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Some standard cube divisor cordial graphs

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Abstract

A cube divisor cordial labeling of a graph G with vertex set $V(G)$ is a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that an edge $e = uv$ is assigned the label 1 if $[f(u)]^3|f(v)$ or $[f(v)]^3|f(u)$ and the label 0 otherwise, then $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a cube divisor cordial labeling is called a cube divisor cordial graph. In this paper we discuss cube divisor cordial labeling of some standard graphs such as path, cycle, wheel, flower and fan.

Keywords: Divisor cordial labeling, square divisor cordial labeling, cube divisor cordial labeling.
AMS Subject Classification(2010): 05C78.

1 Introduction and preliminaries

Throughout this work, by a graph we mean finite, connected, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)|$ and size $|E(G)|$.

Definition 1.1. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices(edges) then the labeling is called a *vertex labeling(an edge labeling).*

The most recent findings on various graph labeling techniques can be found in Gallian[2].

Notation 1.2. $e_f(i)$ = Number of edges with label $i; i = 0, 1$.

Definition 1.3. Let $G = (V(G), E(G))$ be a simple graph and f :V(G)→ $\{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise. The function f is called a *divisor cordial labeling* if $|e_f(0) - e_f(1)| \leq 1$.
A graph which admits a divisor cordial labeling is called a *divisor cordial graph.*

The concept of divisor cordial labeling was introduced by Varatharajan et al.[10] and they proved the following results:

- The path P_n is divisor cordial.
- The cycle C_n is divisor cordial.
- The wheel graph W_n is divisor cordial.
- The star graph $K_{1,n}$ is divisor cordial.
- The complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ are divisor cordial.
- The complete graph K_n is not divisor cordial for $n \geq 7$.

Definition 1.4. Let $G = (V(G), E(G))$ be a simple graph and f :V(G)$\rightarrow \{1, 2, \dots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if $[f(u)]^2 | f(v)$ or $[f(v)]^2 | f(u)$ and the label 0 otherwise. The function f is called a *square divisor cordial labeling* if $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a square divisor cordial labeling is called a *square divisor cordial graph.*

The concept of square divisor cordial labeling was introduced by Murugesan et al.[7] and they proved the following results:

- The path P_n is square divisor cordial if and only if $n \leq 12$.
- The cycle C_n is square divisor cordial if and only if $3 \leq n \leq 11$.
- The wheel graph W_n is square divisor cordial.
- The star graph $K_{1,n}$ is square divisor cordial if and only if $n = 2, 3, 4, 5$ or 7.
- The complete bipartite graph $K_{2,n}$ is square divisor cordial.
- The complete bipartite graph $K_{3,n}$ is square divisor cordial if and only if $n = 1, 2, 3, 5, 6, 7$ or 9.

S. K. Vaidya and N. H. Shah[8] proved that

- Flower graph Fl_n is a square divisor cordial graph for each n.
- Bistar $B_{n,n}$ is a square divisor cordial graph.
- Restricted $B^2_{n,n}$ is a square divisor cordial graph.

The present authors are motivated by two research articles "Divisor Cordial Graphs" by Varatharajan et al.[10] and "Square Divisor Cordial Graphs" by Murugesan et al.[7]. They defined cube divisor cordial labeling as follows.

Definition 1.5. Let $G = (V(G), E(G))$ be a simple graph and f :V(G)→ $\{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if $[f(u)]^3|f(v)$ or $[f(v)]^3|f(u)$ and the label 0 otherwise. The function f is called a *cube divisor cordial labeling* if $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a cube divisor cordial labeling is called a *cube divisor cordial graph.*

Definition 1.6. If $G = (V(G), E(G))$ is a graph and e is an edge of G then $G - e$ denotes *the edge deleted subgraph* of G having $V(G)$ as its vertex set and $E(G) - \{e\}$ as its edge set.

Definition 1.7. The *wheel* W_n is defined as the join $C_n + K_1$. The vertex K_1 is the apex vertex and the vertices on the underlying cycle are called rim vertices. The edges of the underlying cycle are called the rim edges and the edges joining the apex and the rim vertices are called spoke edges.

Definition 1.8. The *flower graph* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.
It contains three types of vertices: an apex of degree $2n$, n vertices of degree 4 and n vertices of degree 2.

Definition 1.9. The *fan graph* f_n is defined as the join $P_n + K_1$. The vertex corresponding to K_1 is said to be the apex vertex.

2 Main Results

Theorem 2.1. Given a positive integer n, there is a cube divisor cordial graph G which has n vertices.

Proof: Let n be any positive integer.
Case 1: $n \equiv 0(mod\ 2)$.
By constructing a path containing $\frac{n}{2}+2$ vertices $v_1, v_2, \ldots, v_{\frac{n}{2}+2}$ which are labeled as $1, 2, \ldots, \frac{n}{2}+2$ respectively and attaching $\frac{n}{2} - 2$ vertices $v_{\frac{n}{2}+3}, v_{\frac{n}{2}+4}, \ldots, v_n$ which are labeled as $\frac{n}{2} + 3, \frac{n}{2} + 4, \ldots, n$ respectively, to the vertex v_1. We see that $e_f(0) = \frac{n}{2}$ and $e_f(1) = \frac{n}{2} - 1$ and hence, $|e_f(0) - e_f(1)| \leq 1$. Thus, the resultant graph G is cube divisor cordial.

Case 2: $n \equiv 1(mod\ 2)$.
By constructing a path containing $\lfloor\frac{n}{2}\rfloor + 2$ vertices $v_1, v_2, \ldots, v_{\lfloor\frac{n}{2}\rfloor+2}$ which are labeled as $1, 2, \ldots, \lfloor\frac{n}{2} + 2\rfloor$ respectively and attaching $\lfloor\frac{n}{2}\rfloor - 1$ vertices $v_{\lfloor\frac{n}{2}\rfloor+3}, v_{\lfloor\frac{n}{2}\rfloor+4}, \ldots, v_n$ which are labeled as $\lfloor\frac{n}{2}\rfloor + 3, \lfloor\frac{n}{2}\rfloor + 4, \ldots, n$ respectively , to the vertex v_1. We see that $e_f(0) = \lfloor\frac{n}{2}\rfloor$ and $e_f(1) = \lfloor\frac{n}{2}\rfloor$ and hence, $|e_f(0) - e_f(1)| \leq 1$. Thus, the resultant graph G is cube divisor cordial. ∎

Illustration 2.2. The cube divisor cordial graph for $n = 13$ is shown in *Figure 1.*

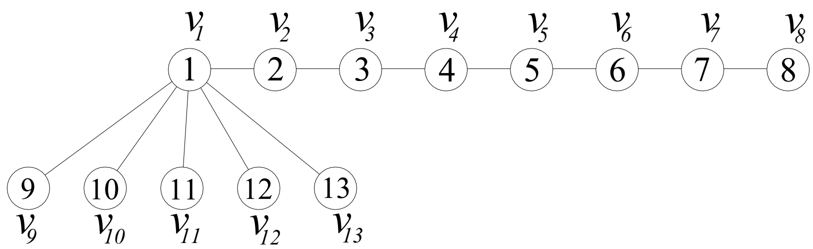

Figure 1: The cube divisor cordial graph for $n = 13$.

Theorem 2.3. If G is a cube divisor cordial graph of even size, then $G - e$ is also cube divisor cordial for all $e \in E(G)$.

Proof: Let G be the cube divisor cordial graph of size n, where n is an even number. Then if follows that $e_f(0) = e_f(1) = \frac{n}{2}$. Let e be any edge in G which is labeled either 0 or 1. Then in $G - e$, we have either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$ and hence, $|e_f(0) - e_f(1)| \leq 1$. Thus, $G - e$ is a cube divisor cordial. ∎

Theorem 2.4. If G is a cube divisor cordial graph of odd size, then $G - e$ is also cube divisor cordial for some $e \in E(G)$.

Proof: Let G be the cube divisor cordial graph of size n, where n is an odd number. Then it follows that either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$. If $e_f(0) = e_f(1) + 1$, then remove the edge e which is labeled as 0 and if $e_f(1) = e_f(0) + 1$ then remove the edge e which is labeled as 1 from G. Then in $G - e$, we have $e_f(0) = e_f(1)$ and hence $|e_f(0) - e_f(1)| \leq 1$. Thus, $G - e$ is a cube divisor cordial for some $e \in E(G)$. ∎

Theorem 2.5. The path P_n is a cube divisor cordial graph if and only if $n = 1, 2, 3, 4, 5, 6, 8$.

Proof: Let P_n be the path of length n with vertices $v_1, v_2, \ldots, v_n$. To define cube divisor cordial labeling $f : V(P_n) \to \{1, 2, \ldots, n\}$ we consider the following cases.

Case 1: $n = 1, 2, 3$.

$f(v_i) = i; i = 1, 2, 3$.

Then, we have $|e_f(0) - e_f(1)| \leq 1$.

Case 2: $n = 4, 5, 6$.

$f(v_1) = n$,

$f(v_i) = i - 1; i = 2, 3, \ldots, n$.

Then, we have $|e_f(0) - e_f(1)| \leq 1$.

Case 3: $n = 8$.

$f(v_1) = n$,

$f(v_2) = 2$,

$f(v_3) = 1$,
$f(v_i) = i - 1; i = 4, 5, \ldots, n$.
Then, we have $|e_f(0) - e_f(1)| \leq 1$.
Thus, path P_n is a cube divisor cordial graph if $n = 1, 2, 3, 4, 5, 6, 8$.

Case 4: $n = 7$.
In path P_7 we have 7 vertices and 6 edges. Then obviously $e_f(i) = 3, i = 0, 1$.
In any labeling pattern we get at most 2 edges having label 1. $e_f(1) \leq 2$.
Therefore, $e_f(0) \geq 4$ and $|e_f(0) - e_f(1)| \geq 2$.
Thus, P_7 is not a cube divisor cordial graph.

Case 5: $n \geq 9$.
If possible, let there exist a cube divisor cordial labeling f.

Subcase 1: $n \equiv 0(mod\ 2)$.
Then obviously either $e_f(0) = \frac{n}{2}$ and $e_f(1) = \frac{n}{2} - 1$ or $e_f(0) = \frac{n}{2} - 1$ and $e_f(1) = \frac{n}{2}$.
In order to get the labels 1, at most two numbers can be assigned to the vertices that are adjacent to the vertices labeled with $1, 2, \ldots, \lfloor \sqrt[3]{n} \rfloor - 1$ and at most one number can be assigned to the vertex that is adjacent to the vertex labeled with $\lfloor \sqrt[3]{n} \rfloor$.
Therefore, we have

$$\begin{aligned} e_f(1) &\leq 2(\lfloor \sqrt[3]{n} \rfloor - 1) + 1 \\ &\leq 2\lfloor \sqrt[3]{n} \rfloor - 1. \\ \text{Hence, } e_f(0) &\geq n - 1 - (2\lfloor \sqrt[3]{n} \rfloor - 1) \\ &= n - 2\lfloor \sqrt[3]{n} \rfloor \\ &> \frac{n}{2} \\ &> \frac{n}{2} - 1. \end{aligned}$$

This is a contradiction. Thus, if n is even, there can not be a cube divisor cordial labeling.

Subcase 2: $n \equiv 1(mod\ 2)$.
Then obviously $e_f(i) = \frac{n-1}{2}, i = 0, 1$.
In order to get the labels 1, at most two numbers can be assigned to the vertices that are adjacent to the vertices labeled with $1, 2, \ldots, \lfloor \sqrt[3]{n} \rfloor - 1$ and at most one number can be assigned to the vertex that is adjacent to the vertex labeled with $\lfloor \sqrt[3]{n} \rfloor$.
Therefore, we have

$$\begin{aligned} e_f(1) &\leq 2(\lfloor \sqrt[3]{n} \rfloor - 1) + 1 \\ &\leq 2\lfloor \sqrt[3]{n} \rfloor - 1. \\ \text{Hence, } e_f(0) &\geq n - 1 - (2\lfloor \sqrt[3]{n} \rfloor - 1) \end{aligned}$$

$$= n - 2\lfloor \sqrt[3]{n} \rfloor$$
$$> \frac{n-1}{2}.$$

This is a contradiction. Thus, if n is odd, there can not be a cube divisor cordial labeling. ■

Illustration 2.6. Cube divisor cordial labeling of the path P_4 is shown in *Figure 2*.

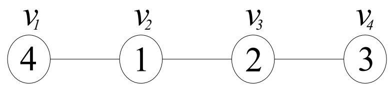

Figure 2: Cube divisor cordial labeling of the path P_4.

Theorem 2.7. The cycle C_n is a cube divisor cordial graph if and only if $n = 3, 4, 5$.

Proof: Let C_n be the cycle with vertices $v_1, v_2, \ldots, v_n$. The proof is divided in to the following two cases.
Case 1: $n = 3, 4, 5$.
We define cube divisor cordial labeling $f : V(C_n) \to \{1, 2, \ldots, n\}$ as $f(v_i) = i, i = 1, 2, \ldots, n$.
Then, we have $|e_f(0) - e_f(1)| \leq 1$.
Thus, cycle C_n is a cube divisor cordial graph if $n = 3, 4, 5$.
Case 2: $n \geq 6$.
If possible, let there exist a cube divisor cordial labeling f.
Subcase 1: $n \equiv 0(mod\ 2)$.
Then obviously $e_f(i) = \frac{n}{2}, i = 0, 1$.

In order to get the labels 1, at most two numbers can be assigned to the vertices that are adjacent to the vertices labeled with $1, 2, \ldots, \lfloor \sqrt[3]{n} \rfloor - 1$ and at most one number can be assigned to the vertex which is adjacent to the vertex labeled with $\lfloor \sqrt[3]{n} \rfloor$.
Therefore, we have

$$e_f(1) \leq 2(\lfloor \sqrt[3]{n} \rfloor - 1) + 1$$
$$\leq 2\lfloor \sqrt[3]{n} \rfloor - 1.$$
$$\text{Hence, } e_f(0) \geq n - (2\lfloor \sqrt[3]{n} \rfloor - 1)$$
$$= n + 1 - 2\lfloor \sqrt[3]{n} \rfloor$$
$$> \frac{n}{2}.$$

This is a contradiction. Thus if n is even, there can not be a cube divisor cordial labeling.
Subcase 2: $n \equiv 1(mod\ 2)$.
Then obviously either $e_f(0) = \frac{n-1}{2}$ and $e_f(1) = \frac{n+1}{2}$ or $e_f(0) = \frac{n+1}{2}$ and $e_f(1) = \frac{n-1}{2}$.

In order to get the labels 1, at most two numbers can be assigned to the vertices that are adjacent to the vertices labeled with $1, 2, \ldots, \lfloor\sqrt[3]{n}\rfloor - 1$ and at most one number can be assigned to the vertex which is adjacent to the vertex labeled with $\lfloor\sqrt[3]{n}\rfloor$.
Therefore, we have

$$\begin{aligned} e_f(1) &\leq 2(\lfloor\sqrt[3]{n}\rfloor - 1) + 1 \\ &\leq 2\lfloor\sqrt[3]{n}\rfloor - 1. \\ \text{Hence, } e_f(0) &\geq n - (2\lfloor\sqrt[3]{n}\rfloor - 1) \\ &= n + 1 - 2\lfloor\sqrt[3]{n}\rfloor \\ &> \frac{n+1}{2} \\ &> \frac{n-1}{2}. \end{aligned}$$

This is a contradiction. Thus, if n is odd, there can not be a cube divisor cordial labeling. ■

Illustration 2.8. Cube divisor cordial labeling of the cycle C_5 is shown in *Figure 3*.

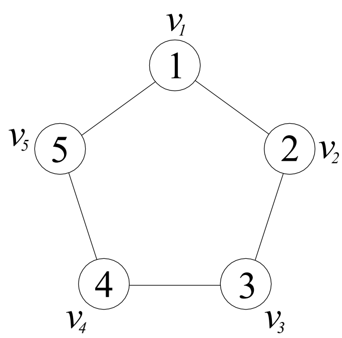

Figure 3: Cube divisor cordial labeling of the cycle C_5.

Theorem 2.9. The wheel W_n is a cube divisor cordial graph for all n.

Proof: Let v_0 be the apex(center) vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of the wheel W_n. We note that $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$.
To define vertex labeling $f : V(W_n) \to \{1, 2, \ldots, n + 1\}$, we consider following cases.
Case 1: $n \equiv 0(mod\ 2)$.
Define $f(v_0) = 1$,
$f(v_i) = i + 1; 1 \leq i \leq n$.
Therefore, $e_f(1) = e_f(0) = n$. Thus, $|e_f(0) - e_f(1)| \leq 1$.
Case 2: $n \equiv 1(mod\ 2)$.
Let p be the largest prime number such that $p \leq n + 1$.
Define $f(v_0) = 1$, $f(v_n) = p$.

Label the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ successively from the set $\{2, 3, \ldots, p-1, p+1, \ldots, n+1\}$.
Therefore, $e_f(0) = e_f(1) = n$. Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, the wheel W_n is a cube divisor cordial graph for all n. ∎

Illustration 2.10. Cube divisor cordial labeling of the wheel W_7 is shown in *Figure 4*.

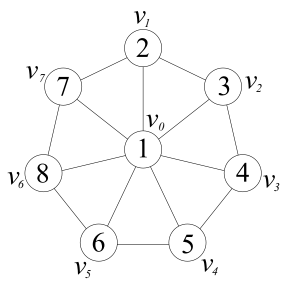

Figure 4: Cube divisor cordial labeling of the wheel W_7

Illustration 2.11. Cube divisor cordial labeling of the wheel W_8 is shown in *Figure 5*.

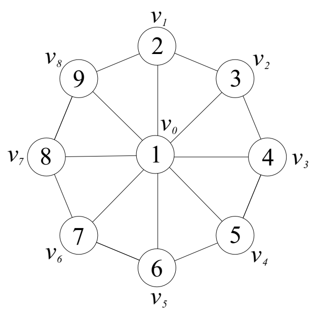

Figure 5: Cube divisor cordial labeling of the wheel W_8

Theorem 2.12. The Flower graph Fl_n is a cube divisor cordial graph for all n.

Proof: Let u_0 be the apex vertex, $u_1, u_2, \ldots, u_n$ be the vertices of degree 4 and $v_1, v_2, \ldots, v_n$ be the vertices of degree 2 of Fl_n. We note that $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$.

Define vertex labeling $f : V(Fl_n) \to \{1, 2, \ldots, 2n+1\}$ as follows.
$f(u_0) = 1,\ f(u_1) = 2$,
$f(v_1) = 3$,
$f(u_i) = 1 + 2i; 2 \leq i \leq n$,
$f(v_i) = 2i; 2 \leq i \leq n$.
In view of the above labeling pattern we have, $e_f(0) = e_f(1) = 2n$.
Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, the flower graph Fl_n is a cube divisor cordial graph for all n. ∎

Illustration 2.13. Cube divisor cordial labeling of the flower Fl_6 is shown in *Figure 6.*

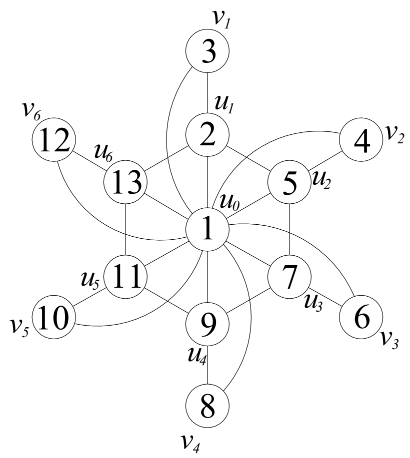

Figure 6: Cube divisor cordial labeling of the flower Fl_6.

Theorem 2.14. The fan graph f_n is a cube divisor cordial graph for all n.

Proof: Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the vertices of path P_n. We note that $|V(f_n)| = n+1$ and $|E(f_n)| = 2n-1$.

Define vertex labeling $f : V(f_n) \to \{1, 2, \ldots, n+1\}$ as follows.
$f(v_0) = 1$,
$f(v_i) = i+1; 1 \leq i \leq n$.
In view of the above labeling pattern we have $e_f(1) = \left\lceil \frac{2n-1}{2} \right\rceil$ and $e_f(0) = \left\lfloor \frac{2n-1}{2} \right\rfloor$.
Thus,$|e_f(0) - e_f(1)| \leq 1$.
Hence, the fan graph f_n is a cube divisor cordial graph for all n. ∎

Illustration 2.15. Cube divisor cordial labeling of the fan f_9 is shown in *Figure 7.*

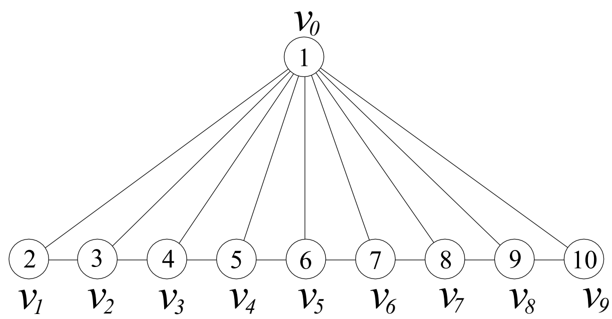

Figure 7: Cube divisor cordial labeling of the fan f_9.

Concluding Remark: Deriving new cube divisor cordial graph families is an open problem.

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