

Theorem 3.6. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two regular fuzzy graphs of same degree with $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Then G_1 and G_2 are edge regular fuzzy graphs of same degree if and only if $G_1 \circ G_2$ is an edge regular fuzzy graph.

Proof: Let $d_{G_1}(u_1) = d_{G_2}(u_2) = m, \forall u_1 \in V_1$ and $u_2 \in V_2$ where m is a constant. Assume that G_1 and G_2 are k – edge regular fuzzy graphs, where k is a constant.

Then $d_{G_1}(u_1v_1) = d_{G_2}(u_2v_2) = k \ \forall u_1v_1 \in E_1$ and $u_2v_2 \in E_2$.

From Theorem 3.5, when $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1[G_2]}((u_1,u_2)(u_1,v_2)) = 2p_2 d_{G_1}(u_1) + d_{G_2}(u_2v_2)$$

$$= 2p_2 m + k \tag{3.1}$$

From Theorem 3.5, when $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1[G_2]}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2d_{G_2}(u_2) + (p_2-1)(d_{G_1}(u_1) + d_{G_1}(v_1))$$

$$= k + 2m + (p_2-1)(m+m)$$

$$= k + 2p_2 m \tag{3.2}$$

From Theorem 3.5, when $u_1v_1 \in E_1$ and $u_2v_2 \notin E_2$,

$$d_{G_1[G_2]}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + (p_2-1)(d_{G_1}(u_1) + d_{G_1}(v_1)) + d_{G_2}(u_2) + d_{G_2}(v_2)$$

$$= k + (p_2-1)(m+m) + m + m$$

$$= k + 2p_2 m \tag{3.3}$$

From (3.1), (3.2) and (3.3), $G_1 \circ G_2$ is an edge regular fuzzy graph.

Conversely, assume that $G_1 \circ G_2$ is an edge regular fuzzy graph.

We have to prove that G_1 and G_2 are edge regular fuzzy graphs of same degree. Let $u_1v_1, w_1x_1 \in E_1$ be any two edges of G_1. Fix $u \in V_2$. Then $(u_1,u)(v_1,u)$ and $(w_1,u)(x_1,u) \in E$, $d_{G_1 \circ G_2}((u_1,u)(v_1,u)) = d_{G_1 \circ G_2}((w_1,u)(x_1,u))$.

$$d_{G_1}(u_1v_1) + 2d_{G_2}(u) + (p_2-1)(d_{G_1}(u_1) + d_{G_1}(v_1)) = d_{G_1}(w_1x_1) + 2d_{G_2}(u) + (p_2-1)(d_{G_1}(w_1) + d_{G_1}(x_1))$$

$$d_{G_1}(u_1v_1)+2m+(p_2-1)(m+m)=d_{G_1}(w_1x_1)+2m+(p_2-1)(m+m)$$

$$d_{G_1}(u_1v_1)=d_{G_1}(w_1x_1)\ \ \forall\ u_1v_1,\ w_1x_1\in E_1.$$

Therefore, G_1 is an edge regular fuzzy graph. Similarly, G_2 is an edge regular fuzzy graph.

Now, to prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Suppose that G_1 is k_1 – edge regular fuzzy graph and G_2 is k_2 – edge regular fuzzy graph with $k_1 \neq k_2$.

$$d_{G_1\circ G_2}((u_1,u_2)(u_1,v_2))=2p_2d_{G_1}(u_1)+d_{G_2}(u_2v_2)$$

$$=2p_2m+k_2 \tag{3.4}$$

Therefore, $d_{G_1\circ G_2}((u_1,u_2)(v_1,u_2))=d_{G_1}(u_1v_1)+2d_{G_2}(u_2)+(p_2-1)(d_{G_1}(u_1)+d_{G_1}(v_1))$

$$=k_1+2m+(p_2-1)(m+m)$$

$$=k_1+2mp_2 \tag{3.5}$$

From (3.4) and (3.5), $d_{G_1\circ G_2}((u_1,u_2)(u_1,v_2))\neq d_{G_1\circ G_2}((u_1,u_2)(v_1,u_2))$, since $k_1\neq k_2$.

This is a contradiction to our assumption that $G_1\circ G_2$ is an edge regular fuzzy graph. Therefore, G_1 and G_2 are edge regular fuzzy graphs of same degree. ∎

Theorem 3.7 [8] Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs.

1. If $\sigma_1\le\mu_2$ and σ_1 is a constant function with $\sigma_1(u)=c_1$ for all $u\in V_1$, then for any $(u_1,u_2)(v_1,v_2)\in E$,

   (a) When $u_1=v_1, u_2v_2\in E_2$, $d_{G_1[G_2]}((u_1,u_2)(u_1,v_2))=2p_2d_{G_1}(u_1)+c_1(d_{G_2^*}(u_2)+d_{G_2^*}(v_2)-2)$,

   (b) When $u_1v_1\in E_1, u_2=v_2$, $d_{G_1[G_2]}((u_1,u_2)(v_1,u_2))=d_{G_1}(u_1v_1)+(p_2-1)(d_{G_1}(u_1)+d_{G_1}(v_1))$ $+2c_1d_{G_2^*}(u_2)$,

   (c) When $u_1v_1\in E_1, u_2v_2\notin E_2$, $d_{G_1[G_2]}((u_1,u_2)(v_1,v_2))=d_{G_1}(u_1v_1)+(p_2-1)(d_{G_1}(u_1)+d_{G_1}(v_1))$ $+c_1(d_{G_2^*}(u_2)+d_{G_2^*}(v_2))$.

2. If $\sigma_2\le\mu_1$ and σ_2 is a constant function with $\sigma_2(u)=c_2$ for all $u\in V_2$, then for any $(u_1,u_2)(v_1,v_2)\in E$,

   (a) When $u_1=v_1, u_2v_2\in E_2$, $d_{G_1[G_2]}((u_1,u_2)(u_1,v_2))=2c_2p_2d_{G_1^*}(u_1)+d_{G_2}(u_2v_2)$,

   (b) When $u_1v_1\in E_1, u_2=v_2$, $d_{G_1[G_2]}((u_1,u_2)(v_1,u_2))=c_2(p_2(d_{G_1^*}(u_1)+d_{G_1^*}(v_1))-2)+2d_{G_2}(u_2)$,

   (c) When $u_1v_1\in E_1, u_2v_2\notin E_2$, $d_{G_1[G_2]}((u_1,u_2)(v_1,v_2))=c_2(p_2(d_{G_1^*}(u_1)+d_{G_1^*}(v_1))-2)+d_{G_2}(u_2)$ $+d_{G_2}(v_2)$.

Theorem 3.8. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs on a regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_1 \le \mu_2$ and μ_1 be a constant function with $\mu_1(e)=c_1$ for all $e\in E_1$. Then $G_1 \circ G_2$ is an edge regular fuzzy graph.

Proof: Given $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ are two fuzzy graphs on regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_1 \le \mu_2$ and $\mu_1(e)=c_1$ for all $e\in E_1$ respectively. Then $G_1:(\sigma_1,\mu_1)$ is both regular and edge regular. (Using Theorem 2.8.) Then $d_{G_1}(u_1v_1)=k, d_{G_1}(u_1)=m$,$d_{G_2^*}(u_2)=n,$ $\forall u_1,v_1\in V_1$ and $u_2\in V_2$, where k,m and n are constants.

We prove the theorem in three cases using Theorem 3.7. Let $(u_1,u_2)(v_1,v_2)\in E$.

Case 1: When $u_1=v_1, u_2v_2\in E_2$.

$$d_{G_1[G_2]}((u_1,u_2)(u_1,v_2)) = 2p_2 d_{G_1}(u_1)+c_1(d_{G_2^*}(u_2)+d_{G_2^*}(v_2)-2).$$
$$= 2p_2m + c_1(n+n-2).$$
$$= 2[p_2m+c_1(n-1)] \tag{3.6}$$

Case 2: When $u_2=v_2, u_1v_1\in E_1$.

$$d_{G_1[G_2]}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1)+(p_2-1)(d_{G_1}(u_1)+d_{G_1}(v_1))+2c_1 d_{G_2^*}(u_2).$$
$$= k+(p_2-1)(m+m)+2c_1n.$$
$$= 2m-2c_1+2p_2m-2m+2c_1n. \text{ (Since, } k=2m-2c_1)$$
$$= 2[p_2m+c_1(n-1)] \tag{3.7}$$

Case 3: When $u_1v_1\in E_1, u_2v_2\notin E_2$.

$$d_{G_1[G_2]}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1)+(p_2-1)(d_{G_1}(u_1)+d_{G_1}(v_1))+c_1(d_{G_2^*}(u_2)+d_{G_2^*}(v_2)).$$
$$= k+(p_2-1)(m+m)+c_1(n+n).$$
$$= 2m-2c_1+2p_2m-2m+2c_1n. \text{ (Since, } k=2m-2c_1)$$
$$= 2[p_2m+c_1(n-1)] \tag{3.8}$$

From (3.6), (3.7) and (3.8), $G_1 \circ G_2$ is an edge regular fuzzy graph. ■

Corollary 3.9. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs with $\sigma_1\le\mu_2$ and σ_1 be a constant function with $\sigma_1(u)=c_1$ for all $u\in V_1$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be regular underlying crisp graphs. If G_1 is strong, then $G_1\circ G_2$ is an edge regular fuzzy graph.

Proof: Given that $G_1:(\sigma_1,\mu_1)$ is strong with $\sigma_1(u)=c_1$ for all $u\in V_1$. Then $\mu_1(e)=c_1$ for all $e\in E_1$. Therefore, the result follows from Theorem 3.8. ■

Theorem 3.10. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs on a regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $\sigma_2 \leq \mu_1$ and let μ_2 be a constant function with $\mu_2(e)=c_2$ for all $e \in E_2$. Then $G_1 \circ G_2$ is an edge regular fuzzy graph.

Proof: The proof is similar to the proof of Theorem 3.8. ∎

Corollary 3.11. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs with $\sigma_2 \leq \mu_1$ and σ_2 be a constant function with $\sigma_2(u)=c_2$ for all $u \in V_2$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be regular underlying crisp graphs. If G_2 is strong, then $G_1 \circ G_2$ is an edge regular fuzzy graph.

Proof: The proof is similar to the proof of Theorem 3.9. ∎

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