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Edge antimagic total labeling of isomorphic copies of subdivided stars

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Abstract

Enomoto, Llado, Nakamigawa and Ringel defined the concept of a super $(a, 0)$-edge-antimagic total labeling and proposed the conjecture that every tree is a super $(a, 0)$-edge-antimagic total labeling. In the support of this conjecture, the present paper deals with different results on antimagicness of isomorphic copies of subdivided stars.

Keywords: Super edge antimagic total graph, subdivided stars.
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1 Introduction

We begin with simple, finite, connected and undirected graph $G(V, E)$ with V and E denote the vertex-set and the edge-set. A labeling of a graph is a mapping that carries the graph elements to numbers (usually to positive or non-negative integers). Some labelings use the vertex-set only or the edge-set. We shall call them vertex-labelings or edge-labelings, respectively. A general reference for graph-theoretic ideas can be found in [22]. For a detailed survey of the graph labeling we refer to Gallian [11]. In this paper the domain will be the set of all vertices and edges and such a labeling is called a total labeling. The notion of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [12, 13] on what they called magic valuations of graphs. The definition of (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [20] as a natural extension of edge-magic labeling defined by Kotzig and Rosa.

Conjecture 1.1. [8] Every tree admits a super edge-magic total labeling.

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To support this conjecture, many authors have considered super edge-magic total labeling for many particular classes of trees for example, [1–7, 9, 10, 15–21]. Lee and Shah [14] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still as an open problem. Ngurah et. al. [15] proved that $T(m,n,k)$ is also super edge-magic if $k = n+3$ or $n+4$. In [19], Salman et. al. found the super edge-magic total labeling of a subdivision of a star S_n^m for $m = 1, 2$. However, super (a,d)-edge-antimagic total labelings of copies of subdivided star $G \cong mT(n_1, n_2, n_3, ..., n_r)$ for different $\{n_i : 1 \le i \le r\}$ and $m \ge 3$ is still an open problem.

Definition 1.2. A graph G is called (a,d)-edge-antimagic total ($(a,d)-EAT$) if there exist integers $a > 0$, $d \ge 0$ and a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, ..., v+e\}$ such that $W = \{w(xy) : xy \in E(G)\}$ forms an arithmetic progression starting from a with the common difference d, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ for any $xy \in E(G)$. W is called the set of edge-weights of the graph G.

Definition 1.3. A (a,d)-edge-antimagic total labeling λ is called super (a,d)-edge-antimagic total labeling if $\lambda(V(G)) = \{1, 2, ..., v\}$.

Lemma 1.4. [7] If g is a super edge-magic total labeling of G with the magic constant a, then the function $g_1 : V(G) \cup E(G) \to \{1, 2, ..., v+e\}$ defined by

$$g_1(x) = \begin{cases} v + 1 - g(x), & \text{for } \ x \in V(G), \\ 2v + e + 1 - g(x), & \text{for } \ x \in E(G). \end{cases}$$

is also a super edge-magic total labeling of G with the magic constant $a_1 = 4v + e + 3 - a$.

Definition 1.5. For $n_i \ge 1$ and $p \ge 3$, let $G \cong T(n_1, n_2, ..., n_p)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the i-th edge of the star $K_{1,p}$, where $1 \le i \le p$.

2 Main Results

We consider the following proposition which will be used frequently in the main results.

Proposition 2.1. [3] If a (v,e)-graph G has a (s,d)-EAV labeling then

- G has a super $(s+v+1, d+1)$-EAT labeling,
- G has a super $(s+v+e, d-1)$-EAT labeling.

Theorem 2.2. Let $G \cong 2T(n+2, n, n)$ be a graph with order v and $n \equiv 1(\mathrm{mod}2)$. Then G admits super $(a,0)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a,2)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 3n + 7$.

Proof: We suppose the vertex-set and the edge-set of G, as follows:
$V(G) = \{c_j \mid 1 \le j \le 2\} \cup \{x_{ij}^{l_i} \mid 1 \le i \le 3 \ ; \ 1 \le l_i \le n_i; \ 1 \le j \le 2\}$,

$E(G) = \{c_j x_{ij}^1 | 1 \le i \le 3 \; ; \; 1 \le j \le 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \le i \le 3 \; ; 1 \le l_i \le n_i - 1 \; ; \; 1 \le j \le 2\}$.
The order and size of the graph G are $v = 6(n+1)$ and $e = 2(3n+2)$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = 2(n+3) + \frac{3n+1}{2} j, \; j = 1, 2.$$

For odd l_i $1 \le l_i \le n_i$ and $i = 1, 2$ and 3, we define

$$\lambda(u) = \begin{cases} \frac{n+4-l_1}{2} + \frac{3n+5}{2}(j-1), & \text{for } \; u = x_{1j}^{l_1}, \\ \frac{n+4+l_2}{2} + \frac{3n+5}{2}(j-1), & \text{for } \; u = x_2^{l_2}, \\ \frac{3n+5}{2}(j-1) & \text{for } \; u = x_{3j}^{l_3}; \; l_3 = 1, \\ \frac{3(n+2)-l_3}{2} + \frac{9(n+1)}{2}(j-1), & \text{for } \; u = x_{3j}^{l_3}; \; l_i \ge 3. \end{cases}$$

For even l_i , $1 \le l_i \le n_i$ and $i = 1, 2$ and 3, we define

$$\lambda(u) = \begin{cases} \frac{7n+13-l_1}{2} + \frac{3n+1}{2}(j-1), & \text{for } \; u = x_{1j}^{l_1}, \\ \frac{7n+13+l_2}{2} + \frac{3n+1}{2}(j-1), & \text{for } \; u = x_{2j}^{l_2}, \\ \frac{9n+13-l_3}{2} - \frac{3(n+1)}{2}(j-1), & \text{for } \; u = x_{3j}^{l_3}. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{(3n+6)+1, (3n+6)+2, ..., (3n+6)+e\}$. Let $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 0)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 15n + 17$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 2)$-edge-antimagic total labeling and we obtain the magic constant $a = v+1+s = 9n+14$. ■

Theorem 2.3. Let $G \cong 2T(n+2, n, n+1)$ be a graph with order v and $n \equiv 1(\text{mod}2)$. Then G admits super $(a, 1)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 3)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 4$.

Proof: We suppose the vertex-set and the edge-set of G, as follows:
$V(G) = \{c_j \mid 1 \le j \le 2\} \cup \{x_{ij}^{l_i} \mid 1 \le i \le 3 \; ; \; 1 \le l_i \le n_i; \; 1 \le j \le 3\}$,
$E(G) = \{c_j x_{ij}^1 | 1 \le i \le 3 \; ; \; 1 \le j \le 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \le i \le 3 \; ; 1 \le l_i \le n_i - 1 \; ; \; 1 \le j \le 2\}$.
The order and size of the graph G are $v = 2(3n+4)$ and $e = 6(n+1)$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = (2n+4) + j, \quad j = 1, 2.$$

For all l_i $1 \leq l_i \leq n_i$ and $i = 1, 2$ and 3, we define

$$\lambda(u) = \begin{cases} (2n+4) + j - 2l_1, & \text{for } u = x_{1j}^{l_1}, \\ (2n+4) + j + 2l_2, & \text{for } u = x_{2j}^{l_2}, \\ (6n+8) + j - 2l_3, & \text{for } u = x_{3j}^{l_3}. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{4, 4+2, ..., 4+2(e-1)\}$. Let $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 1)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 6(2n+3)$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 3)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = (6n + 13)$. ■

Theorem 2.4. Let $G \cong 2T(n+2, n, n+1, 2n+1)$ be a graph with order v and $n \equiv 1 (\text{mod} 2)$. Then G admits super $(a, 0)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 2)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 5n + 9$.

Proof: We suppose the vertex-set and the edge-set of G, as follows:
$V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} \mid 1 \leq i \leq 4 \; ; \; 1 \leq l_i \leq n_i; \; 1 \leq j \leq 2\}$,
$E(G) = \{c_j x_{ij}^1 | 1 \leq i \leq 4 \; ; \; 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \leq i \leq 4 \; ; 1 \leq l_i \leq n_i - 1 \; ; \; 1 \leq j \leq 2\}$.
The order and size of the graph G are $v = 10(n+1)$ and $e = (10n+8)$. Now, we define the labeling $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = 3n + 7 + \frac{5n+3}{2} j, \; j = 1, 2.$$

For odd l_i $1 \leq l_i \leq n_i$ and $i = 1, 2, 3$ and 4, we define

$$\lambda(u) = \begin{cases} \frac{n+4-l_1}{2} + \frac{5n+7}{2}(j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{n+4+l_2}{2} + \frac{5n+7}{2}(j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{3(n+2)-l_3}{2} + \frac{(5n+7)}{2}(j-1), & \text{for } u = x_{3j}^{l_3}, \\ \frac{(5n+7)}{2} j, & \text{for } u = x_{4j}^{l_4}; \; l_4 = 1 \;, \\ \frac{5n+8-l_4}{2} + \frac{15(n+1)}{2}(j-1), & \text{for } u = x_{3j}^{l_3}; \; l_4 \geq 3 \end{cases}$$

For even l_i ,$1 \leq l_i \leq n_i$ and $i = 1, 2, 3$ and 4, we define

$$\lambda(u)=\begin{cases}\frac{11n+17-l_1}{2}+\frac{5n+3}{2}(j-1), & \text{for } u=x_{1j}^{l_1},\\ \frac{11n+17+l_2}{2}+\frac{5n+3}{2}(j-1), & \text{for } u=x_{2j}^{l_2},\\ \frac{13n+19-l_3}{2}-\frac{(5n+3)}{2}(j-1), & \text{for } u=x_{3j}^{l_3},\\ \frac{15n+17}{2}-\frac{5(n+1)}{2}(j-1), & \text{for } u=x_{4j}^{l_4}.\end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S=\{(5n+8)+1,(5n+8)+2,...,(5n+8)+e\}$. Let $s=min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-edge-antimagic total labeling and we obtain the magic constant $a=v+s+e=25n+27$. Similarly by Proposition 2.1, λ can be extended to a super $(a,2)$-edge-antimagic total labeling and we obtain the magic constant $a=v+1+s=5(3n+4)$. ■

Theorem 2.5. Let $G\cong 2T(n+2,n,n+1,2n+2)$ be a graph with order v and $n\equiv 1(\mathrm{mod}2)$. Then G admits super $(a,1)$-edge-antimagic total labeling with $a=v2+s-1$ and super $(a,3)$-edge-antimagic total labeling with $a=v+s+1$, where $s=4$.

Proof: We suppose the vertex-set and the edge-set of G, as follows:
$V(G)=\{c_j\mid 1\le j\le 2\}\cup\{x_{ij}^{l_i}\mid 1\le i\le 4\ ;\ 1\le l_i\le n_i;\ 1\le j\le 2\}$,
$E(G)=\{c_jx_{ij}^{1}|1\le i\le 4\ ;\ 1\le j\le 2\}\cup\{x_{ij}^{l_i}x_{ij}^{l_i+1}\mid 1\le i\le 4\ ;1\le l_i\le n_i-1\ ;\ 1\le j\le 2\}$.
The order and size of the graph G are $v=2(5n+6)$ and $e=10(n+1)$. Now, we define the labeling $\lambda:V(G)\to\{1,2,...,v\}$ as follows:

$$\lambda(c_j)=(2n+4)+j,\quad j=1,2.$$

For all l_i $1\le l_i\le n_i$, where $i=1,2$ and 3, we define

$$\lambda(u)=\begin{cases}(2n+4)+j-2l_1, & \text{for } u=x_{1j}^{l_1},\\ (2n+4)+j+2l_2, & \text{for } u=x_{2j}^{l_2},\\ (6n+8)+j-2l_3, & \text{for } u=x_{3j}^{l_3},\\ (10n+12)+j-2l_4, & \text{for } u=x_{4j}^{l_4}.\end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S=\{4,4+2,...,4+2(e-1)\}$. Let $s=min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,1)$-edge-antimagic total labeling and we obtain the magic constant

$a = v + e + s = 2(10n + 13)$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 3)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = 10n + 17$. ■

Theorem 2.6. Let $G \cong 2T(n+2, n, n+1, 2(n+1), 4n+3)$ be a graph with order v and $n \equiv 1(\text{mod}2)$. Then G admits super $(a, 0)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 2)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 9n + 13$.

Proof: We suppose the vertex-set and the edge-set of G, as follows:
$V(G) = \{c_j \mid 1 \le j \le 2\} \cup \{x_{ij}^{l_i} \mid 1 \le i \le 5\ ;\ 1 \le l_i \le n_i;\ 1 \le j \le 2\}$,
$E(G) = \{c_j x_{ij}^1 | 1 \le i \le 5\ ;\ 1 \le j \le 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \le i \le 5\ ; 1 \le l_i \le n_i - 1\ ;\ 1 \le j \le 2\}$.
The order and size of the graph G are $v = 18(n+1)$ and $e = 2(9n+8)$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = 5n + 9 + \frac{9n+7}{2} j,\ j = 1, 2.$$

For odd l_i $1 \le l_i \le n_i$ and $i = 1, 2, 3, 4$ and 5, we define

$$\lambda(u) = \begin{cases} \frac{n+4-l_1}{2} + \frac{9n+11}{2}(j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{n+4+l_2}{2} + \frac{9n+11}{2}(j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{3(n+2)-l_3}{2} + \frac{(9n+11)}{2}(j-1), & \text{for } u = x_{3j}^{l_3}, \\ \frac{5n+8-l_4}{2} + \frac{(9n+11)}{2}(j-1), & \text{for } u = x_{4j}^{l_4}, \\ \frac{9n+11}{2} + \frac{(9n+11)}{2}, & \text{for } u = x_{5j}^{l_5};\ l_5 = 1, \\ \frac{9n+12-l_5}{2} + \frac{27(n+1)}{2}, & \text{for } u = x_{5j}^{l_5};\ l_5 \ge 3. \end{cases}$$

For even l_i, $1 \le l_i \le n_i$ and $i = 1, 2, 3, 4$ and 5, we define

$$\lambda(u) = \begin{cases} \frac{19n+25-l_1}{2} + \frac{9n+7}{2}(j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{19n+25+l_2}{2} + \frac{9n+7}{2}(j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{21n+27-l_3}{2} - \frac{(9n+7)}{2}(j-1), & \text{for } u = x_{3j}^{l_3}, \\ \frac{23n+29-l_4}{2} + \frac{(9n+7)}{2}(j-1), & \text{for } u = x_{4j}^{l_4}, \\ \frac{27n+31-l_5}{2} - \frac{9(n+1)}{2}(j-1), & \text{for } u = x_{5j}^{l_5}. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{(9n+12)+1, (9n+12)+2, ..., (9n+12)+e\}$. Let $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 0)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 55n + 47$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 2)$-edge-antimagic total labeling and we obtain the magic constant $a = v+1+s = 45n+47$. ■

Theorem 2.7. Let $G \cong 2T(n+2, n, n+1, 2(n+1), 4(n+1))$ be a graph with order v and $n \equiv 1(\text{mod}2)$. Then G admits super $(a, 1)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 3)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 4$.

Proof: We suppose the vertex-set and the edge-set of G, as follows: $V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} \mid 1 \leq i \leq 5\ ;\ 1 \leq l_i \leq n_i;\ 1 \leq j \leq 2\}$,
$E(G) = \{c_j x_{ij}^1 | 1 \leq i \leq 5\ ;\ 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \leq i \leq 5\ ; 1 \leq l_i \leq n_i - 1\ ;\ 1 \leq j \leq 2\}$.
The order and size of the graph G are $v = 2(9n+8)$ and $e = 2(9n+7)$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = (2n+4) + j, \quad j = 1, 2.$$

For all l_i $1 \leq l_i \leq n_i$, where $i = 1, 2$ and 3, we define

$$\lambda(u) = \begin{cases} (2n+4) + j - 2l_1, & \text{for } u = x_{1j}^{l_1}, \\ (2n+4) + j + 2l_2, & \text{for } u = x_{2j}^{l_2}, \\ (6n+9) + j - 2l_3, & \text{for } u = x_{3j}^{l_3}, \\ (10n+12) + j - 2l_4, & \text{for } u = x_{4j}^{l_4}, \\ (18n+20) + j - 2l_5, & \text{for } u = x_{5j}^{l_5}. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{4, 4+2, ..., 4+2(e-1)\}$. Let $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 1)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 36n + 34$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 3)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = 18n + 19$. ■

Theorem 2.8. Let $G \cong T(n+2, n, n+1, 2(n+1), 4(n+1), ..., n_p)$ be a graph with order v and $n \equiv 1(\text{mod}2)$. Then G admits super $(a, 0)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 2)$-edge-antimagic total labeling with $a = v + s + 1$, $s = (n+5) + 2^{p-2}(n+1)$, $n_i = 2^{i-3}(n+1)$ for $i = 4, 5, 6, ..., p-1$ and $n_p = 2^{i-3}(n+1) - 1$.

Proof: We suppose that the vertex-set and the edge-set of G, as follows:
$V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} \mid 1 \leq i \leq p \ ; \ 1 \leq l_i \leq n_i; \ 1 \leq j \leq 2\}$,
$E(G) = \{c_j x_{ij}^1 | 1 \leq i \leq p \ ; \ 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \leq i \leq p \ ; 1 \leq l_i \leq n_i - 1 \ ; \ 1 \leq j \leq 2\}$.
The order and size of the graph G are $v = 2(n+1) + 2^{p-1}(n+1)$ and $e = 2n + 2^{p-1}(n+1)$.
Now, we define the labeling $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = \frac{2(n+2) + 2^{p-2}(n+1)}{2} + \frac{(n-1) + 2^{p-2}(n+1)}{2} j, \ j = 1, 2.$$

For odd l_i $1 \leq l_i \leq n_i$, where $i = 1, 2, 3, 4$ and 5, we define

$$\lambda(u) = \begin{cases} \frac{n+4-l_1}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{n+4+l_2}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{(3n+4)-l_3}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{3j}^{l_3}, \\ \frac{(n+4)+2^{k-2}(n+1)-l_k}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{4j}^{l_4}, \\ & \text{for } 1 \leq l_k \leq n_k, \\ & \text{for } k = 4, 5, ..., p-1, \\ \frac{(n+3)+2^{p-2}(n+1)}{2} j, & \text{for } u = x_{pj}^{l_p}, \\ & \text{for } l_p = 1. \\ \frac{(n+4)+2^{k-2}(n+1)-l_p}{2} + \frac{3(n+1)(1+2^{p-2})}{2}(j-1), & \text{for } u = x_{pj}^{l_p}, \\ & \text{for } 2 \leq l_p \leq n_p. \end{cases}$$

For even l_i , $1 \leq l_i \leq n_i$ and $i = 1, 2, 3, 4$ and 5, we define

$$\lambda(u) = \begin{cases} \frac{3(n+1)+2^{p-1}(n+1)-l_1}{2} & \\ +\frac{(n-1)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{1j}^{l_1}, \\ \frac{3(n+1)+2^{p-1}(n+1)+l_2}{2} & \\ +\frac{(n-1)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{2j}^{l_2}, \\ \frac{3(n+3)+(2^{p-1}+2^{k-2})(n+1)-l_k}{2} & \\ +\frac{(n-1)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{kj}^{l_k}, \\ & \text{for } 2 \leq l_k \leq n_k \ , k = 3, 4, ..., p-1 \\ \frac{(3n+7)+(2^{p-1}+2^{k-2})(n+1)-l_p}{2} & \\ -\frac{(n-1)+2^{p-2}(n+1)}{2}(j-1), & \text{for } u = x_{pj}^{l_p}, \\ & \text{for } 2 \leq l_p \leq n_p. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{(n+4) + 2^{p-2}(n+1) + 1, (n+4) + 2^{p-2}(n+1) + 2, ..., (n+4) + 2^{p-2}(n+1) + e\}$. Let $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 0)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 5n + 7 + 2^{p-2} 5(n+1)$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 2)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = 3n + 8 + 2^{p-2}3(n + 1)$. ∎

Theorem 2.9. Let $G \cong T(n+2, n, n+1, 2(n+1), 4(n+1), ..., n_p)$ be a graph with order v and $n \equiv 1(\text{mod}2)$. Then G admits super $(a, 1)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 3)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 4$.

Proof: We suppose the vertex-set and the edge-set of G, as follows:
$V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} \mid 1 \leq i \leq p\ ;\ 1 \leq l_i \leq n_i;\ 1 \leq j \leq 2\}$,
$E(G) = \{c_j x_{ij}^1 | 1 \leq i \leq p\ ;\ 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} x_{ij}^{l_i+1} \mid 1 \leq i \leq p\ ; 1 \leq l_i \leq n_i - 1\ ;\ 1 \leq j \leq 2\}$.
The order and size of the graph G are $v = 2(n+2) + 2^{p-1}(n+1)$ and $e = 2(n+1) + 2^{p-1}(n+1)$. Now, we define the labeling $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = (2n + 4) + j,\ \ j = 1, 2.$$

For all $l_i \quad 1 \leq l_i \leq n_i$, we define

$$\lambda(u) = \begin{cases} (2n+4) + j - 2l_1, & \text{for } \ u = x_{1j}^{l_1}, \\ (2n+4) + j + 2l_2, & \text{for } \ u = x_{2j}^{l_2}, \\ (6n+8) + j - 2l_3, & \text{for } \ u = x_{3j}^{l_3}, \\ (10n+12) + j - 2l_4, & \text{for } \ u = x_{4j}^{l_4}, \\ (10n+12) + j + \sum\limits_{m=5}^{i} [2^{m-2}(n+1)] - 2l_i, & \text{for } \ u = x_{ij}^{l_i},\ i \geq 5. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{4, 4 + 2, ..., 4 + 2(e - 1)\}$. Let $s = min(S)$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 1)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 2(2n + 5) + 2^p(n + 1)$. Similarly by Proposition 2.1, λ can be extended to a super $(a, 3)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = 2n + 9 + 2^{p-1}(n + 1)$. ∎

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