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On equi independent equitable dominating sets in graphs

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Abstract

The concept of equi independent equitable domination is the combination of two important concepts, namely independent domination and equitable domination. A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. A vertex subset D is said to be equitable independent set if any two vertices of D are either non adjacent or if adjacent then their degrees differ by atleast 2. An equitable dominating set D is said to be an equi independent equitable dominating set if it is also equitable independent set. The equi independent equitable domination number i^e is the minimum cardinality of an equi independent equitable dominating set.

Keywords:equi independent equitable domination number, equitable domination number, domination number.
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1 Introduction

The concept of equitable domination was introduced by Sampathkumar while the concept of independent domination was formalized by Berge [1], Ore [8], Cockayne and Hedetniemi [3, 4]. A brief survey on independent domination is carried out in recent past by Goddard and Henning [5]. Motivated by the concepts of independent domination and equitable domination a new concept of equitable independent equitable domination was conceived by Swaminathan and Dharamlingam [9] and formalized by Vaidya and Kothari [10–12]. We investigate some new results and obtain equi independent equitable domination number of some path related graphs.

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For standard graph theoretic terminology we follow Harary [6] while the terms related to theory of domination are used here in the sense of Haynes *et al.* [7]. A set $D \subseteq V(G)$ is called a *dominating set* if every vertex in $V(G) - D$ is adjacent to at least one vertex in D.

The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset D of $V(G)$ is an *independent set* if no two vertices in D are adjacent. A dominating set D which is also an independent set is called an *independent dominating set.* The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set. Here we summarize the basic definition and existing results.

Definition 1.1. A subset D of $V(G)$ is called an *equitable dominating set* if for every $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set is *equitable domination number* of G which is denoted by γ^e.

Definition 1.2. A vertex $v \in D$ is an *isolate* of D if $N(v) \subseteq V(G) - D$.

Definition 1.3. A vertex $u \in V(G)$ is *degree equitable adjacent* or *equitable adjacent* with a vertex $v \in V(G)$ if $|deg(u) - deg(v)| \leq 1$ and $uv \in E(G)$.

Definition 1.4. A vertex $v \in V(G)$ is called *equitable isolate* if $|d(v) - d(u)| \geq 2$ for every $u \in N(v)$.

It is obvious that, if $v \in V(G)$ is an equitable isolate and D is any equitable dominating set then $v \in D$. The isolated vertices are obviously equitable isolates. Hence $I_s \subseteq I_e \subseteq D$ for every equitable dominating set D, where I_s and I_e are the set of all isolated vertices and the set of all equitable isolates of G respectively.

Definition 1.5. The *equitable neighbourhood* of v denoted by $N^e(v)$ is defined as $N^e(v) = \{u \in V(G)/u \in N(v), |deg(v) - deg(u)| \leq 1\}$.

$\Delta^e(G) = \max\limits_{v \in V(G)} |N^e(v)|$, and $\delta^e(G) = \min\limits_{v \in V(G)} |N^e(v)|$ are known as maximum and minimum equitable degree of graph G respectively.

Remark 1.6. $\delta^e(G) \leq \delta(G)$ and $\Delta^e(G) \leq \Delta(G)$.

Remark 1.7. If G is any k-regular graph or $(k, k+1)$ bi-regular graph then $\delta^e(G) = \delta(G)$ and $\Delta^e(G) = \Delta(G)$.

Remark 1.8. $\Delta^e(G) = \delta^e(G) = 0$ for $K_{1,n}$ with $n \geq 3$.

The concept of equitable independent set was introduced by Swaminathan and Dharamlingam [9].

Definition 1.9. A subset D of $V(G)$ is called an *equitable independent set* if for any $u \in D$, $v \notin N^e(u)$ for all $v \in D - \{u\}$. The maximum cardinality of an equitable independent set is denoted by $\beta^e(G)$.

Remark 1.10. *Every independent set is an equitable independent set.*

Definition 1.11. An equitable dominating set D is said to be *equi independent equitable dominating set* if it is an equitable independent set. The minimum cardinality of an equi independent equitable dominating set is called *equi independent equitable domination number* which is denoted by i^e.

Illustration 1.12. In Figure 1, $D = \{v, v_1, u_2, v_4, u_5, u_6\}$ is an equitable independent set as well as equitable dominating set for gear graph G with $i^e(G) = 6$.

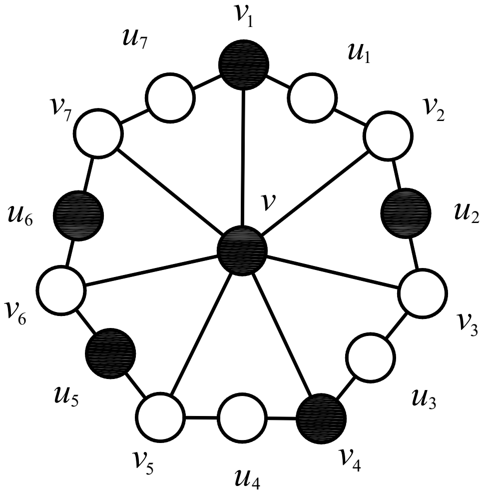

Figure 1

2 Main Results

Definition 2.1. The *corona* $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined to be the graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Theorem 2.2. Let G be a graph with $\delta(G) \geq 3$. Then $\gamma^e(G \odot K_1) = \gamma^e(G) + |V(G)|$.

Proof: Let G be a graph with $\delta(G) \geq 3$. Then every pendant vertex of $G \odot K_1$ is equitable isolate. Therefore they belong to every equitable dominating set of $G \odot K_1$. Observe that removal of all pendant vertices from $G \odot K_1$ leaves G. This implies that

$$\gamma^e(G \odot K_1) \geq \gamma^e(G) + |V(G)|$$

Now if $D = S \cup \{x/x \text{ is pendant vertex of } G \odot K_1\}$ where S be the γ^e-set of G. Then D is an equitable dominating set with $|D| = \gamma^e(G) + |V(G)|$ and consequently $\gamma^e(G \odot K_1) = \gamma^e(G) + |V(G)|$. ∎

Definition 2.3. The *cartesian product* $G \times H$ of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, is the graph with vertex set $V(G \times H) = \{(u, v)/u \in V(G) \, \text{and} \, v \in V(H)\}$ and edge set $E(G \times H) = \{(u, v)(u', v')/u = u', vv' \in E(H) \, \text{or} \, uu' \in E(G), v = v'\}$.

Conjecture 2.4. (Vizing) $\gamma(G \times H) \geq \gamma(G)\gamma(H)$.

For the graphs considered in Illustration 2.5, Bresar *et al.* [2] have stated that the analogous of Vizing's conjecture 2.4 is not true for independent domination number and posed a new conjecture 2.6. In the context of the same graphs we found that the analogous of Vizing's conjecture is also not true for equi independent equitable domination number as shown in Illustration 2.5.

Illustration 2.5. Let G be the graph of order 11 constructed from K_3 by adding 2 leaves adjacent to vertex v_1 of K_3, 3 leaves adjacent to vertex v_2 of K_3 and 3 leaves adjacent to the vertex v_3 of K_3. Then $i^e(G) = 9$. Let $H = \bar{G}$; then $i^e(H) = 4$. This implies that $i^e(G)i^e(H) = 36$. However $i^e(G \times H) = 34$.

Conjecture 2.6. For all graphs G and H, $\gamma(G \times H) \geq min\{i(G)\gamma(H), i(H)\gamma(G)\}$.

We pose the analogous of Conjecture 2.6 for equitable domination and equi independent equitable domination numbers as follows.

Conjecture 2.7. For all graph G and H, $\gamma^e(G \times H) \geq min\{i^e(G)\gamma^e(H), i^e(H)\gamma^e(G)\}$.

Theorem 2.8. $i^e\left(P_n \times P_2\right) = \gamma^e\left(P_n \times P_2\right) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for odd} \, n \\ \frac{n}{2} + 1 & \text{for even} \, n \end{cases}$

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and u_1, u_2 be the vertices of P_2. Then $V(P_n \times P_2) = \{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n), (u_2, v_1), (u_2, v_2), \ldots, (u_2, v_n)\}$. We denote the vertices (u_1, v_i) by x_i and (u_2, v_i) by y_i. The vertices $\{y_i, y_{i+1}, x_i, x_{i+1}\}$ form a cycle C_4 sharing a common edge $y_{i+1}x_{i+1}$ with a cycle C_4 with vertices $\{y_{i+1}, y_{i+2}, x_{i+1}, x_{i+2}\}$. Then atleast one of the vertices from $y_i, y_{i+1}, x_i,$ x_{i+1} must belong to every equitable dominating set, which implies that $\gamma^e(P_n \times P_2) \geq \frac{n}{2}$.

As $N^e[x_{4i+1}] = \{x_{4i}, x_{4i+1}, x_{4i+2}, y_{4i+1}\}$, $N^e[y_{4i+3}] = \{y_{4i+2}, y_{4i+3}, y_{4i}, x_{4i+3}\}$,
$N^e[x_n] = \{x_{n-1}, y_n\}$, $N^e[y_n] = \{y_{n-1}, x_n\}$.
We consider the following subsets based upon the number of vertices of path P_n.
For $n \equiv 0 \, (\text{mod} \, 4)$, $D = \left\{x_{4i+1}, y_{4j+3}, x_n / \, 0 \leq i < \left\lfloor \frac{n}{4} \right\rfloor, 0 \leq j < \left\lfloor \frac{n}{4} \right\rfloor\right\}$, $|D| = \frac{n}{2} + 1$,
for $n \equiv 1, 3 \, (\text{mod} \, 4)$, $D = \left\{x_{4i+1}, y_{4j+3} / \, 0 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor, 0 \leq j \leq \left\lfloor \frac{n}{5} \right\rfloor\right\}$, $|D| = \lceil \frac{n}{2} \rceil$,
for $n \equiv 2 \, (\text{mod} \, 4)$, $D = \left\{x_{4i+1}, y_{4j+3}, y_n / \, 0 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor, 0 \leq j < \left\lfloor \frac{n}{4} \right\rfloor\right\}$, $|D| = \frac{n}{2} + 1$.
Here $N^e[D] = V(P_n \times P_2)$. Therefore D is an equitable dominating set of $P_n \times P_2$. Also D is an independent set as vertices $x_{4i+1}, y_{4j+3}, x_n, y_n$ of D are not adjacent to each other. This implies

that D is an equitable independent set. Hence D is an equi independent equitable dominating set of $P_n \times P_2$ and $i^e\left(P_n \times P_2\right) = \gamma^e\left(P_n \times P_2\right) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for odd}\, n \\ \frac{n}{2}+1 & \text{for even}\, n \end{cases}$ ■

Remark 2.9. The above result presents a graph family which satisfies Conjecture 2.7, since

$$\gamma^e(P_n) = i^e(P_n) = \left\lceil \frac{n}{3} \right\rceil \Rightarrow i^e(P_n)\gamma^e(P_2) = i^e(P_2)\gamma^e(P_n) = \left\lceil \frac{n}{3} \right\rceil$$
$$\Rightarrow \gamma^e(P_n \times P_2) > i^e(P_n)\gamma^e(P_2) \text{ and } \gamma^e(P_n \times P_2) > i^e(P_2)\gamma^e(P_n).$$

Definition 2.10. The closed helm CH_n is the graph obtained from helm H_n by joining each pendant vertex to form a cycle.

Remark 2.11. Let v be the apex vertex of CH_n then induced subgraph of $V(CH_n) - \{v\}$ is $C_n \times P_2$.

Theorem 2.12. $i^e\left(C_n \times P_2\right) = \begin{cases} 2 & \text{for}\, n = 3, 4 \\ i^e(CH_n) - 1 & \text{for } n \geq 5 \end{cases}$

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n and u_1, u_2 be the vertices of P_2. Then $V(C_n \times C_2) = \{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n), (u_2, v_1), (u_2, v_2), \ldots, (u_2, v_n)\}$. We denote vertices (u_1, v_i) by x_i and vertices (u_2, v_i) by y_i.

Case 1: $n = 3, 4$

Let $D = \{x_1, y_3\}$. Note that $N^e[D] = V(C_n \times P_2)$ which implies that D is an equitable dominating set of $C_n \times P_2$. Also x_1 is non adjacent to y_3 which implies that D is an equitable independent set of $C_n \times P_2$. Hence D is an equi independent equitable dominating set of $C_n \times P_2$ and $i^e(C_n \times P_2) = 2$.

Case 2: $n \geq 5$

In this case apex vertex v is an equitable isolate of CH_n. By removing the apex vertex v from CH_n it reduce to $C_n \times P_2$. This implies that $i^e(C_n \times P_2) = i^e(CH_n) - 1$. ■

Remark 2.13. The above result presents a graph family which satisfies Conjecture 2.7, since

$$\gamma^e(C_n) = i^e(C_n) = \left\lceil \frac{n}{3} \right\rceil \text{ and } \gamma^e(P_2) = i^e(P_2) = 1$$
$$\Rightarrow i^e(C_n)\gamma^e(P_2) = i^e(P_2)\gamma^e(C_n) = \left\lceil \frac{n}{3} \right\rceil$$
$$\Rightarrow \gamma^e(C_n \times P_2) > i^e(C_n)\gamma^e(P_2) \text{ and } \gamma^e(C_n \times P_2) > i^e(P_2)\gamma^e(C_n).$$

Definition 2.14. The *Möbius* ladder M_n is a graph obtained from the ladder $P_n \times P_2$ by joining the opposite end vertices of two copies of P_n.

Theorem 2.15. $\gamma\left(M_n\right)=\gamma^e\left(M_n\right)=i^e\left(M_n\right)=\begin{cases}\frac{n}{2}+1 & n\equiv 0,2\ (\mathrm{mod}\,4)\\ \lfloor\frac{n}{2}\rfloor+1 & n\equiv 1\ (\mathrm{mod}\,4)\\ \lceil\frac{n}{2}\rceil+1 & n\equiv 3\ (\mathrm{mod}\,4)\end{cases}$

Proof: Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of *Möbius* ladder M_n. Then $E(M_n)=\{v_1u_n, v_nu_1, v_iu_i/2\leq i\leq n-1\}$. Observe that any pair of adjacent vertices of M_n is equitable adjacent. This implies that $\gamma^e(M_n)=\gamma(M_n)$.

As $N^e[v_{4i+1}]=\{v_{4i}, v_{4i+1}, v_{4i+2}, u_{4i+1}\}$, $N^e[u_{4j+3}]=\{u_{4i+2}, u_{4i+3}, u_{4i}, v_{4i+3}\}$.
We consider the following subsets depending upon the number of vertices of path P_n.
For $n\equiv 0\,(\mathrm{mod}\,4)$, $D=\{v_{4i+1}, v_n, u_{4i+3}/0\leq i<\frac{n}{4}\}$, with $|D|=\frac{n}{2}+1$,
for $n\equiv 1\,(\mathrm{mod}\,4)$, $D=\{v_{4i+1}, u_{4i+3}/0\leq i<\lfloor\frac{n}{4}\rfloor\}$, with $|D|=\lfloor\frac{n}{2}\rfloor+1$,
for $n\equiv 2\,(\mathrm{mod}\,4)$, $D=\{v_{4i+1}, u_1, u_{4j+3}/0\leq i\leq\lfloor\frac{n}{4}\rfloor, 0\leq j<\lfloor\frac{n}{4}\rfloor\}$, with $|D|=\frac{n}{2}+1$,
for $n\equiv 3\,(\mathrm{mod}\,4)$, $D=\{v_{4i+1}, v_n, u_{4j+3}, u_{n-1}/0\leq i\leq\lfloor\frac{n}{4}\rfloor, 0\leq j<\lfloor\frac{n}{4}\rfloor\}$, with $|D|=\lceil\frac{n}{2}\rceil+1$.
Here for any choice of n, $N^e[D]=N[D]=V(M_n)$. Therefore D is an γ^e-set as well as $\gamma-$set of M_n. Also for any choice of n, every vertex of D is non adjacent to any other vertex of D, which implies that D is an independent set, consequently D is an equitable independent set of M_n. Hence D is an equi independent equitable dominating set of M_n and

$$\gamma\left(M_n\right)=\gamma^e\left(M_n\right)=i^e\left(M_n\right)=\begin{cases}\frac{n}{2}+1 & n\equiv 0,2\ (\mathrm{mod}\,4)\\ \lfloor\frac{n}{2}\rfloor+1 & n\equiv 1\ (\mathrm{mod}\,4)\\ \lceil\frac{n}{2}\rceil+1 & n\equiv 3\ (\mathrm{mod}\,4)\end{cases}$$ ■

Definition 2.16. For a graph G the splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v)=N(v')$.

Theorem 2.17. $i^e(S'(P_n))=\gamma^e(S'(P_n))=\gamma(P_{n-2})+n$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and $u_1, u_2, \ldots, u_n$ be the corresponding vertices which are added to obtain $S'(P_n)$. Then $V(S'(P_n))=\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Observe that $u_3, u_4, \ldots, u_{n-3}, u_{n-2}, u_1, u_n$ are equitable isolates, which implies that they must belong to any equitable dominating set of $S'(P_n)$. Also $\{v_1, u_2\}$ and $\{v_n, u_{n-1}\}$ are pair of equitable adjacent vertices. Therefore atleast one vertex from each pair must belong to any equitable dominating set of $S'(P_n)$. Note that subgraph induced by $V(S'(P_n))-\{u_1, u_3, u_4, \ldots, u_{n-3}, u_{n-2}, u_n, v_1, v_n\}$ is a path with $n-2$ vertices, which implies that $\gamma^e(S'(P_n))\geq\gamma(P_{n-2})+n$.

Let S be the $\gamma-$set of $V(P_n)-\{v_1, v_n\}$ and $D=S\cup\{u_1, u_2, \ldots, u_n\}$. Therefore $N^e[D]=V(S'(P_n))$. While D is an equitable independent set of $S'(P_n)$ as vertices $u_1, u_3, u_4, \ldots, u_{n-3}, u_{n-2}, u_n$ are equitable isolates and no two vertices of S are adjacent to each other. Hence, D is an equi independent equitable dominating set of $S'(P_n)$ and $i^e(S'(P_n))=\gamma^e(S'(P_n))=\gamma(P_{n-2})+n$. ■

Definition 2.18. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G)\cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

Theorem 2.19. *$i^e(M(P_n)) = \gamma^e(M(P_n)) = \gamma(P_{n-5}) + n$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of P_n. Note that $V(M(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\}$. Observe that vertices $v_3, v_4, \ldots, v_{n-2}, v_1, v_n$ are equitable isolates of $M(P_n)$. Therefore they must belong to any equitable dominating set of $M(P_n)$. While vertex v_2 is equitably adjacent to only e_1 and vertex v_{n-1} is equitably adjacent to only e_{n-1}. But vertex e_1 can equitably dominate v_2, e_2 and vertex e_{n-1} can equitably dominate v_n, e_{n-2}. Therefore e_1 and e_{n-1} must belong to every equitable dominating set. On other hand subgraph induced by $V(M(P_n)) - \{v_1, v_2, \ldots, v_n, e_1, e_2, e_{n-2}, e_{n-1}\}$ is a path with $n-5$ vertices, which implies that $\gamma^e(M(P_n)) \geq \gamma(P_{n-5}) + n$.

Let S be the $\gamma-$set of $V(M(P_n)) - \{v_1, v_3, v_4, \ldots, v_{n-2}, v_n, e_1, e_2, e_{n-2}, e_{n-1}\}$ and $D = S \cup \{v_1, v_3, v_4, \ldots, v_{n-1}, v_{n-2}, v_n, e_1, e_{n-1}\}$. Then D is equitable dominating set of $M(P_n)$ with $|D| = \gamma(P_{n-5}) + n$. Also D is an equitable independent set of $M(P_n)$ as vertices $v_1, v_3, v_4, \ldots, v_{n-1}, v_{n-2}$ are equitable isolates and the remaining vertices of D are not equitably adjacent to each other. Hence, D is an equi independent equitable dominating set of $M(P_n)$ and $i^e(M(P_n)) = \gamma^e(M(P_n)) = \gamma(P_{n-5}) + n$. ■

Definition 2.20. The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Theorem 2.21. *$i^e(T(P_n)) = \gamma^e(T(P_n)) = \gamma(T(P_n))$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of P_n. Then $V(T(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\}$. Observe that any pair of adjacent vertices of $T(P_n)$ is also equitably adjacent. Therefore any $\gamma-$set D of $T(P_n)$ is an equitable dominating set of $T(P_n)$. Also any two vertices of D are not adjacent to each other. Therefore D is an equitable independent set of $T(P_n)$. Hence $i^e(T(P_n)) = \gamma^e(T(P_n)) = \gamma(T(P_n))$. ■

Definition 2.22. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.

Theorem 2.23. *$\gamma^e(D_2(P_n)) = \gamma(D_2(P_{n-2})) + 4$.*

Proof: Consider two copies of P_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of P_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of P_n. Observe that v_1, v_n, u_1, u_n are equitable isolates of $D_2(P_n)$. Therefore they must belong to any equitable dominating set of $D_2(P_n)$. Observe that the remaining vertices are equitably adjacent to each other which are dominated by $n-2$ vertices of P_{n-2}. This implies that $\gamma^e(D_2(P_n)) \geq \gamma(D_2(P_{n-2})) + 4$.

As $N^e[v_{4i-1}] = \{v_{4i-2}, v_{4i-1}, v_{4i}, u_{4i-2}, u_{4i}\}$. We consider the following subsets depending upon the number of vertices of path P_n.
For $n \equiv 0 \,(\mathrm{mod}\, 4)$, $D = \{v_1, v_n, u_1, u_n, v_{4i-1}, v_{4i}, v_{n-1}, v_{n-2}\}$ for $1 \leq i < \frac{n}{4}$,
for $n \equiv 1, 2 \,(\mathrm{mod}\, 4)$, $D = \{v_1, v_n, u_1, u_n, v_{4i-1}, v_{4i}\}$ for $1 \leq i \leq \lfloor \frac{n}{4} \rfloor$,
for $n \equiv 3 \,(\mathrm{mod}\, 4)$, $D = \{v_1, v_n, u_1, u_n, v_{4i-1}, v_{4i}, v_{n-2}\}$ for $1 \leq i \leq \lfloor \frac{n}{4} \rfloor$.
Note that $N^e[D] = V(D_2(P_n))$. Therefore D is an equitable dominating set of $D_2(P_n)$ with $|D| = \gamma(D_2(P_n)) + 4$. Hence, $\gamma^e(D_2(P_n)) = \gamma(D_2(P_{n-2})) + 4$.

■

Theorem 2.24. $i^e(D_2(P_n)) = \begin{cases} \frac{2n}{3} + 4 & n \equiv 0 \; (\mathrm{mod}\, 3) \\ \left\lfloor \frac{2n}{3} \right\rfloor + 4 & n \equiv 1, 2 \; (\mathrm{mod}\, 3) \end{cases}$

Proof: Consider two copies of P_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of P_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of P_n. Observe that v_1, v_n, u_1, u_n are equitable isolate vertices of $D_2(P_n)$. Therefore they belong to every equitable dominating set of $D_2(P_n)$. Let S be the γ^e-set of $D_2(P_n)$. Observe that all the vertices of S are equitably adjacent to each other. Therefore D is not an equitable independent set of $D_2(P_n)$, which implies that $i^e(D_2(P_n)) > \gamma^e(D_2(P_n))$.

As $N^e[v_{3i}] = \{v_{3i-1}, v_{3i}, v_{3i+1}, u_{3i-1}, u_{3i+1}\}$ and $N^e[u_{3i}] = \{u_{3i-1}, u_{3i}, u_{3i+1}, v_{3i-1}, v_{3i+1}\}$. We consider the following subsets depending upon the number of vertices of path P_n.
For $n \equiv 0 \,(\mathrm{mod}\, 3)$, $D = \{v_1, v_{n-1}, v_n, u_1, u_{n-1}, u_n, v_{3i}, u_{3i}\}$ where $0 \leq i < \frac{n}{3} - 1$,
for $n \equiv 1 \,(\mathrm{mod}\, 3)$, $D = \{v_1, v_{n-1}, v_n, u_1, u_{n-1}, u_n, v_{3i}, u_{3i}\}$ where $0 \leq i < \lfloor \frac{n}{3} \rfloor - 1$,
for $n \equiv 2 \,(\mathrm{mod}\, 3)$, $D = \{v_1, v_n, u_1, u_n, v_{3i}, u_{3i}\}$, where $0 \leq i < \lfloor \frac{n}{3} \rfloor$.
Here $N^e[D] = V(D_2(P_n))$. Therefore D is an equitable dominating set of $D_2(P_n)$. Also vertices of D are not adjacent to each other. This implies that, D is an independent set. Consequently D is an equitable independent set of $D_2(P_n)$. Hence D is an equi independent equitable dominating set of $D_2(P_n)$ and $i^e(D_2(P_n)) = \begin{cases} \frac{2n}{3} + 4 & n \equiv 0 \; (\mathrm{mod}\, 3), \\ \left\lfloor \frac{2n}{3} \right\rfloor + 4 & n \equiv 1, 2 \; (\mathrm{mod}\, 3). \end{cases}$ ■

Definition 2.25. Duplication of a vertex v_i by a new edge $e = v_i' v_i''$ in graph G produces a new graph G' such that $N(v_i') \cap N(v_i'') = \{v_i\}$.

Theorem 2.26. Let G be a graph obtained by duplication of each vertex of P_n by an edge then $i^e(G) = \gamma^e(G) = \gamma(P_{n-2}) + n$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and G be a graph obtained by duplication of each vertex v_i of P_n by an edge $u_i u_{i+1}$. Observe that the vertices u_{2i-1} are equitable adjacent to only u_{2i} and vice versa for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. Therefore at least one vertex from every pair $\{u_{2i-1}, u_{2i}\}$ belong to every equitable dominating set of G. At least one vertex form each of u_1, v_1, v_2 and u_{2n-1}, u_{2n}, v_n belong to every equitable dominating set of G as they are equitably adjacent to each other. On the other hand the subgraph induced by $V(G) - \{u_1, u_2, \ldots, u_{2n}, v_1, v_2\}$ is a path with $n-2$ vertices, which implies that $\gamma^e(G) \geq \gamma(P_{n-2}) + n$.

Let S be the $\gamma-$set of $V(P_n) - \{v_1, v_n\}$ and $D = S \cup \{u_1, u_3, u_5, \ldots, u_{2n-1}\}$. Therefore D is an equitable dominating set of G with $|D| = \gamma(P_{n-2}) + n$. Also the vertices u_{2i-1} are not adjacent to each other and they are not equitably adjacent to any vertex of set S, which implies that D is an equitable independent set of G. Hence, D is an equi independent equitable dominating set of G and $i^e(G) = \gamma^e(G) = \gamma(P_{n-2}) + n$. ■

Definition 2.27. Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

Theorem 2.28. Let G be a graph obtained by duplication of each edge of P_n by a vertex then $i^e(G) = \gamma^e(G) = \gamma(P_{n-2}) + n - 1$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and G be a graph obtained by duplication of each edge v_iv_{i+1} of P_n by a vertex u_i. Observe that $u_2, u_3, \ldots, u_{n-2}$ are equitable isolates of G. Therefore they belong to every equitable dominating set of G. At least one vertex from each of $\{u_1, v_1\}$ and $\{u_{n-1}, v_n\}$ must belong to any equitable dominating set of G as they are equitably adjacent vertices. Also the subgraph induced by $V(G) - \{u_1, u_2, \ldots, u_{n-1}, v_1, v_n\}$ is a path with $n-2$ vertices and this implies that $\gamma^e(G) \geq \gamma(P_{n-2}) + n - 1$.

Let S be the $\gamma-$set of P_{n-2} and $D = S \cup \{u_1, u_2, \ldots, u_{n-1}\}$. From the above argument D is an equitable dominating set G with $|D| = \gamma(P_{n-2}) + n - 1$. Also D is an equitable independent set of G as vertices u_i's are not adjacent to each other as well as not equitably adjacent to any vertex of S. Hence, D is an equi independent equitable dominating set of G and $i^e(G) = \gamma^e(G) = \gamma(P_{n-2}) + n - 1$. ■

3 Concluding Remarks

The equi independent equitable domination is a combination of two concepts namely equitable independent and equitable domination. We prove some new results in the context of the above concept. A Vizing type Conjecture is posed and two families of graphs satisfying the conjecture are also investigated.

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