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On horizontal and complete lifts of $(1, 1)$ tensor fields F satisfying the structure equation $F(2K + S, S) = 0$

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Abstract

The horizontal and complete lifts from a manifold M^n to its cotangent bundles $T^*(M^n)$ were studied by several authors. The purpose of this paper is to deal with some problems on horizontal and complete lifts tensor fields satisfying structure equation $F^{2K+S} + F^S = 0$. We also discuss the integrability conditions and prolongations in the third tangent space $T_3(M^n)$.

Keywords: Horizontal and complete lifts, distribution, integrability conditions and prolongations.
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1 Introduction

Let M^n be a differentiable manifold of class C^∞ and F be a non-null tensor field of type $(1, 1)$ satisfying

$$F^{2K+S} + F^S = 0 \tag{1}$$

where K is a fixed integer greater than or equal to 1 and S is a fixed odd integer greater than or equal to 1. F is of constant rank r everywhere in M^n. We call such a structure an $F(2K + S, S)-$ structure of rank $2r$.

Let the operators l and m be defined as

$$l = -F^{2K} \tag{2}$$

$$m = I + F^{2K} \tag{3}$$

where I denotes the identity operator on M^n.
The operators l and m defined by (2) and (3) satisfy the following:

$$\left.\begin{array}{l} l + m = I \\ l^2 = l,\ m^2 = m,\ lm = ml = 0 \\ Fl = lF = F,\ Fm = mF = 0 \end{array}\right\} \tag{4}$$

Thus there exist in M^n two complementary distributions D_l and D_m corresponding to the projection tensors l and m, respectively. If the rank of F is r, then D_l is $r-$ dimensional and D_m is $(n-r)-$ dimensional, where $\dim M^n = n$.

2 The complete lift of F in the tangent bundle $T(M^n)$

Let M^n be a $n-$ dimensional differentiable manifold of class C^∞ and $T_P(M^n)$ the tangent space at a point p of M^n . Then

$$T(M^n) = \bigcup_{P \in M^n} T_P(M^n)$$

is the tangent bundle over the manifold M^n. The tangent bundle $T(M^n)$ of M^n is a differentiable manifold of dimension $2n$.

Let $\Im_s^r$ denote the set of tensor field of class C^∞ and type (r,s) in M^n and let $\Im_s^r(T(M^n))$ denote the corresponding set of tensor fields in $T(M^n)$.

The complete lift F^C of an element F of $\Im_1^1(M^n)$ with local components F_i^h has components of the form [11]

$$F^C = \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}. \tag{5}$$

Let $F, G, \ \epsilon \ \Im_1^1(M^n)$. Then we have [11]

$$(FG)^C = F^C G^C \tag{6}$$

Putting $F = G$ in (6), we obtain

$$(F^2)^C = (F^C)^2 \tag{7}$$

Putting $G = F^2$ in (6) and making use of (7), we get

$$(F^3)^C = (F^C)^3$$

Continuing the above process of replacing G in equation (6) by some higher powers of F, we obtain

$$\left.\begin{array}{c} (F^K)^C = (F^C)^K \\ (F^S)^C = (F^C)^k \\ (F^{S+2K})^C = (F^C)^{2K+S} \end{array}\right\} \tag{8}$$

where k is any positive integer. Also if G and H are tensors of the same type then

$$(G+H)^C = G^C + H^C$$

Taking complete lift on both sides of equation (1) we get

$$(F^{2K+S} + F^S)^C = 0$$

$$(F^{2K+S})^C + (F^S)^C = 0$$

Using (2.4) and $I^C = I$, we get

$$(F^C)^{2K+S} + (F^C)^S = 0 \quad (9)$$

We see that the equation (1) implies (9) and by page 34 in [11] the rank of F^C is $2r$ if and only if the rank of F is r. Thus we have the following theorem.

Theorem 2.1. Let F be an element of $\Im_1^1(M^n)$. Then F satisfies the structure equation (1) if F^C satisfies the structure equation (9) Furthermore, F is of rank r if and only if F^C is of rank $2r$.

Proof: Let F be an $F(2K+S,S)$-structure of rank r in M^n. Then the complete lift l^C of l and m^C of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distribution D_l^C and D_m^C determined by l^C and m^C respectively. The distributions D_l^C and D_m^C are respectively the complete lifts D_l^C and D_m^C of D_l and D_m. ■

3 Integrability conditions of $F(2K+S,S)-$ structures in a tangent bundle

Let $F \in \Im_1^1(M^n)$ and suppose that F satisfies (1). Then the Nijenhuis tensor N_F of F is a tensor field of type $(1,2)$ given by [11]

$$(i) \quad N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y] \quad (10)$$

Let N^C be the Nijenhuis tensor of F^C in $T(M^n)$, where F^C is the complete lift of F in M^n. Then we have

$$(ii)\ N^C(X^C,Y^C) = [F^CX^C,F^CY^C] - F^C[F^CX^C,Y^C] - F^C[X^C,F^CY^C] + (F^2)^C[X^C,Y^C]$$

For any $X,Y \in \Im_0^1(M^n)$ and $F \in \Im_1^1(M^n)$ we have [11]

$$(i) \quad [X^C,Y^C] = [X,Y]^C \quad \text{and} \quad (X+Y)^C = X^C + Y^C, \quad (11)$$

$$(ii) \quad F^CX^C = (FX)^C$$

From (4) and (11) we have

$$F^C l^C = (Fl)^C = F^C \tag{12}$$

and $\quad F^C m^C = (Fm)^C = 0.$

Theorem 3.1. The following identities hold:

$$N^C(m^C X^C, m^C Y^C) = (F^C)^2[m^C X^C, m^C Y^C], \tag{13}$$

$$m^C N^C(X^C, Y^C) = m^C[F^C X^C, F^C Y^C], \tag{14}$$

$$m^C N^C(l^C X^C, l^C Y^C) = m^C[F^C X^C, F^C Y^C] \tag{15}$$

$$m^C N^C((F^{2K+S} + F^S)^C X^C, (F^{2K+S} + F^S)^C Y^C)) = m^C N^C[l^C X^C, l^C Y^C]. \tag{16}$$

Proof: The proofs of (13) to (16) follow by virtue of (2),(3),(10) *and* (12) ∎

Theorem 3.2. For any $X, Y \in \Im_0^1(M^n)$ the following conditions are equivalent.

(i) $\quad m^C N^C(X^C, Y^C) = 0$

(ii) $\quad m^C N^C(l^C X^C, l^C Y^C) = 0$

and (iii) $\quad m^C N^C((F^{2K+S} + F^S)^C X^C, (F^{2K+S} + F^S)^C Y^C) = 0.$

Proof: In consequence of equation (16) we have,

$N^C(l^C X^C, l^C Y^C) = 0$ if and only if $N^C((F^{2K+S} + F^S)^C X^C, (F^{2K+S} + F^S)^C Y^C)) = 0$ for all $X, Y \in \Im_0^1(M^n)$.

Now the right hand sides of the quations (14) and (15) are equal and in view of the last equation shows that conditions $(i), (ii)$ and (iii) are equivalent. ∎

Theorem 3.3. The complete lift D_m^C in $T(M^n)$ of a distribution D_m in M^n is integrable if D_m is integrable in M^n.

Proof: The distribution D_m is integrable if and only if [11]

$$l(mX, mY) = 0 \tag{17}$$

for all $X, Y \in \Im_0^1(M^n),$ where $l = I - m$.

Taking complete lift of both sides and using (11) we get

$$l^C(m^C X^C, m^C Y^C) = 0$$

for all $X, Y \in \Im_0^1(M^n)$, where $l^C = (I - m)^C = I - m^C$ is the projection tensor complementary to m^C. Thus the condition (17) implies (3). ∎

Theorem 3.4. The complete lift D_m^C in $T(M^n)$ of a distribution D_m in M^n is integrable if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$ for all $X, Y \in \Im_0^1(M^n)$.

Proof: The distribution D_m *is integrable in* M^n if and only if $N(mX, mY) = 0$ for any $X, Y \in \Im_0^1(M^n)$ [11].
By virtue of condition (13), we have

$$N^C(m^C X^C, m^C Y^C) = (F^C)^2[m^C X^C, m^C Y^C],$$

Multiplying throughout by l^C, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^C)^2 l^C[m^C X^C, m^C Y^C],$$

In view of (3) the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0 \tag{18}$$

Also we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{19}$$

Adding (18) and (19), we get $(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0$.
Since $l^C + m^C = I^C = I$, we have $N^C(m^C X^C, m^C Y^C) = 0$. ■

Theorem 3.5. Let the distribution D_l be integrable in M^n, that is $mN(X,Y) = 0$ for all $X, Y \in \Im_0^1(M^n)$. Then the distribution D_l^C is integrable in $T(M^n)$ if and only if any one of the conditions of Theorem (3.2) is satisfied, for all $X, Y \in \Im_0^1(M^n)$.

Proof: The distribution D_l is integrable in M^n if and only if $mN(lX, lY) = 0$.
Thus distribution D_l^C is integrable in $T(M^n)$ if and only if $m^C N^C(l^C X^C, l^C Y^C) = 0$. Hence the theorem follows by making use of the equation (16). ■

Theorem 3.6. The complete lift F^C of an $(F^{2K+S} + F^S)-$ structure F in M^n is partially integrable in $T(M^n)$ if and only if F is partially integrable in M^n.

Proof: The $(F^{2K+S} + F^S)-$ structure F in M^n is partially integrable if and only if

$$N(lX, lY) = 0 \text{ for any } X, Y \in \Im_0^1(M^n). \tag{20}$$

In view of the equations (2),(3),(10) we obtain $N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$ which implies $N^C(l^C X^C, l^C Y^C) = 0$ if and only if $N(lX, lY) = 0$.

Also from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to $N^C((F^{2K+S}+F^S)^C X^C, (F^{2K+S}+F^S)^C Y^C)) = 0$. ∎

Theorem 3.7. The complete lift F^C of an $(F^{2K+S}+F^S)-$ structure F is partially integrable in $T(M^n)$ if and only if F is partially integrable in M^n.

Proof: A necessary and sufficient condition for an $(F^{2K+S}+F^S)-$ structure in M^n to be integrable is that

$$N(X,Y) = 0 \tag{21}$$

for any $X, Y \in \Im_0^1(M^n)$.

In view of equation (3.1), we get $N^C(X^C, Y^C) = (N(X,Y))^C$. Therefore with the help of (21) we obtain the result. ∎

4 The horizontal lift of an $(F^{2K+S}+F^S)-$ structure

The horizontal lift S^H of a tensor field S of arbitrary type in $T(M^n)$ is defined by

$$S^H = S^C - \nabla_\gamma S \tag{22}$$

where S is a tensor field defined by

$$S = S^{i........h}_{k.......j}(\frac{\partial}{\partial x^i} \otimes ...... \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes .... \otimes dx^j,$$

in M^n with affine connection ∇ and $\nabla_\gamma S$ is a tensor field in $T(M^n)$ give

$$\nabla_\gamma S = (y^l \nabla_l S^{i........h}_{k.......j})(\frac{\partial}{\partial y^i} \otimes ...... \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes .... \otimes dx^j$$

with respect to the induced coordinates (x^h, y^k) in $\Pi^{-1}(U)$ [12].

The horizontal lift F^H of a tensor field F of type $(1,1)$ in M^n with components F_i^H in M^n has components

$$F^H = \begin{pmatrix} F_i^h & 0 \\ 0 & F_i^h \end{pmatrix}. \tag{23}$$

Now, we prove some theorems on horizontal lift satisfying the structure (1).

Theorem 4.1. *Let $F \in \Im_1^1(M^n)$ be an $F(2K+S,S)-$ structure in M^n. Then its horizontal lift F^H is also an $F(2K+S,S)-$ structure in $T(M^n)$.*

Proof: If $P(t)$ is a polynomial in one variable t, then we have [11]

$$(P(F))^H = P(F^H) \tag{24}$$

for any $F \in \Im_1^1(M^n)$.

Also let I be the identity tensors field of type $(1,1)$ in M^n. Then

$$I^H = I \tag{25}$$

where the right-hand side denote the field of the identity tenors in $T(M^n)$.

Taking horizontal lift on both sides of equation (1) and using (24), (25), we obtain

$$(F^H)^{2K+S} + (F^H)^S = 0$$

which shows that F^H is an $F(2K+S,S)-$ structure in $T(M^n)$. ■

Now from the local expression of F^H, we see that, if F is of rank r, then F^H is of rank $2r$. Thus we have the following result.

Theorem 4.2. *If* F *is an* $F(2K+S,S)-$ *structure of rank* r *in* M^n, *then its horizontal lift* F^H *is also* $F(2K+S,S)-$ *structure of rank* $2r$ *in* $T(M^n)$.

Proof: Let m be a projection tensor field of type $(1,1)$ in M^n defined by (2) and (3).

Then there exists in M^n a distribution D determined by m. Also

$$m^2 = m.$$

With the help of the equation (24), we find

$$(m^H)^2 = m^H.$$

Thus, m^H is also a projection tensor in $T(M^n)$. Hence there exists in M^n a distribution D^H corresponding to m^H, which is called the horizontal lift of the distribution D. ■

Theorems (4.1) and (4.2) also hold in the cotangent bundle $T^*(M^n)$.

In $T^*(M^n)$ the connection ∇ is a symmetric affine connection.

5 Prolongation of an $F(2K+S,S)-$ structure in third tangent space T_3M^n

Let us denote by $T_3(M^n)$, the third order tangent bundle over M^n and let F^{III} be the third lift on F in $T_3(M^n)$. Then for any $F, G \in \Im_1^1(M^n)$, we have

$$\left.\begin{aligned}(G^{III}F^{III})X^{III} &= G^{III}(F^{III}X^{III})\\ &= G^{III}(FX)^{III}\\ &= (G(FX))^{III}\\ &= (GF)^{III}X^{III}\end{aligned}\right\} \tag{26}$$

for any $F \in \Im_0^1(M^n)$, Thus we have

$$G^{III}F^{III} = (GF)^{III}$$

If $P(t)$ is a polynomial of in t, then we have

$$(P(F))^{III} = P(F)^{III} \tag{27}$$

where $F \in \Im_1^1(M^n)$.

Theorem 5.1. *The third lift F^{III} defines an $F(2K+S,S)-$ structure in $T_3(M^n)$ if and only if* F defines an $F(2K+S,S)-$ *structure* in M^n.

Proof: Let F satisfy $(1,1)$. Then F defines an $F(2K+S,S)-$ structure in M^n satisfying

$$F^{2K+S} + F^S = 0$$

Taking third lift of $F(2K+S,S)-$ structure

$$(F^{2K+S} + F^S)^{III} = 0$$

which in view of equation (27), shows that

$$((F)^{III})^{2K+S} + ((F)^{III})^S = 0 \tag{28}$$

Therefore F^{III} defines an $F(2K+S,S)-$ structure in $T_3(M^n)$. ∎

Theorem 5.2. *The third lift F^{III} is integrable in $T_3(M^n)$ if and only if F is integrable in M^n.*

Proof: Let us denote by N^{III} and N, the nijenhuis tensor of F^{III} and F respectively. Then we have [11]

$$N^{III}(X,Y) = (N(X,Y))^{III} \tag{29}$$

We know that an $F(2K+S,S)-$ structure is integrable in M^n if and only if

$$N(X,Y)=0$$

and so, from (29), we obtain

$$N^{III}(X,Y)=0 \tag{30}$$

Thus F^{III} is integrable iff F is integrable in M^n. ■

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