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Hyers-Ulam stability of Cubic and Quartic Functional Equations in matrix paranormed spaces

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Abstract

In this paper, we prove the Hyers-Ulam stability of the cubic and quartic functional equations in matrix paranormed spaces.

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1 Introduction and preliminaries

The stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam's problem see ([17] and [7]). Thereafter, Rassias [14] attempted to solve the stability problem of the cauchy additive functional equation in a more general setting. The concept introduced by Rassias's theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations see ([1], [7], [8], [9], [13]).

Choonkil Park and Dong Yun Shin [2] investigated functional equation in paranormed spaces. Choonkil Park and Jung Rye Lee [3] proved the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation in paranormed spaces. Recently, Choonkil Park and Dong Yun Shin [4] prove the Hyers-Ulam stability of Cauchy additive functional inequality, the Cauchy additive functional equation and quadratic functional equation in matrix paranormed spaces.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [5] and Steinhaus [16] independently and since then several generalizations and applications of this notion have been investigated by various authors [6], [10], [12], [15]. This notion was defined in normed spaces by Kolk [11].
We recall some basic facts concerning Frechet spaces.

Definition 1.1. [18] Let X be a vector space. A paranorm $P(.) : X \to [0, \infty)$ is a function on X such that

1. $P(0) = 0$;

2. $P(-x) = P(x)$;

3. $P(x + y) \leq P(x) + P(y)$(triangle inequality);

4. If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$(continuity of multiplication).
   The pair $(X, P(.))$ is called a paranormed space if $P(.)$ is a paranorm on X.
   The paranorm is called total if, in addition, we have

5. $P(x) = 0$ implies $x = 0$.
   A Frechet space is a total and complete paranormed space.

We use the following notations:
$M_n(X)$ is that set of all $n \times n$ matrices in X;
$e_j \in M_{1,n}(\mathbb{C})$ is that j^{th} component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{C})$ is that (i, j)- component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i, j)- component is x and the other components are zero.
For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

Note that $(X, \{\|.\|_n\})$is a matrix normed space if and only if $(M_n(X), \|.\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\| \|B\| \|x\|_n$ holds for $A \in M_{k,n}, x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$ and that $(X, \{\|.\|_n\})$is a matrix Banach space if and only if X is a Banach space and $(X, \{\|.\|_n\})$ is a matrix normed space.

Definition 1.2. Let $(X, P(.))$ be a paranormed space.

1. $(X, \{P_n(.)\})$ is a matrix paranormed space if $(M_n(X), P_n(.))$ is a paranormed space for each positive integer n, $P_n(E_{kl} \otimes x) = P(x)$ for $x \in X$, and $P(x_{kl}) \leq P_n([x_{ij}])$ for $[x_{ij}] \in M_n(X)$.

2. $(X, \{P_n(.)\})$ is a matrix Frechet space if X is a Frechet spaces and $(X, \{P_n(.)\})$ is a matrix paranormed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.
Throughout this paper, let $(X, \{\|.\|_n\})$ be a matrix Banach space and $(Y, \{P_n(.)\})$ be a matrix

Frechet space.
Note that $P(2x) \leq 2P(x)$ for all $x \in Y$.

Lemma 1.3. Let $(X, \{P_n(.)\})$ be a matrix paranormed space. Then

1. $P(x_{kl}) \leq P_n([x_{ij}]) \leq \sum_{i,j=1}^{n} P(x_{ij})$ for $[x_{ij}] \in M_n(X)$.

2. $\lim_{s\to\infty} x_s = x$ if and only if $\lim_{s\to\infty} x_{sij} = x_{ij}$ for $x_s = [x_{sij}]$, $x = [x_{ij}] \in M_k(X)$.

Proof: 1. By Definition 1.2, $P(x_{kl}) \leq P_n([x_{ij}])$.
Since $[x_{ij}] = \sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}$,

$$P_n([x_{ij}]) = P_n\left(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}\right) \leq \sum_{i,j=1}^{n} P_n\left(E_{ij} \otimes x_{ij}\right) = \sum_{i,j=1}^{n} P(x_{ij}).$$

2. By (1), we have

$$P\left(x_{skl} - x_{kl}\right) \leq P_n\left([x_{sij} - x_{ij}]\right) = P_n\left([x_{sij}] - [x_{ij}]\right) \leq \sum_{i,j=1}^{n} P(x_{sij} - x_{ij}).$$

So, we get the result.

■

Lemma 1.4. Let $(X, \{\|.\|_n\})$ be a matrix normed space. Then

1. $\|E_{kl} \otimes x\|_n = \|x\|$ for $x \in X$;

2. $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^{n} \|x_{ij}\|$ for $[x_{ij}] \in M_n(X)$;

3. $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_{ijn} = x_{ij}$ for $x_n = [x_{ijn}]$, $x = [x_{ij}] \in M_k(X)$.

Proof: (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\|E_{kl} \otimes x\|_n \leq \|x\|$. Since $e_k\left(E_{kl} \otimes x\right) e_l^* = x$, $\|x\| \leq \|E_{kl} \otimes x\|_n$. So, $\|E_{kl} \otimes x\|_n = \|x\|$.
(2) Since $e_k x e_l^* = x_{kl}$ and $\|e_k\| = \|e_l^*\| = 1$, $\|x_{kl}\| \leq \|[x_{ij}]\|_n$. Since $[x_{ij}] = \sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}$,

$$\|[x_{ij}]\|_n = \left\|\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}\right\|_n \leq \sum_{i,j=1}^{n} \|E_{ij} \otimes x_{ij}\|_n = \sum_{i,j=1}^{n} \|x_{ij}\|.$$

(3) By $\|x_{kln} - x_{kl}\| \leq \|[x_{ijn} - x_{ij}]\|_n = \|[x_{ijn}] - [x_{ij}]\|_n \leq \sum_{i,j=1}^{n} \|x_{ijn} - x_{ij}\|$, we get the result. ■

2 Hyers-Ulam stability of the cubic functional equation in matrix paranormed spaces

In this section, we prove the Hyers-Ulam stability of the cubic functional equation in Matrix Paranormed spaces. For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$Df(a,b) = \frac{1}{2}f(2a+b) + \frac{1}{2}f(2a-b) - f(a+b) - f(a-b) - 6f(a)$$

$$Df_n([x_{ij}],[y_{ij}]) = \frac{1}{2}f_n(2[x_{ij}]+[y_{ij}]) + \frac{1}{2}f_n(2[x_{ij}]-[y_{ij}]) - f_n([x_{ij}+y_{ij}]) - f_n([x_{ij}-y_{ij}]) - 6f_n[x_{ij}]$$

for all $a, b \in X$ and all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$

Theorem 2.1. Let r, θ be positive real numbers with $r > 3$. Let $f : X \to Y$ be a mapping such that

$$P_n(Df_n([x_{ij}],[y_{ij}])) \leq \sum_{i,j=1}^{n} \theta\left(\|x_{ij}\|^r + \|y_{ij}\|^r\right) \tag{2.1}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exist a unique cubic mapping $C : X \to Y$ such that

$$P_n(f_n([x_{ij}]) - C_n([x_{ij}])) \leq \sum_{i,j=1}^{n} \frac{\theta}{2^r - 8}\|x_{ij}\|^r \tag{2.2}$$

for all $x = [x_{i,j}] \in M_n(X)$.

Proof: Let $n = 1$ in (2.1). Then (2.1) is equivalent to

$$P\left(\frac{1}{2}f(2a+b) + \frac{1}{2}f(2a-b) - f(a+b) - f(a-b) - 6f(a)\right) \leq \theta\left(\|a\|^r + \|b\|^r\right) \tag{2.3}$$

for all $a, b \in X$.
Letting $b = 0$ in (2.3), we get $P(f(2a) - 8f(a)) \leq \theta\|a\|^r$ and so $P\left(f(a) - 8f(\frac{a}{2})\right) \leq \frac{1}{2^r}\theta\|a\|^r$ for all $a \in X$. One can easily show that

$$P\left(8^p f(\frac{a}{2^p}) - 8^q f(\frac{a}{2^q})\right) \leq \sum_{l=p}^{q-1} P\left(8^l f\left(\frac{a}{2^l}\right) - 8^{l+1} f\left(\frac{a}{2^{l+1}}\right)\right)$$

$$\leq \frac{1}{2^r}\sum_{l=p}^{q-1}\frac{8^l}{2^{rl}}\theta\|a\|^r \tag{2.4}$$

for all $a, b \in X$ and nonnegative integers p, q with $p < q$. It follows from (2.4) that the sequence $\left\{8^l f(\frac{a}{2^l})\right\}$ is Cauchy for all $a \in X$. Since Y is complete, the sequence $\left\{8^l f(\frac{a}{2^l})\right\}$ converges. So, one can define the mapping $C : X \to Y$ by

$$C(a) = \lim_{l\to\infty} 8^l f\left(\frac{a}{2^l}\right)$$

for all $a \in X$.
Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (2.4), we get

$$P\left(f(a) - C(a)\right) \leq \frac{\theta}{2^r - 8}\|a\|^r \tag{2.5}$$

for all $a \in X$. It follows from 2.3 that

$$\begin{aligned}
&P\left(8^l\left(\frac{1}{2}f\left(\frac{2a+b}{2^l}\right)+\frac{1}{2}f\left(\frac{2a-b}{2^l}\right)-f\left(\frac{a+b}{2^l}\right)\left(\frac{a-b}{2^l}\right)-6f\left(\frac{a}{2^l}\right)\right)\right)\\
&\leq 8^l P\left(\frac{1}{2}f\left(\frac{2a+b}{2^l}\right)+\frac{1}{2}f\left(\frac{2a-b}{2^l}\right)-f\left(\frac{a+b}{2^l}\right)\left(\frac{a-b}{2^l}\right)-6f\left(\frac{a}{2^l}\right)\right)\\
&\leq \frac{8^l}{2^{lr}}\theta\left(\|a\|^r+\|b\|^r\right)
\end{aligned}$$

which tends to zero as $l \to \infty$. So,

$$P\left(\frac{1}{2}C\left(2a+b\right)+\frac{1}{2}C\left(2a-b\right)-C\left(a+b\right)-C\left(a-b\right)-6C(a)\right)=0$$

That is, $\frac{1}{2}C\left(2a+b\right)+\frac{1}{2}C\left(2a-b\right)=C\left(a+b\right)+C\left(a-b\right)+6C(a)$

for all $a, b \in X$. Hence $C : X \to Y$ is cubic. Now, let $T : X \to Y$ be another cubic mapping satisfying (2.5). Then we have

$$\begin{aligned}
P\left(C(a)-T(a)\right) &= P\left(8^l\left(C\left(\frac{a}{2^l}\right)-T\left(\frac{a}{2^l}\right)\right)\right)\\
&\leq 8^l P\left(C\left(\frac{a}{2^l}\right)-T\left(\frac{a}{2^l}\right)\right)\\
&\leq 8^l\left(P\left(C\left(\frac{a}{2^l}\right)-f\left(\frac{a}{2^l}\right)\right)+P\left(T\left(\frac{a}{2^l}\right)-f\left(\frac{a}{2^l}\right)\right)\right)\\
&\leq \frac{2.8^l}{\left(2^r-8\right)2^{lr}}\theta\|a\|^r,
\end{aligned}$$

which tends to zero as $l \to \infty$ for all $a \in X$. So, we conclude that $C(a) = T(a)$ for all $a \in X$.

By Lemma 1.3 and (2.5)

$$P_n\left(f_n\left([x_{ij}]\right)-C_n\left([x_{ij}]\right)\right) \leq \sum_{i,j=1}^{n} P\left(f(x_{ij})-C(x_{ij})\right)$$
$$\leq \sum_{i.j=1}^{n} \frac{\theta}{2^r-8}\left\|x_{ij}\right\|^r$$

for all $x=[x_{i,j}]\in M_n(X)$. Thus $C:X\to Y$ is the unique cubic mapping satisfying (2.2) as desired. ∎

Theorem 2.2. Let r,θ be positive real numbers $r<3$. Let $f:Y\to X$ be a mapping such that

$$\left\|Df_n\left([x_{ij},y_{ij}]\right)\right\|_n \leq \sum_{i,j=1}^{n}\theta\left(P(x_{ij})^r+P(y_{ij})^r\right) \tag{2.6}$$

for all $x=[x_{ij}], y=[y_{ij}]\in M_n(Y)$. Then there exist a unique cubic mapping $C:Y\to X$ such that

$$\left\|f_n\left([x_{ij}]\right)-C_n\left([x_{ij}]\right)\right\|_n \leq \sum_{i,j=1}^{n}\frac{\theta}{8-2^r}P(x_{ij})^r \tag{2.7}$$

for all $x=[x_{ij}]\in M_n(Y)$.

Proof: Let $n=1$ in (2.6). Then (2.6) is equivalent to

$$\left\|\frac{1}{2}f\left(2a+b\right)+\frac{1}{2}f\left(2a-b\right)-f\left(a+b\right)-f\left(a-b\right)-6f(a)\right\| \leq \theta\left(P(a)^r+P(b)^r\right) \tag{2.8}$$

for all $a,b\in Y$.
Letting $b=0$ in (2.8), we get

$$\|f(2a)-8f(a)\|\leq \theta P(a)^r \text{ and so } \quad \left\|f(a)-\frac{1}{8}f(2a)\right\|\leq\frac{\theta}{8}P(a)^r \quad \text{for all } a\in Y.$$

One can easily show that

$$\left\|\frac{1}{8^p}f(2^pa)-\frac{1}{8^q}f(2^qa)\right\|\leq\sum_{l=p}^{q-1}\left\|\frac{1}{8^l}f(2^la)-\frac{1}{8^{l+1}}f(2^{l+1}a)\right\|$$

$$\leq\frac{1}{8}\sum_{l=p}^{q-1}\frac{2^{rl}}{8^l}\theta P(a)^r \tag{2.9}$$

for all $a\in Y$ and non-negative integers p,q with $p<q$. It follows from (2.9) that the sequence $\left\{\frac{1}{8^l}f(2^la)\right\}$ is a Cauchy for all $a\in Y$. Since X is complete, the sequence $\left\{\frac{1}{8^l}f(2^la)\right\}$ converges.

So, one can define the mapping $C : Y \to X$ by

$$C(a) = \lim_{l\to\infty} \frac{1}{8^l} f(2^l a), \text{ for all } a \in Y.$$

Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (2.9), we get

$$\|f(a) - C(a)\| \leq \frac{1}{8 - 2^r}\theta P(a)^r \tag{2.10}$$

for all $a \in Y$. It follows from (2.8) that

$$\left\|\frac{1}{8^l}\left(\frac{1}{2}f\left(2^l(2a+b)\right) + \frac{1}{2}f\left(2^l(2a-b)\right) - f\left(2^l(a+b)\right) - f\left(2^l(a-b)\right) - 6f(2^l a)\right)\right\|$$
$$\leq \frac{2^{lr}}{8^l}\theta\left(P(a)^r + P(b)^r\right).$$

which tends to zero as $l \to \infty$. So,

$$\left\|\frac{1}{2}C(2a+b) + \frac{1}{2}C(2a-b) - C(a+b) - C(a-b) - 6C(a)\right\| = 0$$

That is,

$$\frac{1}{2}C(2a+b) + \frac{1}{2}C(2a-b) = C(a+b) + C(a-b) + 6C(a)$$

for all $a, b \in Y$. Hence $C : Y \to X$ is cubic.

Now, let $T : Y \to X$ be another cubic mapping satisfying (2.10). Then we have

$$\begin{aligned}\|C(a) - T(a)\| &= \frac{1}{8^l}\left\|C(2^l a) - T(2^l a)\right\| \\ &\leq \frac{1}{8^l}\left(\left\|C(2^l a) - f(2^l a))\right\| + \left\|T(2^l a) - f(2^l a)\right\|\right) \\ &\leq \frac{2.2^{lr}}{(8-2^r)\,8^l}\theta P(a)^r\end{aligned}$$

which tends to zero as $n \to \infty$ for all $a \in Y$. So we conclude that $C(a) = T(a)$ for all $a \in Y$. This proves the uniquness of C.

By Lemma 1.4 and (2.10),

$$\begin{aligned}\|f_n([x_{ij}]) - C_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \|f(x_{ij}) - C(x_{ij})\| \\ &\leq \sum_{i,j=1}^{n} \frac{\theta}{8-2^r} P(x_{ij})^r\end{aligned}$$

for all $x = [x_{ij}] \in M_n(Y)$. Thus $C : Y \to X$ is the unique cubic mapping satisfying (2.7). ■

3 Hyers-Ulam stability of the quartic functional equation

In this section, we prove the Hyers-Ulam Stability of the quartic functional equation in Matrix Paranormed spaces. For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$Df(a,b) = \tfrac{1}{2}f(2a+b) + \tfrac{1}{2}f(2a-b) - 2f(a+b) - 2f(a-b) - 12f(a) + 3f(b)$$

$$\begin{aligned} Df_n([x_{ij}],[y_{ij}]) = \frac{1}{2}f_n(2[x_{ij}]+[y_{ij}]) + \frac{1}{2}f_n(2[x_{ij}]-[y_{ij}]) - 2f_n([x_{ij}]+[y_{ij}]) \\ - 2f_n([x_{ij}]-[y_{ij}]) - 12f_n[x_{ij}] + 3f_n[y_{ij}] \end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.1. Let r, θ be positive real numbers with $r > 4$. Let $f : X \to Y$ be a mapping such that

$$P_n(Df_n([x_{ij}],[y_{ij}])) \leq \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \tag{3.1}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exist a unique quartic mapping $Q : X \to Y$ such that

$$P_n(f_n([x_{ij}]) - Q_n([x_{ij}])) \leq \sum_{i,j=1}^{n} \frac{\theta}{2^r - 16} \|x_{ij}\|^r \tag{3.2}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof: Let $n = 1$ in (3.1). Then (3.1) is equivalent to

$$P\left(\frac{1}{2}f(2a+b) + \frac{1}{2}f(2a-b) - 2f(a+b) - 2f(a-b) - 12f(a) + 3f(b)\right) \leq \theta(\|a\|^r + \|b\|^r) \tag{3.3}$$

for all $a, b \in X$.
Letting $b = 0$ in (3.3), we get

$$P(f(2a) - 16f(a)) \leq \theta\|a\|^r$$

for all $a \in X$. So

$$P(f(a) - 16f(\frac{a}{2})) \leq \frac{1}{2^r}\theta\|a\|^r$$

for all $a \in X$. Hence

$$P\left(16^p f(\frac{a}{2^p}) - 16^q f(\frac{a}{2^q})\right) \leq \sum_{l=p}^{q-1} P\left(16^l f(\frac{a}{2^l}) - 16^{l+1} f(\frac{a}{2^{l+1}})\right) \leq \frac{1}{2^r} \sum_{l=p}^{q-1} \frac{16^l}{2^{rl}} \theta \|a\|^r \tag{3.4}$$

for all $a \in X$ and nonnegative integers p, q with $p < q$. It follows from (3.4) that the sequence $\left\{16^l f(\frac{a}{2^l})\right\}$ is a Cauchy for all $a \in X$. Since Y is complete, the sequence $\left\{16^l f(\frac{a}{2^l})\right\}$ converges. So, one can define the mapping $Q : X \to Y$ by

$$Q(a) = \lim_{l \to \infty} 16^l f(\frac{a}{2^l})$$

for all $a \in X$. Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (3.4), we get

$$P\left(f(a) - Q(a)\right) \leq \frac{\theta}{2^r - 16} \|a\|^r \tag{3.5}$$

for all $a \in X$. It follows from (3.3) that

$$\begin{aligned} & P\left(16^l \left(\frac{1}{2} f(\frac{2a+b}{2^l}) + \frac{1}{2} f(\frac{2a-b}{2^l}) - 2f(\frac{a+b}{2^l}) - 2f(\frac{a-b}{2^l}) - 12 f(\frac{a}{2^l}) + 3f(\frac{b}{2^l})\right)\right) \\ & \leq 16^l P\left(\frac{1}{2} f(\frac{2a+b}{2^l}) + \frac{1}{2} f(\frac{2a-b}{2^l}) - 2f(\frac{a+b}{2^l}) - 2f(\frac{a-b}{2^l}) - 12 f(\frac{a}{2^l}) + 3f(\frac{b}{2^l})\right) \\ & \leq \frac{16^l}{2^{lr}} \theta \left(\|a\|^r + \|b\|^r\right) \end{aligned}$$

which tends to zero as $l \to \infty$. So,

$$P\left(\frac{1}{2} Q(2a+b) + \frac{1}{2} Q(2a-b) - 2Q(a+b) - 2Q(a-b) - 12Q(a) + 3Q(b)\right) = 0$$

That is,

$$\frac{1}{2} Q(2a+b) + \frac{1}{2} Q(2a-b) = 2Q(a+b) + 2Q(a-b) + 12Q(a) - 3Q(b)$$

for all $a, b \in X$. Hence $Q : X \to Y$ is quartic. The proof of the uniqueness of Q is similar to the proof of Theorem 2.1.

By Lemma 1.3 and (3.5),

$$\begin{aligned} P_n\left(f_n([x_{ij}]) - Q_n([x_{ij}])\right) & \leq \sum_{i,j=1}^{n} P\left(f(x_{ij}) - Q(x_{ij})\right) \\ & \leq \sum_{i,j=1}^{n} \frac{\theta}{2^r - 16} \|x_{ij}\|^r \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $Q : X \to Y$ is the unique quartic mapping satisfying (3.2). ∎

Theorem 3.2. Let r, θ be positive real numbers with $r < 4$. Let $f : Y \to X$ be a mapping such that

$$\|Df_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \theta\left(P(x_{ij})^r + P(y_{ij})^r\right) \tag{3.6}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(Y)$. Then there exists a unique quartic mapping $Q : Y \to X$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\theta}{16 - 2^r} P(x_{ij})^r \tag{3.7}$$

for all $x = [x_{ij}] \in M_n(Y)$.

Proof: Let $n = 1$ in (3.6). Then (3.6) is equivalent to

$$\left\|\frac{1}{2}f(2a+b) + \frac{1}{2}f(2a-b) - 2f(a+b) - 2f(a-b) - 12f(a) + 3f(b)\right\| \leq \theta\left(P(a)^r + P(b)^r\right) \tag{3.8}$$

for all $a, b \in Y$.
Letting $b = 0$ in (3.8), we get

$$\|16f(a) - f(2a)\| \leq \theta P(a)^r$$

and so

$$\left\|f(a) - \frac{1}{16}f(2a)\right\| \leq \frac{\theta}{16} P(a)^r$$

for all $a \in Y$.
The rest of the proof is similar to the proof of Theorem 2.2. ■

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