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Some Properties of Operations on $\alpha O(X)$

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Abstract

In this paper, we introduce the notions of α_γ-interior, α_γ-neighbourhood, α_γ-derived, α_γ-boundary, α_γ-kernel and α-γ-g.closed set defined by γ-operation on $\alpha O(X)$ and investigate some of their properties.

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1 Introduction

The notion of α-open sets was introduced by Nj$a$stad [6] and he denoted the family of all α-open sets in a topological space (X, τ) by $\alpha O(X, \tau)$ or $\alpha O(X)$. Ibrahim [1] defined the concept of an operation on $\alpha O(X)$ and introduced the notion of α_γ-open sets. Kasahara [2] defined the concept of an operation on topological spaces and introduced α-closed graphs of an operation. Ogata [7] called the operation α as γ operation and introduced the notion of τ_γ which is the collection of all γ-open sets in a topological space (X, τ). The aim of this paper is to continue the study of topological properties by means of operations on $\alpha O(X)$.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X, τ) is said to be α-open [6] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. An operation $\gamma : \alpha O(X, \tau) \to P(X)$ [1] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$. A subset A of X is called an α_γ-open set [1] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. The complement of an α_γ-open set is said to be α_γ-closed. We denote the set of all α_γ-open (resp., α_γ-closed) sets of (X, τ) by $\alpha O(X, \tau)_\gamma$ (resp., $\alpha C(X, \tau)_\gamma$). The α_γ-closure [1] of a subset A of X with an operation γ on $\alpha O(X)$ is denoted by $\alpha_\gamma Cl(A)$ and is defined to be the intersection of all α_γ-closed sets containing A. A point $x \in X$ is in αCl_γ-closure [1] of a set $A \subseteq X$, if $U^\gamma \cap A \neq \phi$ for each α-open set U containing x. The αCl_γ-closure of A is denoted by $\alpha Cl_\gamma(A)$. An operation γ on $\alpha O(X, \tau)$ is said to be α-open [1] if for every α-open set U of X containing $x \in X$, there exists an α_γ-open set V of X such that $x \in V$ and $V \subseteq U^\gamma$.

3 Some Properties of γ-operations on $\alpha O(X)$

Definition 3.1. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. A point $a \in A \subseteq X$ is said to be α_γ-interior point of A if there exists an α-open set N of X containing a such that $N^\gamma \subseteq A$. We denote the set of all such points by $\alpha Int_\gamma(A)$.
Thus $\alpha Int_\gamma(A) = \{x \in A : x \in N \in \alpha O(X) \text{ and } N^\gamma \subseteq A\} \subseteq A$.

Theorem 3.2. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. If A and B are two subsets of X, then the following statements are true:

1. If $A \subseteq B$, then $\alpha Int_\gamma(A) \subseteq \alpha Int_\gamma(B)$.
2. $\alpha Int_\gamma(A) \cup \alpha Int_\gamma(B) \subseteq \alpha Int_\gamma(A \cup B)$.
3. If γ is α-regular, then $\alpha Int_\gamma(A) \cap \alpha Int_\gamma(B) = \alpha Int_\gamma(A \cap B)$.

Proof: Follows from Definition 3.1 and 2.14 [1]. ∎

Theorem 3.3. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. If A is a subset of X, then

1. $\alpha Int_\gamma(X \setminus A) = X \setminus \alpha Cl_\gamma(A)$.
2. $\alpha Cl_\gamma(X \setminus A) = X \setminus \alpha Int_\gamma(A)$.
3. $\alpha Int_\gamma(A) = X \setminus \alpha Cl_\gamma(X \setminus A)$.
4. $\alpha Cl_\gamma(A) = X \setminus \alpha Int_\gamma(X \setminus A)$.

Proof: We prove (1) only and the other parts can be proved similarly.

Let $x \in \alpha Int_\gamma(X \setminus A)$, then there exists an α-open sets U containing x such that $U^\gamma \subseteq X \setminus A$. This implies that $U^\gamma \cap A = \phi$. This gives that $x \notin \alpha Cl_\gamma(A)$ and so $x \in X \setminus \alpha Cl_\gamma(A)$.

Conversely, let $x \in X \setminus \alpha Cl_\gamma(A)$ implies that $x \notin \alpha Cl_\gamma(A)$, then there exists an α-open sets V containing x such that $V^\gamma \cap A = \phi$ implies that $x \in V \subseteq V^\gamma \subseteq X \setminus A$. It follows that $x \in \alpha Int_\gamma(X \setminus A)$. ∎

The proof of the following theorem is obvious and hence omitted.

Theorem 3.4. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then for $A \subseteq X$, we have

1. $\alpha Int_\gamma(A)$ is an α-open set.
2. A is α_γ-open if and only if $\alpha Int_\gamma(A) = A$.

Theorem 3.5. If a subset A of X is α_γ-open, then there exists an α-open set O such that $O \subseteq A \subseteq O^\gamma$.

Proof: If A is an α_γ-open set, then $\alpha Int_\gamma(A) = A$. By taking $O = \alpha Int_\gamma(A)$, we obtain that $O \subseteq A \subseteq O^\gamma$. ■

Definition 3.6. [3] A topological space (X, τ) is said to be α_γ-regular if for each $x \in X$ and for each α-open set V in X containing x, there exists an α-open set U in X containing x such that $U^\gamma \subseteq V$.

Theorem 3.7. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then the following statements are equivalent.

1. $\alpha O(X, \tau) = \alpha O(X, \tau)_\gamma$.
2. (X, τ) is an α_γ-regular space.
3. For every $x \in X$ and every α-open set U of X containing x there exists an α_γ-open set W of X such that $x \in W$ and $W \subseteq U$.

Proof: (1) $\Rightarrow$ (2): Let $x \in X$ and V be an α-open set containing x. Then by assumption, V is an α_γ-open set. This implies that for each $x \in V$, there exists an α-open set U such that $U^\gamma \subseteq V$. Therefore (X, τ) is an α_γ-regular space.

(2) $\Rightarrow$ (3): Let $x \in X$ and U be an α-open set containing x. Then by (2), there is an α-open set W containing x and $W \subseteq W^\gamma \subseteq U$. Applying (2) to set W shows that W is α_γ-open. Hence W is an α_γ-open set containing x such that $W \subseteq U$.

(3) $\Rightarrow$ (1): By (3) and [[1], Proposition 2.13], it follows that every α-open set is α_γ-open, that is, $\alpha O(X, \tau) \subseteq \alpha O(X, \tau)_\gamma$. Also from [[1], Remark 2.6], $\alpha O(X, \tau)_\gamma \subseteq \alpha O(X, \tau)$. Hence we have the result. ■

Remark 3.8. For any topological space (X, τ), we have

1. If $\alpha O(X)$ is indiscrete, then $\alpha O(X)_\gamma$ is also indiscrete.
2. If $\alpha O(X)_\gamma$ is discrete, then $\alpha O(X)$ is discrete.

Remark 3.9. Let (X, τ) be a topological space and $x \in X$. If $\{x\} \in \alpha O(X)_\gamma$, then $\{x\}^\gamma = \{x\}$.

Definition 3.10. Let (X, τ) be a topological space and $x \in X$, then a subset N of X is said to be α_γ-neighbourhood (resp., α-neighbourhood [4]) of x, if there exists an α_γ-open (resp., α-open) set U in X such that $x \in U \subseteq N$.

Proposition 3.11. In a topological space (X, τ), a subset A of X is α_γ-open if and only if it is an α_γ-neighbourhood of each of its points.

Proof: Let $A \subseteq X$ be an α_γ-open set, since for every $x \in A$, $x \in A \subseteq A$ and A is α_γ-open. This shows A is an α_γ-neighbourhood of each of its points.

Conversely, suppose that A is an α_γ-neighbourhood of each of its points. Then for each $x \in A$, there exists $B_x \in \alpha O(X)_\gamma$ such that $B_x \subseteq A$. Then $A = \cup\{B_x : x \in A\}$. Since each B_x is α_γ-open. It follows that A is α_γ-open set. ■

Proposition 3.12. If $A \subseteq B$ in a topological space (X, τ) and A is an α_γ-neighbourhood of a point $x \in X$, then B is also α_γ-neighbourhood of the same point x.

Proof: Obvious. ■

Remark 3.13. Since every α_γ-open set is α-open, then every α_γ-neighbourhood of a point is an α-neighbourhood of the same point.

Definition 3.14. Let A be a subset of a topological space (X, τ) and γ be an operation on $\alpha O(X)$. The union of all α_γ-open sets contained in A is called the α_γ-interior of A and denoted by $\alpha_\gamma Int(A)$.

Theorem 3.15. Let (X, τ) be a topological space and γ be an operation on $\alpha O(X)$. For any subsets A, B of X we have the following:

1. $\alpha_\gamma Int(A)$ is an α_γ-open set in X.
2. A is α_γ-open if and only if $A = \alpha_\gamma Int(A)$.
3. $\alpha_\gamma Int(\alpha_\gamma Int(A)) = \alpha_\gamma Int(A)$.
4. $\alpha_\gamma Int(\phi) = \phi$ and $\alpha_\gamma Int(X) = X$.
5. $\alpha_\gamma Int(A) \subseteq A$.
6. If $A \subseteq B$, then $\alpha_\gamma Int(A) \subseteq \alpha_\gamma Int(B)$.
7. $\alpha_\gamma Int(A \cup B) \supseteq \alpha_\gamma Int(A) \cup \alpha_\gamma Int(B)$.
8. $\alpha_\gamma Int(A \cap B) \subseteq \alpha_\gamma Int(A) \cap \alpha_\gamma Int(B)$.

Proof: Straight forward. ■

Definition 3.16. Let (X, τ) be a topological space with an operation γ on $\alpha O(X)$. A point $x \in X$ is said to be α_γ-limit point of a set A if for each α_γ-open set U containing x, then $U \cap (A \setminus \{x\}) \neq \phi$. The set of all α_γ-limit points of A is called an α_γ-derived set of A and is denoted by $\alpha_\gamma D(A)$.

Some properties of α_γ-derived sets are stated in the following proposition.

Proposition 3.17. Let A, B be any two subsets of a space X, and γ be an operation on $\alpha O(X)$. Then we have the folloing properties:

1. $\alpha_\gamma D(\phi) = \phi$.
2. If $x \in \alpha_\gamma D(A)$, then $x \in \alpha_\gamma D(A \setminus \{x\})$.
3. $\alpha_\gamma D(A \cup B) \supseteq \alpha_\gamma D(A) \cup \alpha_\gamma D(B)$.
4. $\alpha_\gamma D(A \cap B) \subseteq \alpha_\gamma D(A) \cap \alpha_\gamma D(B)$.
5. $\alpha_\gamma D(\alpha_\gamma D(A)) \setminus A \subseteq \alpha_\gamma D(A)$.
6. $\alpha_\gamma D(A \cup \alpha_\gamma D(A)) \subseteq A \cup \alpha_\gamma D(A)$.

Proof: Obvious. ∎

The proofs of Propositions 3.18 and 3.19 are clear.

Proposition 3.18. A subset A of a topological space X is α_γ-closed if and only if it contains the set of its α_γ-limit points.

Proposition 3.19. Let A be any subset of a topological space (X, τ) and γ be an operation on $\alpha O(X)$, then $\alpha_\gamma Cl(A) = A \cup \alpha_\gamma D(A)$.

Proposition 3.20. Let A be any subset of a topological space (X, τ) and γ be an operation on $\alpha O(X)$. Then $\alpha_\gamma Int(A) = A \setminus \alpha_\gamma D(X \setminus A)$.

Proof: If $x \in A \setminus \alpha_\gamma D(X \setminus A)$, then $x \notin \alpha_\gamma D(X \setminus A)$ and so there exists an α_γ-open set U containing x such that $U \cap (X \setminus A) = \phi$. Then $x \in U \subseteq A$ and hence $x \in \alpha_\gamma Int(A)$, that is $A \setminus \alpha_\gamma D(X \setminus A) \subseteq \alpha_\gamma Int(A)$. On the other hand, if $x \in \alpha_\gamma Int(A)$, then $x \notin \alpha_\gamma D(X \setminus A)$ since $\alpha_\gamma Int(A)$ is α_γ-open and $\alpha_\gamma Int(A) \cap (X \setminus A) = \phi$. Hence $\alpha_\gamma Int(A) = A \setminus \alpha_\gamma D(X \setminus A)$. ∎

Proposition 3.21. Let A be any subset of a topological space (X, τ) and γ be an operation on $\alpha O(X)$. Then the following statements are true:

1. $X \setminus \alpha_\gamma Int(A) = \alpha_\gamma Cl(X \setminus A)$.
2. $X \setminus \alpha_\gamma Cl(A) = \alpha_\gamma Int(X \setminus A)$.

3. $\alpha_\gamma Int(A) = X \setminus \alpha_\gamma Cl(X \setminus A)$.

4. $\alpha_\gamma Cl(A) = X \setminus \alpha_\gamma Int(X \setminus A)$.

Proof: We only prove (1), the other parts can be proved similarly.
$X \setminus \alpha_\gamma Int(A) = X \setminus (A \setminus \alpha_\gamma D(X \setminus A)) = (X \setminus A) \cup \alpha_\gamma D(X \setminus A) = \alpha_\gamma Cl(X \setminus A)$. ∎

Definition 3.22. Let A be a subset of a space X, then the α_γ-boundary of A is defined as $\alpha_\gamma Cl(A) \setminus \alpha_\gamma Int(A)$ and is denoted by $\alpha_\gamma Bd(A)$.

Some properties of α_γ-boundary sets are stated in the following proposition.

Proposition 3.23. Let A be any subset of a topological space (X, τ) and γ be an operation on $\alpha O(X)$. Then the following statements hold:

1. $\alpha_\gamma Cl(A) = \alpha_\gamma Int(A) \cup \alpha_\gamma Bd(A)$.

2. $\alpha_\gamma Int(A) \cap \alpha_\gamma Bd(A) = \phi$.

3. $\alpha_\gamma Bd(A) = \alpha_\gamma Cl(A) \cap \alpha_\gamma Cl(X \setminus A)$.

4. $\alpha_\gamma Bd(A) = \alpha_\gamma Bd(X \setminus A)$.

5. $\alpha_\gamma Bd(A)$ is an α_γ-closed set.

Proof: Obvious. ∎

Definition 3.24. Let (X, τ) be a topological space. A mapping $\gamma : \alpha O(X) \to P(X)$ is said to be:

1. α-monotone on $\alpha O(X)$ if for all $A, B \in \alpha O(X)$, $A \subseteq B$ implies $A^\gamma \subseteq B^\gamma$.

2. α-idempotent on $\alpha O(X)$ if $A^{\gamma\gamma} = A^\gamma$ for all $A \in \alpha O(X)$.

3. α-additive on $\alpha O(X)$ if $(A \cup B)^\gamma = A^\gamma \cup B^\gamma$ for all $A, B \in \alpha O(X)$.

If $\bigcup_{i \in I} A_i^\gamma \subseteq (\bigcup_{i \in I} A_i)^\gamma$ for any collection $\{A_i\}_{i \in I} \subseteq \alpha O(X)$, then γ is said to be α-subadditive on $\alpha O(X)$.

Proposition 3.25. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then, γ is α-monotone on $\alpha O(X)$ if and only if γ is α-subadditive on $\alpha O(X)$.

Proof: Let γ be α-monotone on $\alpha O(X)$ and $\{A_i\}_{i \in I} \subseteq \alpha O(X)$. Then for each $i \in I$, $A_i^\gamma \subseteq (\bigcup_{i \in I} A_i)^\gamma$ and thus $\bigcup_{i \in I} A_i^\gamma \subseteq (\bigcup_{i \in I} A_i)^\gamma$.
Conversely, if γ is α-subadditive on $\alpha O(X)$ and $A, B \in \alpha O(X)$ with $A \subseteq B$, then $A^\gamma \subseteq A^\gamma \cup B^\gamma \subseteq (A \cup B)^\gamma = B^\gamma$. Thus γ is α-monotone on $\alpha O(X)$. ∎

Remark 3.26. The α-regularity of operation γ in [1] follows from the α-monotonicity of operation γ.

Remark 3.27. It is easy to verify that if γ is α-additive on $\alpha O(X)$ then γ is α-monotone on $\alpha O(X)$.

The following result shows that the family of α_γ-open sets may be a topology on X.

Theorem 3.28. Let (X, τ) be a topological space. If γ is an α-monotone operation on $\alpha O(X)$, then the family of α_γ-open is a topology on X.

Proof: Clearly $\phi, X \in \alpha O(X)_\gamma$ and by [[1], Theorem 2.11], the union of any family of α_γ-open sets is α_γ-open. To complete the proof it is enough to show that the finite intersection of α_γ-open sets is α_γ-open. Let A and B be two α_γ-open sets and let $x \in A \cap B$, then $x \in A$ and $x \in B$, so there exist α-open sets U and V such that $x \in U \subseteq U^\gamma \subseteq A$ and $x \in V \subseteq V^\gamma \subseteq B$, since γ is an α-monotone operation and $U \cap V$ is α-open set such that $U \cap V \subseteq U$ and $U \cap V \subseteq V$, this implies that $(U \cap V)^\gamma \subseteq U^\gamma \cap V^\gamma \subseteq A \cap B$. Thus $A \cap B$ is α_γ-open set. This completes the proof. ■

Theorem 3.29. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. If $(\bigcup_{i\in I} W_i)^\gamma \subseteq \bigcup_{i\in I} W_i^\gamma$ for any collection $\{W_i\}_{i\in I} \subseteq \alpha O(X)$, then for every α_γ-open set U we have $U^\gamma = U$.

Proof: Let U be an α_γ-open set. Then for every $x \in U$ there exists an α-open set W containing x such that $W \subseteq W^\gamma \subseteq U$. Therefore $\bigcup_{x\in U} W \subseteq \bigcup_{x\in U} W^\gamma \subseteq U$, so $\bigcup_{x\in U} W \subseteq (\bigcup_{x\in U} W)^\gamma \subseteq U$. Therefore, $U \subseteq U^\gamma \subseteq U$ and so $U^\gamma = U$. ■

Theorem 3.30. For any operation γ on $\alpha O(X)$, the map $\alpha_\gamma Cl : \alpha O(X) \to P(X)$ is an operation on $\alpha O(X)$ satisfying (1) and (2) of Definition 3.24, but not (3).

Proof: By [[1], Theorem 2.22 (5)], $\alpha_\gamma Cl$ is an α-monotone on $\alpha O(X)$. We show that $\alpha_\gamma Cl$ is α-idempotent on $\alpha O(X)$. Given $A \in \alpha O(X)$, it is obvious that $\alpha_\gamma Cl(A) \subseteq \alpha_\gamma Cl(\alpha_\gamma Cl(A))$. Let $x \in \alpha_\gamma Cl(\alpha_\gamma Cl(A))$ and V be any α_γ-open set containing x, then by [[1], Theorem 2.23], there is $z \in V \cap \alpha_\gamma Cl(A)$. Since $z \in \alpha_\gamma Cl(A)$ and V is an α_γ-open set containing z, we have that $V \cap A \neq \phi$, thus $x \in \alpha_\gamma Cl(A)$. Therefore $\alpha_\gamma Cl(\alpha_\gamma Cl(A)) \subseteq \alpha_\gamma Cl(A)$ and hence $\alpha_\gamma Cl(\alpha_\gamma Cl(A)) = \alpha_\gamma Cl(A)$.
In general, $\alpha_\gamma Cl$ is not an α-additive operation on $\alpha O(X)$ as shown in the following example. ■

Example 3.31. let $X = \{a, b, c\}$ equipped with the discrete topology on X. We define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise.} \end{cases}$$

Then, the α_γ-open subsets of (X,τ) are ϕ, $\{a,b\}$, $\{a,c\}$ and X. Now, if we let $A=\{b\}$ and $B=\{c\}$, then $\alpha_\gamma Cl(A)=A$, $\alpha_\gamma Cl(B)=B$ and $\alpha_\gamma Cl(A\cup B)=X$, where $A\cup B=\{b,c\}$, this implies that $\alpha_\gamma Cl(A\cup B)=X\neq\{b,c\}=\alpha_\gamma Cl(A)\cup\alpha_\gamma Cl(B)$.

Suppose that A is a subset of a topological space (X,τ), then we have the following properties:

Theorem 3.32. Let $\gamma:\alpha O(X)\to P(X)$ be an operation on $\alpha O(X)$, A and B subsets of a topological space (X,τ). Then, we have the following properties:

1. $A\subseteq \alpha Cl_\gamma(A)$.
2. $\alpha Cl_\gamma(\phi)=\phi$ and $\alpha Cl_\gamma(X)=X$.
3. A is α_γ-closed (that is, $X\setminus A$ is α_γ-open) in (X,τ) if and only if $\alpha Cl_\gamma(A)=A$ holds.
4. If $A\subseteq B$, then $\alpha Cl_\gamma(A)\subseteq\alpha Cl_\gamma(B)$.
5. $\alpha Cl_\gamma(A)\cup\alpha Cl_\gamma(B)\subseteq\alpha Cl_\gamma(A\cup B)$ holds.
6. If γ is α-regular, then $\alpha Cl_\gamma(A\cup B)=\alpha Cl_\gamma(A)\cup\alpha Cl_\gamma(B)$ holds.
7. $\alpha Cl_\gamma(A\cap B)\subseteq\alpha Cl_\gamma(A)\cap\alpha Cl_\gamma(B)$ holds.

Proof: $(1),(2),(4)$: Obviously, by [[1], Definition 2.20], we have $A\subseteq\alpha Cl_\gamma(A)$.

(3): Suppose that $X\setminus A$ is α_γ-open in (X,τ). We claim that $\alpha Cl_\gamma(A)\subseteq A$. Let $x\notin A$. There exists an α-open set U containing x such that $U^\gamma\subseteq X\setminus A$, that is, $U^\gamma\cap A=\phi$. Hence, using [[1], Definition 2.20], we have $x\notin\alpha Cl_\gamma(A)$ and so $\alpha Cl_\gamma(A)\subseteq A$. By (1), it is proved that $A=\alpha Cl_\gamma(A)$.

Conversely, suppose that $A=\alpha Cl_\gamma(A)$. Let $x\in X\setminus A$. Since $x\notin\alpha Cl_\gamma(A)$, there exists an α-open set U containing x such that $U^\gamma\cap A=\phi$, that is, $U^\gamma\subseteq X\setminus A$. Namely, $X\setminus A$ is α_γ-open in (X,τ) and so A is α_γ-closed.

$(5),(7)$: Followed from (4).

(6): Let $x\notin\alpha Cl_\gamma(A)\cup\alpha Cl_\gamma(B)$. Then, there exist two α-open sets U and V containing x such that $U^\gamma\cap A=\phi$ and $V^\gamma\cap B=\phi$. By [[1], Definition 2.14], there exists an α-open set W containing x such that $W^\gamma\subseteq U^\gamma\cap V^\gamma$. Thus, we have that $W^\gamma\cap(A\cup B)\subseteq(U^\gamma\cap V^\gamma)\cap(A\cup B)\subseteq[U^\gamma\cap A]\cup[V^\gamma\cap B]=\phi$, that is, $W^\gamma\cap(A\cup B)=\phi$. Namely, we have $x\notin\alpha Cl_\gamma(A\cup B)$ and so $\alpha Cl_\gamma(A\cup B)\subseteq\alpha Cl_\gamma(A)\cup\alpha Cl_\gamma(B)$. We can obtain (6) by using (5). ∎

Theorem 3.33. Let (X,τ) be a topological space and γ an α-monotone operation on $\alpha O(X)$. If A is a subset of X, then

1. For every α_γ-open set G of X, we have that $\alpha Cl_\gamma(A)\cap G\subseteq\alpha Cl_\gamma(A\cap G)$.

2. For every α_γ-closed set F of X, we have that $\alpha Int_\gamma(A \cup F) \subseteq \alpha Int_\gamma(A) \cup F$.

Proof: (1) Let $x \in \alpha Cl_\gamma(A) \cap G$ and let U be an α-open set containing x. Since $x \in \alpha Cl_\gamma(A)$, implies that $U^\gamma \cap A \neq \phi$. Since G is an α_γ-open set, there exists an α-open set V of X containing x such that $V^\gamma \subseteq G$. Thus $(U \cap V)^\gamma \cap A \neq \phi$, this implies that $U^\gamma \cap (A \cap G) \neq \phi$ by α-monotone and hence $x \in \alpha Cl_\gamma(A \cap G)$. Therefore $\alpha Cl_\gamma(A) \cap G \subseteq \alpha Cl_\gamma(A \cap G)$.

(2) Follows from (1) and Theorem 3.3 (3). ∎

The following example shows that the condition γ is α-monotone is necessary for the above theorem.

Example 3.34. Consider $X = \{a, b, c\}$ with the discrete topology on X. We define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X & \text{otherwise.} \end{cases}$$

Since γ is not α-monotone, so if we let $A = \{a, c\}$ and $G = \{b, c\}$, then $\alpha Cl_\gamma(A) = X$ and $\alpha Cl_\gamma(A) \cap G = \{b, c\}$, this implies that $\alpha Cl_\gamma(A) \cap G = \{b, c\} \not\subseteq \alpha Cl_\gamma(A \cap G) = \alpha Cl_\gamma(\{c\}) = \{c\}$.

Remark 3.35. Let (X, τ) be a topological space and γ an α-regular operation on $\alpha O(X)$. If A is a subset of X, then

1. For every α_γ-open set G of X, we have $\alpha_\gamma Cl(A) \cap G \subseteq \alpha_\gamma Cl(A \cap G)$.

2. For every α_γ-closed set F of X, we have $\alpha_\gamma Int(A \cup F) \subseteq \alpha_\gamma Int(A) \cup F$.

Theorem 3.36. Let (X, τ) be a topological space, N a subset of X and γ an α-open operation on $\alpha O(X)$. Then, $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$ if and only if any one of the following conditions hold:

1. $\alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N)) = X$.

2. $N \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$.

Proof: (1) $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$ if and only if $X \setminus \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N)) = \phi$ by Theorem 3.3 (3) if and only if $X \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$ if and only if $X = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$.

(2) $N \subseteq X = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$ by (1). Conversely, $N \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$, implies that $\alpha Cl_\gamma(N) \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$ by [[1], Theorem 2.26 (2)]. Since $X = \alpha Cl_\gamma(N) \cup (X \setminus \alpha Cl_\gamma(N))$, implies that $X \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N)) \cup (X \setminus \alpha Cl_\gamma(N)) = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$. Hence $X = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$. ∎

Theorem 3.37. Let (X, τ) be a topological space, N a subset of X and γ be both α-regular and α-open operation on $\alpha O(X)$. If $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$ then every non empty α_γ-open set U contains a non empty α_γ-open set A disjoint with N.

Proof: Given $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$. This implies that $\alpha Cl_\gamma(N)$ does not contain any non empty α_γ-open set. Hence for any non empty α_γ-open set U, $U \cap (X \setminus \alpha Cl_\gamma(N)) \neq \phi$. Thus by [[1], Theorem 2.26 (2) and Proposition 2.18] $A = U \cap (X \setminus \alpha Cl_\gamma(N)) = U \setminus \alpha Cl_\gamma(N)$ is a non empty α_γ-open set contained in U and disjoint with N. ■

Definition 3.38. Let A be a subset of a topological space (X, τ) and γ an operation on $\alpha O(X)$. The α_γ-kernel of A, denoted by $\alpha_\gamma Ker(A)$ is defined to be the set $\alpha_\gamma Ker(A) = \cap\{V : A \subseteq V, V \in \alpha O(X, \tau)_\gamma\}$.

Proposition 3.39. Let (X, τ) be a topological space with an operation γ on $\alpha O(X)$ and $x \in X$. Then $y \in \alpha_\gamma ker(\{x\})$ if and only if $x \in \alpha_\gamma Cl(\{y\})$.

Proof: Suppose that $y \notin \alpha_\gamma ker(\{x\})$. Then there exists an α_γ-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \alpha_\gamma Cl(\{y\})$. The proof of the converse case can be done similarly. ■

Proposition 3.40. Let (X, τ) be a topological space with an operation γ on $\alpha O(X)$ and A be a subset of X. Then, $\alpha_\gamma ker(A) = \{x \in X\colon \alpha_\gamma Cl(\{x\}) \cap A \neq \phi\}$.

Proof: Let $x \in \alpha_\gamma ker(A)$ and suppose $\alpha_\gamma Cl(\{x\}) \cap A = \phi$. Hence $x \notin X \setminus \alpha_\gamma Cl(\{x\})$ which is an α_γ-open set containing A. This is impossible, since $x \in \alpha_\gamma ker(A)$. Consequently, $\alpha_\gamma Cl(\{x\}) \cap A \neq \phi$. Let $x \in X$ such that $\alpha_\gamma Cl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \alpha_\gamma ker(A)$. Then, there exists an α_γ-open set V containing A and $x \notin V$. Let $y \in \alpha_\gamma Cl(\{x\}) \cap A$. Hence, V is an α_γ-open set containing y which does not contain x. By this contradiction $x \in \alpha_\gamma ker(A)$ and the claim. ■

Proposition 3.41. The following properties hold for the subsets A, B of a topological space (X, τ) with an operation γ on $\alpha O(X)$:

1. $A \subseteq \alpha_\gamma ker(A)$.
2. $A \subseteq B$ implies that $\alpha_\gamma ker(A) \subseteq \alpha_\gamma ker(B)$.
3. If A is α_γ-open in (X, τ), then $A = \alpha_\gamma ker(A)$.
4. $\alpha_\gamma ker(\alpha_\gamma ker(A)) = \alpha_\gamma ker(A)$.

Proof: (1), (2) and (3): Immediate consequences of $\alpha_\gamma ker(A) = \cap\{U \in \alpha O(X)_\gamma\colon A \subseteq U\}$.

(4): First observe that by (1) and (2), we have $\alpha_\gamma ker(A) \subseteq \alpha_\gamma ker(\alpha_\gamma ker(A))$. If $x \notin \alpha_\gamma ker(A)$, then there exists $U \in \alpha O(X, \tau)_\gamma$ such that $A \subseteq U$ and $x \notin U$. Hence $\alpha_\gamma ker(A) \subseteq U$, and so we have $x \notin \alpha_\gamma ker(\alpha_\gamma ker(A))$. Thus $\alpha_\gamma ker(\alpha_\gamma ker(A)) = \alpha_\gamma ker(A)$. ■

Proposition 3.42. The following statements are equivalent for any points x and y in a topological space (X,τ) with an operation γ on $\alpha O(X)$:

1. $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$.

2. $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$.

Proof: (1) $\Rightarrow$ (2): Suppose that $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$, then there exists a point z in X such that $z \in \alpha_\gamma ker(\{x\})$ and $z \notin \alpha_\gamma ker(\{y\})$. From $z \in \alpha_\gamma ker(\{x\})$ it follows that $\{x\} \cap \alpha_\gamma Cl(\{z\}) \neq \phi$ which implies $x \in \alpha_\gamma Cl(\{z\})$. By $z \notin \alpha_\gamma ker(\{y\})$, we have $\{y\} \cap \alpha_\gamma Cl(\{z\}) = \phi$. Since $x \in \alpha_\gamma Cl(\{z\})$, $\alpha_\gamma Cl(\{x\}) \subseteq \alpha_\gamma Cl(\{z\})$ and $\{y\} \cap \alpha_\gamma Cl(\{x\}) = \phi$. Therefore, it follows that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. Now $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$ implies that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$.

(2) $\Rightarrow$ (1): Suppose that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. Then there exists a point z in X such that $z \in \alpha_\gamma Cl(\{x\})$ and $z \notin \alpha_\gamma Cl(\{y\})$. Then, there exists an α_γ-open set containing z and therefore x but not y, namely, $y \notin \alpha_\gamma ker(\{x\})$ and thus $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$. ■

Proposition 3.43. Let (X,τ) be a topological space and γ be an operation on $\alpha O(X)$. Then, $\cap\{\alpha_\gamma Cl(\{x\}) : x \in X\} = \phi$ if and only if $\alpha_\gamma ker(\{x\}) \neq X$ for every $x \in X$.

Proof: Necessity, suppose that $\cap\{\alpha_\gamma Cl(\{x\}) : x \in X\} = \phi$. Assume that there is a point y in X such that $\alpha_\gamma ker(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every α_γ-open set V containing y and hence $y \in \alpha_\gamma Cl(\{x\})$ for any $x \in X$. This implies that $y \in \cap\{\alpha_\gamma Cl(\{x\}) : x \in X\}$. But this is a contradiction.

Sufficiency, assume that $\alpha_\gamma ker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap\{\alpha_\gamma Cl(\{x\}) : x \in X\}$, then every α_γ-open set containing y must contain every point of X. This implies that the space X is the unique α_γ-open set containing y. Hence $\alpha_\gamma ker(\{y\}) = X$ which is a contradiction. Therefore, $\cap\{\alpha_\gamma Cl(\{x\}) : x \in X\} = \phi$. ■

Definition 3.44. [1] A subset A of a topological space (X,τ) is called an $\alpha_\gamma D$-set if there are two $U, V \in \alpha O(X,\tau)_\gamma$ such that $U \neq X$ and $A = U \setminus V$.

Proposition 3.45. If a singleton $\{x\}$ is an $\alpha_\gamma D$-set of (X,τ), then $\alpha_\gamma ker(\{x\}) \neq X$.

Proof: Since $\{x\}$ is an $\alpha_\gamma D$-set of (X,τ), then there exist two subsets $U_1, U_2 \in \alpha O(X,\tau)_\gamma$ such that $\{x\} = U_1 \setminus U_2$, $\{x\} \subseteq U_1$ and $U_1 \neq X$. Thus, we have that $\alpha_\gamma ker(\{x\}) \subseteq U_1 \neq X$ and so $\alpha_\gamma ker(\{x\}) \neq X$. ■

Definition 3.46. A subset A of a topological space (X,τ) is said to be α-γ-generalized closed (α-γ-g.closed) set if $\alpha Cl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is an α_γ-open set of (X,τ).

Definition 3.47. [1] A subset A of the space (X, τ) is said to be α_γ-generalized closed (briefly, α_γ-g.closed) if $\alpha_\gamma Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an α_γ-open set in (X, τ). The complement of an α_γ-g.closed set is called an α_γ-g.open set.

Definition 3.48. [5] A subset A of a topological space (X, τ) is called an (α, α)-generalized closed set (briefly, (α, α)-g-closed) if $\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open.

Theorem 3.49. Let A be a subset of a topological space (X, τ) and γ an operation on $\alpha O(X)$. Then, the following statements are true:

1. If A is α-γ-g.closed in X, then A is (α, α)-g-closed.

2. If A is α_γ-g.closed in X, then A is α-γ-g.closed.

Proof: Follows from Theorem 2.24 [1]. ■

Remark 3.50. By Theorem 3.49, every α_γ-g.closed is (α, α)-g-closed.

Remark 3.51. It is clear that every α_γ-closed set is α-γ-g.closed, but the converse is not true in general as shown in the following example.

Example 3.52. Let $X = \{1, 2, 3\}$ with the topology $\tau = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, X\}$. We define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{2\} \text{ or } \{1, 3\} \\ X & \text{otherwise.} \end{cases}$$

Now, if we let $A = \{1\}$, since the only α_γ-open supersets of A are $\{1, 3\}$ and X, then A is α-γ-g.closed. But it is easy to see that A is not α_γ-closed.

Theorem 3.53. If A is α_γ-open and α-γ-g.closed, then A is α_γ-closed.

Proof: Suppose that A is α_γ-open and α-γ-g.closed. Since $A \subseteq A$, we have $\alpha Cl_\gamma(A) \subseteq A$, also $A \subseteq \alpha Cl_\gamma(A)$, therefore $\alpha Cl_\gamma(A) = A$. That is, A is α_γ-closed. ■

Theorem 3.54. Let $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$ and A a subset of a topological space (X, τ). Then the following statements are equivalent:

1. A is α-γ-g.closed in (X, τ).

2. $\alpha_\gamma Cl(\{x\}) \cap A \neq \phi$ for every $x \in \alpha Cl_\gamma(A)$.

3. $\alpha Cl_\gamma(A) \subseteq \alpha_\gamma Ker(A)$ holds.

Proof: (1) $\Rightarrow$ (2): Let A be an α-γ-g.closed set of (X,τ). Suppose that there exists a point $x \in \alpha Cl_\gamma(A)$ such that $\alpha_\gamma Cl(\{x\}) \cap A = \phi$. By [[1], Theorem 2.22 (2)], $\alpha_\gamma Cl(\{x\})$ is α_γ-closed. Put $U = X \setminus \alpha_\gamma Cl(\{x\})$. Then, we have $A \subseteq U$, $x \notin U$ and U is an α_γ-open set of (X,τ). Since A is an α-γ-g.closed set, $\alpha Cl_\gamma(A) \subseteq U$. Thus, we have $x \notin \alpha Cl_\gamma(A)$. This is a contradiction.
(2) $\Rightarrow$ (3): Follows from Proposition 3.40.
(3) $\Rightarrow$ (1): Let U be any α_γ-open set such that $A \subseteq U$. Let x be a point such that $x \in \alpha Cl_\gamma(A)$. By (3), $x \in \alpha_\gamma Ker(A)$ holds. Namely, we have that $x \in U$, because $A \subseteq U$ and $U \in \alpha O(X,\tau)_\gamma$. ■

Theorem 3.55. Let (X,τ) be a topological space and γ an operation on $\alpha O(X)$. If a subset A of X is α-γ-g.closed, then $\alpha Cl_\gamma(A) \setminus A$ does not contain any non-empty α_γ-closed set.

Proof: Suppose that there exists a non-empty α_γ-closed set F such that $F \subseteq \alpha Cl_\gamma(A) \setminus A$. Then we have $A \subseteq X \setminus F$ and $X \setminus F$ is α_γ-open. It follows from the assumption that $\alpha Cl_\gamma(A) \subseteq X \setminus F$ and so $F \subseteq (\alpha Cl_\gamma(A) \setminus A) \cap (X \setminus \alpha Cl_\gamma(A))$. Therefore, we have $F = \phi$. ■

Remark 3.56. In the above theorem, if γ is an α-open operation, then the converse of the above theorem is true.

Proof: Let U be an α_γ-open set such that $A \subseteq U$. Since γ is an α-open operation, it follows from [[1], Theorem 2.26] that $\alpha Cl_\gamma(A)$ is α_γ-closed in (X,τ). Thus by [[1], Definition 2.2 and Theorem 2.11], we have $\alpha Cl_\gamma(A) \cap (X \setminus U) = F$ is α_γ-closed in (X,τ). Since $X \setminus U \subseteq X \setminus A$, $F \subseteq \alpha Cl_\gamma(A) \setminus A$. Using the assumptions of the converse of Theorem 3.55 above, $F = \phi$ and hence $\alpha Cl_\gamma(A) \subseteq U$. ■

Theorem 3.57. Let (X,τ) be a topological space and γ an operation on $\alpha O(X)$. Then for each $x \in X$, $\{x\}$ is α_γ-closed or $X \setminus \{x\}$ is α-γ-g.closed in (X,τ).

Proof: Suppose that $\{x\}$ is not α_γ-closed, then $X \setminus \{x\}$ is not α_γ-open. Let U be any α_γ-open set such that $X \setminus \{x\} \subseteq U$. Then $U = X$. Hence, $\alpha Cl_\gamma(X \setminus \{x\}) \subseteq U$. Thereore, $X \setminus \{x\}$ is an α-γ-g.closed set. ■

Proposition 3.58. A subset A of X is α_γ-g.open if and only if $F \subseteq \alpha_\gamma Int(A)$ whenever $F \subseteq A$ and F is α_γ-closed in X.

Proof: Let A be α_γ-g.open and $F \subseteq A$ where F is α_γ-closed. Since $X \setminus A$ is α_γ-g.closed and $X \setminus F$ is an α_γ-open set containing $X \setminus A$ implies $\alpha_\gamma Cl(X \setminus A) \subseteq X \setminus F$. By Proposition 3.21 (1), $X \setminus \alpha_\gamma Int(A) \subseteq X \setminus F$. That is $F \subseteq \alpha_\gamma Int(A)$.

Conversely, suppose that F is α_γ-closed and $F \subseteq A$ implies $F \subseteq \alpha_\gamma Int(A)$. Let $X \setminus A \subseteq U$ where U is α_γ-open. Then $X \setminus U \subseteq A$ where $X \setminus U$ is α_γ-closed. By hypothesis $X \setminus U \subseteq \alpha_\gamma Int(A)$. That is $X \setminus \alpha_\gamma Int(A) \subseteq U$. By Proposition 3.21 (1), $\alpha_\gamma Cl(X \setminus A) \subseteq U$. This implies $X \setminus A$ is α_γ-g.closed and A is α_γ-g.open. ■

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