

New results on edge rotation distance graphs

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Abstract

The concept of edge rotations and distance between graphs was introduced by Gary Chartrand et al. A graph G can be transformed into a graph H by an edge rotation if G contains distinct vertices u, v and w such that $uv \in E(G)$, $uw \notin E(G)$ and $H \cong G - uv + uw$. In this paper we consider rotations on some snake related graphs followed by some general results.

Keywords: Edge rotation, edge rotation distance graphs, r -distance graph, triangular snake, double triangular snake, alternating double triangular snake, quadrilateral snake, double quadrilateral snake, alternating double quadrilateral snake.

AMS Subject Classification(2010): 05C12.

1 Introduction

Unless mentioned otherwise, for terminology and notation the reader may refer Buckley and Harary [2] and Chartrand and Zhang [5], new ones will be introduced as and when found necessary.

In this paper, by a graph G , we mean a simple, undirected, connected graph without self-loops. The *order* and *size* are respectively the number of vertices denoted by n and the number of edges denoted by m .

The *distance* $d(u, v)$ between any two vertices u and v , of G , is the length of a shortest path between u and v . The *eccentricity* $e(u)$ of a vertex u is the distance to a farthest vertex from u . The maximum and the minimum eccentricity amongst the vertices of G are respectively called the *diameter* $diam(G)$ and *radius* $rad(G)$. If $d(u, v) = e(u)$, ($v \neq u$) then we say that v is an *eccentric vertex* of u .

The distance between isomorphism classes of graphs was introduced by Zelinka in [14] which was also studied for trees by Zelinka in [15]. 'Edge Rotations' or the concept of rotation between edges of the graphs and the distance between such graphs was introduced by Chartrand et al. [3] which were based on [14] and [15]. A graph G can be transformed into a graph H by an edge rotation given by $H \cong G - uv + uw$ where u, v and w are distinct vertices of G such that

$uv \in E(G)$ and $uw \notin E(G)$. Later, Zelinka [16] gave a comparison of various distances for the isomorphism classes of graphs and trees, which was based on the concept of edge rotations.

Zelinka studied various aspects by using the concept of distance between graphs and edge rotations in [17], [18] and [19].

The rotation distance between graphs G and H is denoted by $d_r(G, H)$, if there exists a sequence of graphs G_1, G_2, \dots, G_{k-1} such that G_1 is obtained by an edge rotation on G , and for each $1 \leq i \leq k$, G_{i+1} is obtained by an edge rotation on G_i , with H obtained from G_{k-1} by one edge rotation. In this case we denote the rotation distance from G to H as $d_r(G, H)$ and it is equal to k .

Definition 1.1. [3] Let $S = \{ G_1, G_2, \dots, G_k \}$ be a set of graphs all of the same order and the same size. Then the rotation distance graph $D(S)$ of S has S as its vertex set and vertices (graphs) G_i and G_j are adjacent if $d_r(G_i, G_j) = 1$, where $d_r(G_i, G_j)$ is the rotation distance between G_i and G_j .

A graph G is an edge rotation distance graph(ERDG) (or r - distance graph) if $G \cong D(S)$ for some set S of graphs.

In 1990, Chartrand et al. [4] showed that the cycles, the complete bipartite graphs $K_{3,3}$, and $K_{2,p}$ ($p \geq 1$) are edge rotation distance graphs. In 1997, Jarrett [10] gave a proof using different technique and showed complete graphs, trees, wheel ($W_{1,n}$) and the complete bipartite graph $K_{m,n}$ ($3 \leq m \leq n$) are edge rotation distance graphs. In [8], Huilgol et al. showed that the generalized Petersen graph, $G_p(n, 1)$, the generalized star, $K_{(1,n)}$ are edge rotation distance graphs.

In this paper we consider the edge rotations on ladder graph, triangular snake, quadrilateral snake, double triangular snake, double quadrilateral snake, alternate triangular snake, alternate quadrilateral snake. A triangular snake is a connected graph in which all blocks are triangles and the block cut point graph is a path[12]. Since these graphs contain cycles as subgraphs, to generate them we use the method used by Jarrett [10] with slight modifications to prove all of the above specified graphs are Edge Rotation Distance Graphs(ERDG). Here the number of vertices and edges are denoted by n' and m' , in order to avoid confusion.

Definition 1.2. [13] A triangular snake T_n is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n - 1$.

Definition 1.3. [13] A double triangular snake $D(T_n)$ consists of two triangular snakes that have a common path.

Definition 1.4. [13] An alternate triangular snake AT_n is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to a new vertex v_i .

Definition 1.5. [13] An alternate double triangular snake ADT_n consists of two alternate triangular snakes that have a common path.

Definition 1.6. [13] A quadrilateral snake Q_n is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} to new vertices v_i and w_i respectively and then joining v_i and w_i .

Definition 1.7. [13] An alternate quadrilateral snake AQ_n is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and then joining v_i and w_i .

Definition 1.8. [13] An alternate double quadrilateral snake $A(D(Q_n))$ consists of two alternate quadrilateral snakes that have a common path.

Definition 1.9. [13] A polygonal chain $G_{m,n}$ is a connected graph all of whose m blocks are polygons on n sides.

Definition 1.10. A ladder, L_n is defined as the cartesian product of a path and K_2 , that is, $L_n = P_n \times K_2$.

2 Edge Rotations on Snakes

We use the method by Jarrett [10] with some modifications to prove the following snake related theorems.

Theorem 2.1. Every triangular snake is an ERDG.

Proof: We first generate a T_2 . Since the same pattern is repeated we just change the labeling and thus generate a T_n . Since a T_2 is nothing but a triangle, the construction is as follows.

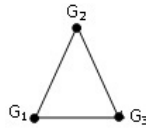


Figure 1: The triangular snake graph - T_2 .

Let $S = \{G_1, G_2, G_3\}$. Consider the graphs G_1, G_2 and G_3 as shown in Figure 2. We first show that the $d_r(G_1, G_2) = d_r(G_2, G_3) = d_r(G_3, G_1) = 1$.

We see that the edge xu_2 is rotated to xu_6 , thus resulting in one rotation between the graphs G_1 and G_2 . Similarly, we observe the edge yu_4 rotated to yu_2 between the graphs G_2 and G_3 to show one rotation. Also, the edge yu_2 in G_3 is rotated to yu_4 in G_1 . Thus, the rotation distance between each of these graphs is equal to one resulting in a T_2 . Thus $D(S) \cong T_2$.

To generate a T_3 , we use graph(vertex), i.e., G_3 , by just changing the labels of the vertices.

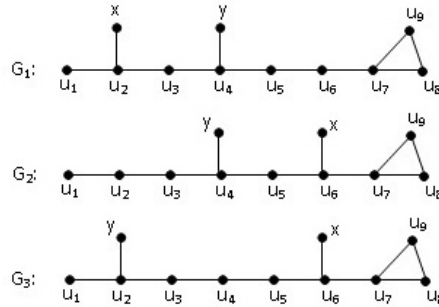


Figure 2: Rotations on a triangular snake graph.

That is we perform the rotations in the reverse directions, viz., G_3 to G_2 , G_2 to G_1 and then finally G_3 to G_1 , thus forming one more C_3 .

In a similar way a T_n is generated using $2n - 1$ graphs. Hence, a T_n is an edge rotation distance graph. ■

Theorem 2.2. Every double triangular snake is an ERDG.

Proof: We first prove $D(T_2)$ is an ERDG by fixing the value of $n' = 4$. We generate a cycle of length 4 and then show that the rotation distance between the first and third vertex is one thus forming a $D(T_2)$. Let $S = \{G_1, G_2, G_3, G_4\}$. Consider the graphs G_1, G_2, G_3 and G_4 as shown in Figure 4.

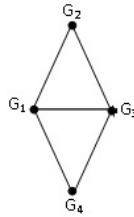


Figure 3: A double triangular snake - $D(T_2)$.

We first show that the $d_r(G_1, G_2) = d_r(G_2, G_3) = d_r(G_3, G_4) = d_r(G_4, G_1) = 1$. Also, we show that $d_r(G_1, G_3) = 1$.

We see that the edge xu_2 is rotated to xu_6 , thus resulting in one rotation between the graphs G_1 and G_2 . Similarly, we observe the edge yu_4 rotated to yu_8 between the graphs G_2 and G_3 to show one rotation. The edge xu_6 in G_3 is rotated to xu_2 in G_4 and yu_8 in G_4 is rotated to yu_4 in G_1 to show one rotation between G_4 and G_1 . Also, u_1u_3 is rotated to u_1u_4 between the graphs G_1 and G_3 to show one rotation between them. In order to equalize the size between the remaining graphs G_2 and G_4 we add an edge u_2u_4 . Thus, the rotation distance between

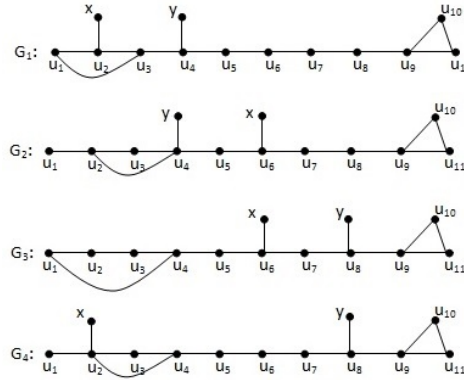


Figure 4: Rotations on a double triangular snake.

each of these graphs is equal to one resulting in a $D(T_2)$. Thus $D(S) \cong D(T_2)$.

In a similar way a $D(T_n)$ is generated. Hence, a $D(T_n)$ is an edge rotation distance graph. ■

Theorem 2.3. Every alternating double triangular snake is an ERDG.

Proof: The alternating double triangular snake is denoted by ADT_n . Here we will generate an ADT_2 followed by a path. To generate this through edge rotations we shall first generate a C_4 and then from the third vertex(i.e., G_3) we use edge rotation between graphs G_3 and a new graph G_1 and thus generate a path of length 1. Let $S = \{G_1, G_2, G_3, G_4, G_1\}$. Consider the graphs G_1, G_2, G_3 and G_4 as shown in Figure 6.

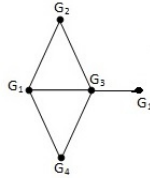


Figure 5: An alternate double triangular snake - ADT_3 .

Now we show that the $d_r(G_1, G_2) = d_r(G_2, G_3) = d_r(G_3, G_4) = d_r(G_4, G_1) = 1$. Also, we show $d_r(G_1, G_3) = 1$. To generate a path of length one from the graph G_3 we consider one more new graph G_1 and thus show the rotation distance between G_3 and G_1 is one.

We see that the edge xu_2 is rotated to xu_6 , thus resulting in one rotation between the graphs G_1 and G_2 . Similarly, we observe the edge yu_4 rotated to yu_8 between the graphs G_2 and G_3 to show one rotation. The edge xu_6 in G_3 is rotated to xu_2 in G_4 and yu_8 in G_4 is rotated to yu_4 in G_1 to show one rotation between G_4 and G_1 . Also, u_1u_3 is rotated to u_1u_4 between the graphs G_1 and G_3 to show one rotation between them. In order to equalize the size between the remaining graphs G_2 and G_4 we add an edge u_2u_4 . Thus, the rotation distance between

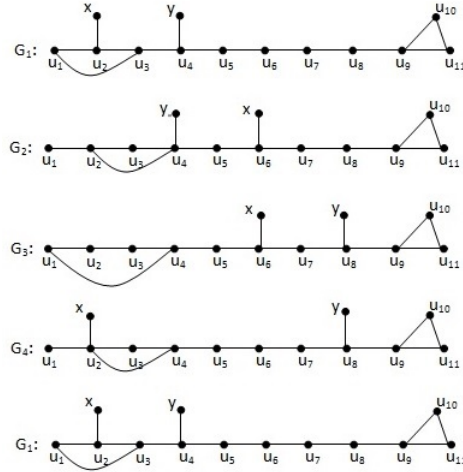


Figure 6: Rotations on an alternate double triangular snake.

each of these graphs is equal to one resulting in a $D(T_2)$. Thus $D(S) \cong D(T_2)$.

Now, to form ADT_2 (a path of length 1, from graph G_3), we consider the graph G_1 once again and show the rotation distance between G_3 and G_1 is one. The edge u_1u_4 in G_3 is rotated to u_1u_3 in the fifth graph (G_1) to form an ADT_2 .

Thus, in a similar way an $AD(T_n)$ is generated. Hence, an $AD(T_n)$ is an edge rotation distance graph. ■

Theorem 2.4. Every quadrilateral snake is an ERDG.

Proof: A quadrilateral snake is denoted by Q_n . If $n = 2$, then it is a C_4 . Since the same pattern is repeated, we generate a Q_2 , and thus by changing the order of labeling we generate a Q_n .

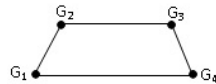


Figure 7: A quadrilateral snake.

Let $S = \{G_1, G_2, G_3, G_4\}$. Consider the graphs G_1, G_2, G_3 and G_4 as shown in Figure 8.

We first show that the $d_r(G_1, G_2) = d_r(G_2, G_3) = d_r(G_3, G_4) = d_r(G_4, G_1) = 1$. Here in the above set of graphs, to show the rotation between the graph G_1 and G_2 , the edge xu_2 is rotated to edge xu_6 . The edge yu_4 is rotated to yu_8 between the graphs G_2 and G_3 . The edge xu_6 is rotated to xu_2 between the graphs G_3 and G_4 . The edge yu_8 is rotated to yu_4 between the

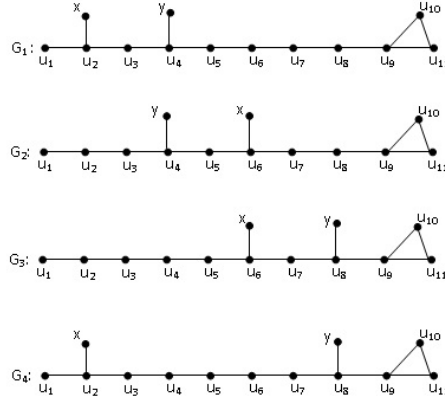


Figure 8: Rotations on a quadrilateral snake.

graphs G_4 and G_1 to show one rotation. Thus, the rotation distance between each of these graphs is equal to one resulting in a Q_2 . Thus $D(S) \cong Q_2$.

Similarly we show that Q_n is an edge rotation distance graph. ■

Theorem 2.5. Every double quadrilateral snake is an ERDG.

Proof: We generate a cycle of length 6 and then show that the rotation distance between the first and fourth vertex is one and thus forming a $D(Q_2)$.

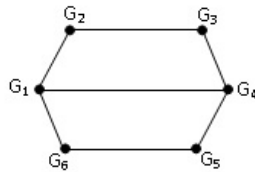


Figure 9: A double quadrilateral snake $D(Q_2)$.

Let $S = \{G_1, G_2, G_3, G_4, G_5, G_6\}$. Consider the graphs G_1, G_2, G_3, G_4, G_5 and G_6 as shown in Figure 10. We first show that the $d_r(G_1, G_2) = d_r(G_2, G_3) = d_r(G_3, G_4) = d_r(G_4, G_5) = d_r(G_5, G_6) = d_r(G_6, G_1) = 1$. Here in the above set of graphs, to show the rotation between the graph G_1 and G_2 , the edge xu_2 is rotated to edge xu_6 . The edge yu_4 is rotated to yu_8 between the graphs G_2 and G_3 . The edge xu_6 is rotated to xu_{10} between the graphs G_3 and G_4 . The edge yu_8 is rotated to yu_{12} between the graphs G_4 and G_5 to show one rotation. Similarly, the edge xu_{10} is rotated to xu_2 between the graphs G_5 and G_6 . And the edge yu_{12} is rotated to yu_4 between the graphs G_6 and G_1 to show one rotation.

To show the rotation distance between G_1 and G_4 is one we add the edge $u_1 u_3$ to G_1 and the edge $u_1 u_4$ to G_4 . Since the size of the graphs G_1 and G_4 changes we add an extra edge

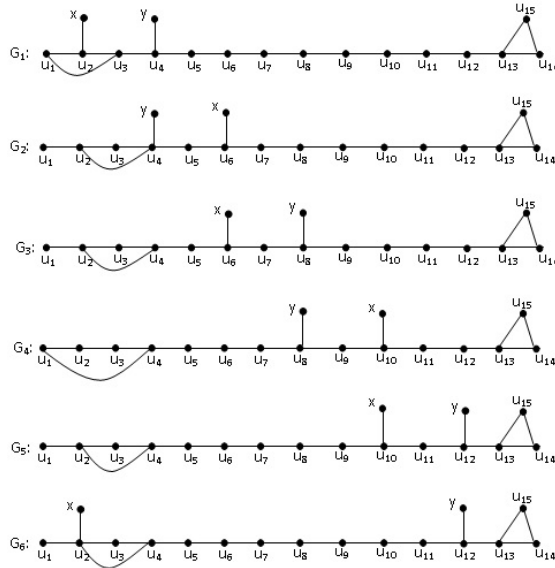


Figure 10: Rotations on a double quadrilateral snake.

u_2u_4 to the remaining graphs namely G_2, G_3, G_5 and G_6 .

Thus, the rotation distance between each of these graphs is equal to one resulting in a $D(Q_2)$. Thus $D(S) \cong D(Q_2)$. Extending the construction we get a $D(Q_n)$ by considering $5n - 4$ graphs. ■

Theorem 2.6. Every alternating double quadrilateral snake is an ERDG.

Proof: To prove this theorem, we use the proof of Theorem 2.5, with a slight change. We first generate a DQ_2 and then show that the rotation distance between the fourth graph (G_4 , (vertex)) and the new graph (one again G_1 to be considered) is one, thus forming a path of length 1.

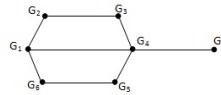


Figure 11: An alternating double quadrilateral snake.

For this we need to show the rotation distance between the graph G_4 and G_1 (considered once again) is one. Since we have already added an edge u_1u_4 to the graph G_4 , we add an edge u_1u_3 to G_1 , and thus by performing this rotation shows the distance between them is one, and forming a path of required length.

Thus, the above mentioned procedure generates an $AD(Q_2)$. Since, this pattern is repeated the basic number of graphs to generate such a snake is $6 + 1$. To generate $AD(Q_n)$, the number of graphs required is $6 * (n/2)$, for n , even and $6 * \lfloor (n/2) \rfloor + 1$, for n , odd. ■

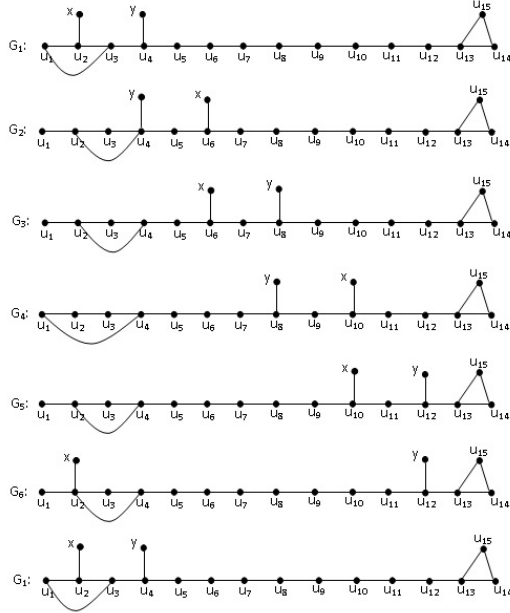


Figure 12: Rotations on an alternating double quadrilateral snake.

Theorem 2.7. A ladder graph, L_n is an ERDG.

Proof: Let $P: u_1, u_2, \dots, u_{n+2}$ be a path and G be a graph obtained by adding two new vertices u_{n+3}, u_{n+4} and three new edges $u_{n+2}u_{n+3}, u_{n+3}u_{n+4}, u_{n+4}u_{n+2}$. Then, for $i = 1, 2, \dots, n - 1$, define G_i to be a graph obtained from G by adding one new vertex x adjacent only to u_i . We also define G_n as the graph obtained from G by adding one new vertex x adjacent only to u_1 . For all n , we add a new edge $u_1 u_3$ for all G_n . For $n = 3$, the graphs G_1, G_2 and G_3 are shown in Figure 14.

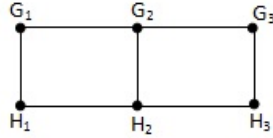


Figure 13: A ladder graph.

Every graph G_i has exactly one vertex of degree one, and an edge rotation changes the degrees of exactly two vertices, $d_r(G_i, G_j) > 1$. On the other hand for $i = 1, 2, 3, \dots, n - 2$, $G_{i+1} \cong G_i - xu_i + xu_{i+1}$ and consequently $d_r(G_i, G_{i+1}) = 1$ and $d_r(G_n, G_{n-1}) = 1$, since $G_n \cong G_{n-1} - xu_{n-1} + xu_1$; thus $D(\{G_1, G_2, \dots, G_n\}) \cong P_n$.

Similarly we generate one more path from the set of graphs $\{H_1, H_2, \dots, H_n\}$.

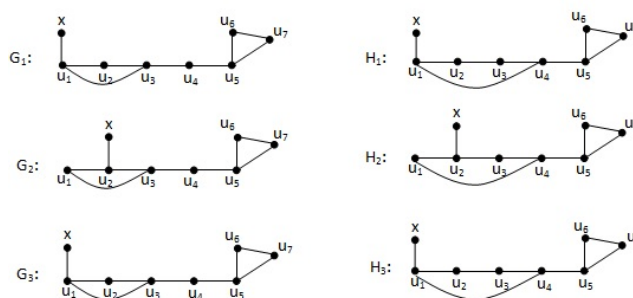


Figure 14: Rotations on a ladder graph.

Now we show that the rotation distance between each of these graphs G_i and H_i is one. Since $G_i \cong H_i - u_1u_3 + u_1u_4$, $d_r(G_i, H_i) = 1$.

Hence, $D(\{G_1, G_2, \dots, G_n, H_1, H_2, \dots, H_n\}) \cong L_n$. ■

Remark 2.8. A polygonal chain $G_{m,n}$ is an ERDG.

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