VARIOUS GRAPH OPERATIONS ON SEMI SMOOTH GRACEFUL GRAPHS

V J Kaneria

Department of Mathematics, Saurashtra University, Rajkot−360005, India.

E−mail: *kaneria_vinodray_j@yahoo.co.in*

H M Makadia

Govt. Engineering College, Rajkot−360005, India.

E−mail: *makadia.hardik@yahoo.com*

R V Viradia

A D Patel Institute of Technology, V V Nagar−388121, India.

E−mail: *rajiv.viradia@gmail.com*

Abstract

In this paper we have done some graph operations on smooth graceful and semi smooth graceful graphs. By applying path union of graphs, star of a graph and cycle of a graph we have generated new graceful families. We have proved that star of a semi smooth graceful graph is graceful. We also proved that $K_{m,n}$, $P(t \cdot H)$ are semi smooth graceful, where H is a semi smooth graceful graph, step grid graph and cycle graph $C(t \cdot H)$ are smooth graceful, when $t \equiv$ (mod 4), H is as above, every semi smooth graceful graph is odd graceful and $C^t(m \cdot C_n)$, $P^t(k \cdot T)$, $< C_{n_1}, P_{n_2}, C_{n_3}, \dots, P_{n_{2t}}, C_{n_{2t+1}} >$, $< K_{m_1,n_1}, P_{r_1}, K_{m_2,n_2}, P_{r_2}, \dots, P_{r_{t-1}}, K_{m_t,n_t} >$, $< P_{n_1} \times P_{m_1}, P_{r_1}, P_{n_2} \times P_{m_2}, \dots, P_{r_{t-1}}, P_{n_t} \times P_{m_t} >$ are graceful, when T is semi smooth graceful tree.

Key words : Graceful graph, smooth graceful graph, odd graceful graph, step grid graph and consecutive graph operations on a graph.
AMS **subject classification number :** 05C78.

1 Introduction :

In 1966 Rosa [1] defined α-labeling as a graceful labeling with an additional property. A graph which admits α-labeling is necessarily bipartite. A natural generalization of graceful graph is the notion of k-graceful graph. Obviously 1-graceful is graceful and a graph which admits α-labeling is always k-graceful graph, $\forall\ k \in N$. Ng [2] has identified some graphs that are k-graceful, $\forall\ k \in N$, but do not have α-labeling.

Kaneria and Jariya [3,4] define smooth graceful labeling and semi smooth graceful labeling. Every smooth graceful graph is also a semi smooth graceful graph. They proved cycle C_n ($n \equiv 0 \pmod 4$), path P_n, grid graph $P_n \times P_m$ and complete bipartite graph $K_{2,n}$ are smooth graceful graphs.

For a comprehensive bibliography of papers on various graph labelings are given in Gallian [5]. The present paper is focused on various graph operations on semi smooth graceful graph to generate new families of graceful graph.

We will consider a simple undirected finite graph $G = (V, E)$ on $|V| = p$ vertices and $|E| = q$ edges. For all terminology and standard notations we follows Harary [6]. Here we will recall some definitions which are used in this paper.

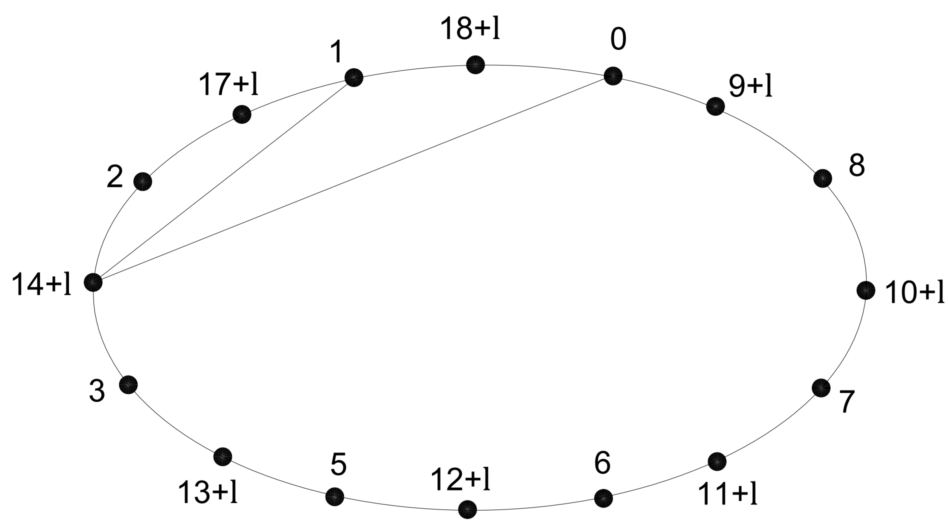
Definition$-$1.1 : A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \dots, q\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{1, 2, \dots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition$-$1.2 : A function f is called k-*graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \dots, k + q - 1\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{k, k + 1, k + 2, \dots, k + q - 1\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective, for every edge $e = (u, v) \in E(G)$. A graph G is called k-*graceful graph* if it admits a k-graceful labeling.

Definition−1.3 : A function f is called *odd graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \ldots, 2q-1\}$ is injective and the induced function $f^{\star} : E(G) \longrightarrow \{1, 3, 5, \ldots, 2q-1\}$ defined as $f^{\star}(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called *odd graceful graph* if it admits an odd graceful labeling.

Definition−1.4 : A smooth graceful graph G, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all non-negative integer l, there is a function $g : V(G) \longrightarrow \{0, 1, \ldots, \lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{q+1}{2} \rfloor + l, \lfloor \frac{q+3}{2} \rfloor + l, \ldots, q + l\}$ such that the induced edge labeling function $g^{\star} : E(G) \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $f^{\star}(e) = |f(u) - f(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

Example−1.5 : A cycle C_{16} with twin chords and its smooth graceful labeling shown in *figure*−1.

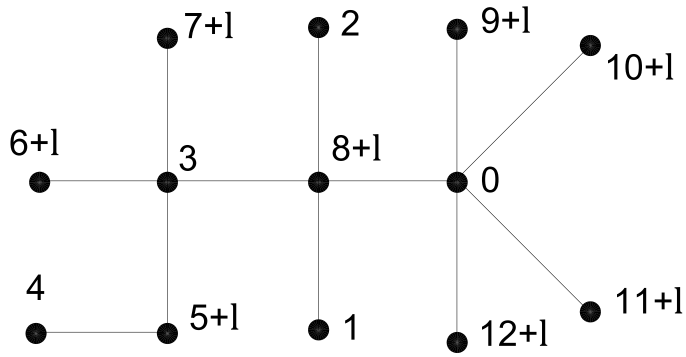


Figure−1 Smooth graceful labeling for a cycle C_{16} with twin chords.

Definition−1.6 : A semi smooth graceful graph G, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all non-negative integer l, there is an integer t $(0 \leq t < q)$ and an injective function $g : V(G) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$ such that the induced edge labeling function $g^{\star} : E(G) \longrightarrow \{1+l, 2+l, \ldots, q+l\}$ defined as $f^{\star}(e) = |f(u) - f(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

If we take $l = 0$ in above both definitions−1.4, 1.6 the labeling functions g will become graceful labeling for the graph G. Every smooth graceful graph is also a semi smooth graceful graph by taking $t = \lfloor \frac{q+1}{2} \rfloor$.

Example−1.7 **:** A tree on 12 edges and its semi smooth graceful labeling shown in *figure*−2.



Figure−2 Semi smooth graceful labeling for a tree with $|E(T)| = 12$.

Definition−1.8 **:** Let G be a graph and $G_1, G_2, \ldots, G_n$ $(n \geq 2)$ be n copies of G. Then the graph obtained by an edge from G_i to G_{i+1} (for $i = 1, 2, \ldots, n-1$) is called path union of G and we will denote it by $P(G_1, G_2, \ldots, G_n)$.

Definition−1.9 **:** A graph obtained by replacing each vertex of the star $K_{1,n}$ by a connected graph G of n vertices is called star of G and we will denote it by $G^{\star}$. The graph G which replaced at the center of $K_{1,n}$ we will call it as central copy of $G^{\star}$.

Definition−1.10 **:** For a cycle C_n, each vertices of C_n are replaced by connected graphs $G_1, G_2, \ldots, G_n$ is known as *cycle of graphs* and we will denote such graph by $C(G_1, G_2, \ldots, G_n)$. If we replace each vertices of C_n by a connected graph G (i.e. $G_1 = G = G_2 = \ldots = G_n$), such cycle of graph G we will denote it by $C(n \cdot G)$.

If we replace each vertices of C_n by $C(n \cdot G)$, such cycle graph $C(n \cdot (n \cdot G))$, we will denote it by $C^2(n \cdot G)$. In general for any $t \geq 2$ $C^t(n \cdot G) = C(n \cdot C^{t-1}(n \cdot G))$.

Definition−1.11 **:** Take $P_n, P_n, P_{n-1}, P_{n-2}, \ldots, P_3, P_2$ paths on $n, n, n-1, n-2, \ldots, 3, 2$ vertices and arranged them vertically. A graph obtained by joining horizontal vertices of given successive paths is known as a *step grid graph* of size n $(n \geq 3)$ and we will denote it by St_n.

Obviously $|V(St_n)| = \frac{1}{2}(n^2 + 3n - 2)$ and $|E(St_n)| = n^2 + n - 2$. Above definition was introduced by Kaneria and Makadia [7].

Definition-1.12 **:** Let $G_1, G_2, \ldots, G_t$ be any connected graphs. The graph $< G_1, P_{n_1}, G_2, P_{n_2}, \ldots, G_{t-1}, P_{n_{t-1}}, G_t >$ obtained by joining two consecutive graphs G_i and G_{i+1} by P_{n_i}, a path of of length n_i and $n_i \in N$, $\forall\ i = 1, 2, \ldots, t-1$ is called arbitrary path union of graphs $G_1, G_2, \ldots, G_t$ join by arbitrary paths $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$. In other words consecutive graphs $G_1, G_2, \ldots, G_t$ join by arbitrary paths $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$ is known as arbitrary path union of graphs G_i $(1 \leq i \leq t)$.

If we replace each $P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}}$ by a path P_n of length n such path union of graphs, we will denote by $P_n(G_1, G_2 \ldots, G_t)$ and if we take $G_i = G$ $(1 \leq i \leq t)$,where G is a connected graph, we will denote such graph (arbitrary path union of a graph G) by $P_n(t \cdot G)$. Obviously $P_1(G_1, G_2, \ldots, G_n) = P(G_1, G_2, \ldots, G_n)$, simple path union of $G_1, G_2, \ldots, G_n$ and $P_1(t \cdot G) = P(t \cdot G) = P(G_1, G_2, \ldots, G_n)$, where $G_1 = G = \ldots = G_n$.

If we replace $G = P(t \cdot H)$ in $P(t \cdot G)$, such graph $P(t \cdot P(t \cdot H))$, we will denote it by $P^2(t \cdot H)$. In general for any $s \geq 2$ $P^s(t \cdot G) = P(t \cdot P^{s-1}(t \cdot G))$ or $P^{s-1}(t \cdot P(t \cdot G))$.

2 Main Results :

Theorem-2.1 **:** $K_{m,n}$ is a semi smooth graceful graph.

Proof : Let $v_1, v_2, \ldots, v_m,\ u_1, u_2, \ldots, u_n$ be vertices of the complete bipartite graph $K_{m,n}$. Obviously $K_{m,n}$ is a bipartite graph with the vertex graceful labeling function $f : V(K_{m,n}) \longrightarrow \{0, 1, \ldots, q = mn\}$ defined by

$$f(v_i) = m - i \text{ or } i - 1, \qquad \forall\ i = 1, 2, \ldots, m;$$
$$f(u_j) = q - m(j-1), \qquad \forall\ j = 1, 2, \ldots, n.$$

Let l be any non-negative integer. Define the vertex labeling function $g : V(K_{m,n}) \longrightarrow \{0, 1, \ldots, m-1, m+l, m+1+l, \ldots, mn+l\}$ such that its induced edge labeling function $g^\star : E(K_{m,n}) \longrightarrow \{1+l, 2+l, \ldots, mn+l\}$ with $g^\star(e) = |g(u) - g(v)|$, $\forall\, e = (u, v) \in E(K_{m,n})$ defined by

$$g(w) = f(w), \qquad \text{when } w \in \{v_1, v_2, \ldots, v_m\}$$
$$= f(w) + l, \qquad \text{when } w \in \{u_1, u_2, \ldots, u_n\}.$$

Now for each $e = (u, v) \in E(K_{m,n})$, we see that

$$
\begin{aligned}
g^\star(e) &= g^\star((u, v)) \\
&= |g(u) - g(v)| \\
&= |f(u) + l - f(v)| \\
&= |f(u) - f(v)| + l \\
&= f^\star(e) + l
\end{aligned}
$$

Therefore $g^\star$ is a bijection as $f^\star$ and $g^\star(E) = \{1 + l, 2 + l, \ldots, q + l\}$. Hence $K_{m,n}$ is semi smooth graceful.

Theorem-2.2 **:** Step grid graph St_n is a smooth graceful graph.

Proof : Let $G = St_n$ be a step grid graph of size n. Where mention each vertices of n^{th} column like $u_{1,j}$ $(1 \leq j \leq n)$, $(n-1)^{th}$ column like $u_{2,j}$ $(1 \leq j \leq n)$, $(n-2)^{th}$ column like $u_{3,j}$ $(1 \leq j \leq n-1)$, $(n-3)^{th}$ column like $u_{4,j}$ $(1 \leq j \leq n-2)$, similarly the first column like $u_{n,j}$ $(j = 1, 2)$. Here we recall that $p = |V(G)| = \frac{1}{2}(n^2 + 3n - 2)$ and $q = |E(G)| = n^2 + n - 2$. Moreover St_n is a bipartite graceful graph (proved by Kaneria and Makadia [7]) with vertex labeling function $f : V(St_n) \longrightarrow \{0, 1, \ldots, q\}$ defined by

$$
\begin{aligned}
f(u_{1,j}) &= \tfrac{q}{2} - \tfrac{1}{8} + (-1)^{j+1}\left[\tfrac{j^2}{4} - \tfrac{1}{8}\right], && \forall\, j = 1, 2, \ldots, n; \\
f(u_{i,j}) &= f(u_{i-1,j-1}) + (-1)^j, && \forall\, i = 2, 3, \ldots, \lfloor \tfrac{n}{2} \rfloor, \\
& && \forall\, j = 1, 2, \ldots, n + i - 1; \\
f(u_{i,1}) &= (n - i + 1)^2 + 1, && \forall\, i = n, n-1, \ldots, \lceil \tfrac{n}{2} \rceil; \\
f(u_{i,2}) &= q - (n - i + 1)(n - i), && \forall\, i = n, n-1, \ldots, \lceil \tfrac{n}{2} \rceil; \\
f(u_{i,j}) &= f(u_{i+1,j-2}) + (-1)^{j-1}, && \forall\, i = n-1, n-2, \ldots, 2, \\
& && \forall\, j = 3, 4, \ldots, n + 2 - i.
\end{aligned}
$$

Let l be any non-negative integer. Define the vertex labeling function $g : V(St_n) \longrightarrow \{0, 1, \ldots, \frac{q}{2} - 1, \frac{q}{2} + l, \frac{q}{2} + 1 + l, \ldots, q + l\}$ such that its induced edge labeling function $g^\star : E(St_n) \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ with $g^\star(e) = |g(u) - g(v)|$, $\forall\, e = (u, v) \in E(St_n)$ defined by

$$
\begin{aligned}
g(w) &= f(w), && \text{when } f(w) < \tfrac{q}{2} \\
&= f(w) + l, && \text{when } f(w) \geq \tfrac{q}{2}.
\end{aligned}
$$

Now for each $e = (u, v) \in E(St_n)$, we see that

$$\begin{aligned} g^\star(e) &= g^\star((u, v)) \\ &= |g(u) - g(v)| \\ &= |f(u) - f(v)| + l, \quad \text{as for any } e = (u, v) \in E(St_n) \text{ one of } \{f(u), f(v)\} \text{ is} \\ &\qquad\qquad \text{less than } \tfrac{q}{2} \text{ and another is greater than or equal to } \tfrac{q}{2}. \end{aligned}$$

$\Rightarrow g^\star(e) = f^\star(e) + l, \forall\ e \in E(St_n)$.

Therefore $g^\star$ is a bijection as $f^\star$ and $g^\star(E) = \{1 + l, 2 + l, \dots, q + l\}$. Thus St_n is smooth graceful.

Theorem-2.3 **:** Path union of t copies of a semi smooth graceful graph H is graceful.

Proof : Let G be a path union of t copies of a semi smooth graceful graph H with $p = |V(H)|$ and $q = |E(H)|$. Let l be an arbitrary non-negative integer and $f : V(H) \longrightarrow \{0, 1, \dots, t-1, t+l, t+l+1, \dots, q+l\}$ be a semi smooth graceful labeling for some $t \in \{0, 1, \dots, q\}$. Then its induced edge labeling function $f^\star : E(H) \longrightarrow \{1+l, 2+l, \dots, q+l\}$ with $f^\star(e) = |f(u) - f(v)|, \forall\ e = (u, v) \in E(H)$ is a bijection.

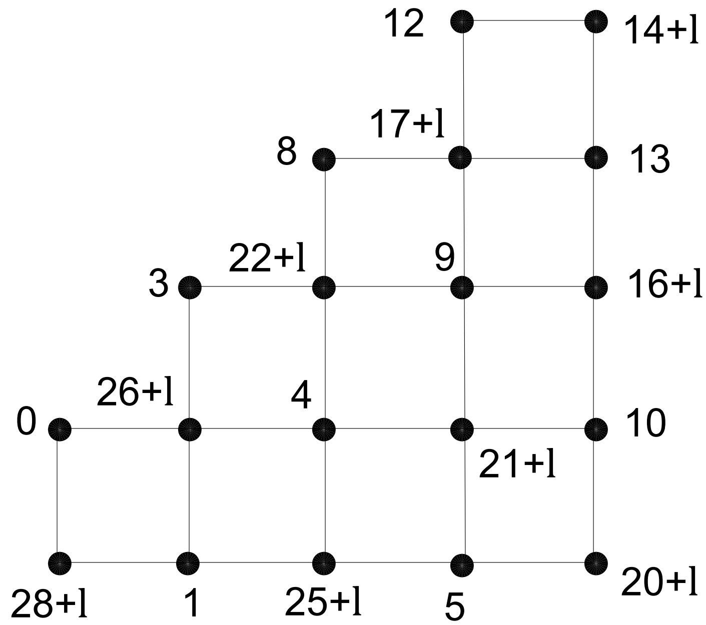
Let $V(H) = \{v_1, v_2, \dots, v_p\}$ and take $v_{i,1}, v_{i,2}, \dots, v_{i,p}$ as vertices for i^{th} copy of H in G, $\forall\ i = 1, 2, \dots, t$ with $v_{1,j} = v_j$, $\forall\ j = 1, 2, \dots, p$ in first copy of H in G. Obviously $P = |V(G)| = tp$ and $Q = |E(G)| = tq + t - 1$.

Define the vertex labeling function $g : V(G) \longrightarrow \{0, 1, \dots, Q\}$ as follows

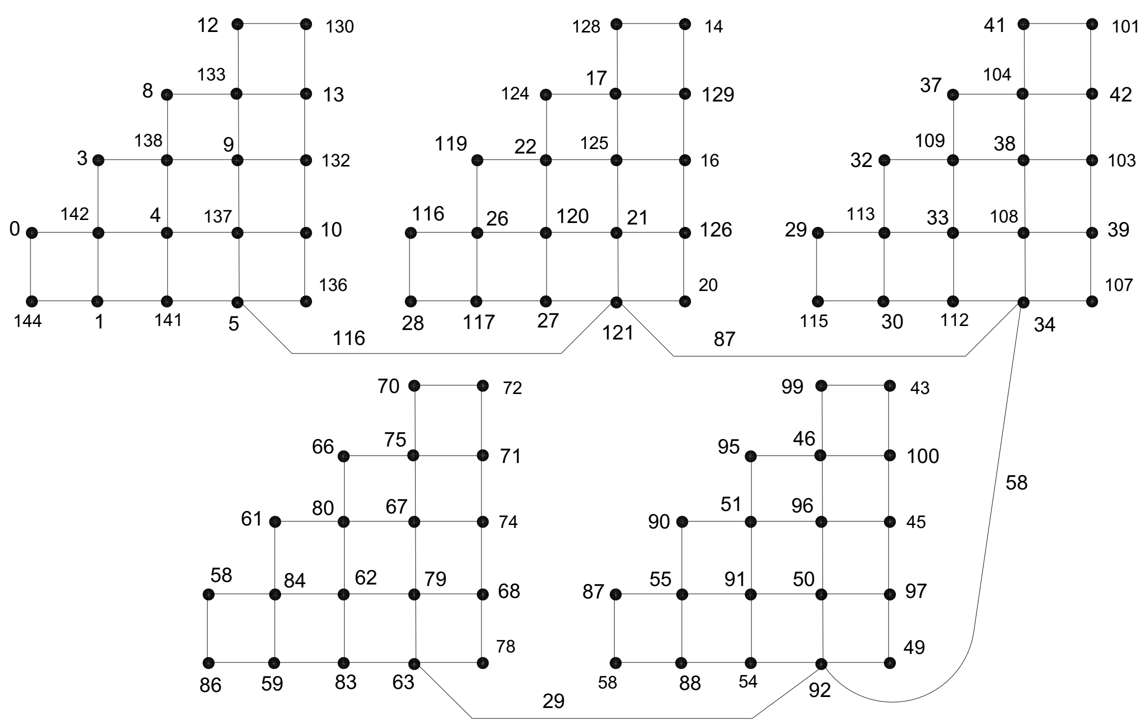
$$\begin{aligned} g(v_{1,j}) &= f(v_j), && \text{if } f(v_j) < t \\ &= f(v_j) + (Q - q) - l, && \text{if } f(v_j) \geq t, \forall\ j = 1, 2, \dots, p; \\ g(v_{2,j}) &= g(v_{1,j}) + (Q - q), && \text{if } g(v_{1,j}) < \tfrac{Q}{2} \\ &= g(v_{1,j}) - (Q - q), && \text{if } g(v_{1,j}) > \tfrac{Q}{2}, \forall\ j = 1, 2, \dots, p; \\ g(v_{i,j}) &= g(v_{i-2,j}) + (q + 1), && \text{if } g(v_{i-2,j}) < \tfrac{Q}{2} \\ &= g(v_{i-1,j}) - (q + 1), && \text{if } g(v_{i-2,j}) > \tfrac{Q}{2}, \forall\ i = 3, 4, \dots, t, \forall\ j = 1, 2, \dots, p. \end{aligned}$$

Now choose a vertex v of $H^{(i)}$ and each corresponding vertex to v in each copy $H^{(i+1)}$ join by an edge to form path union G, $\forall\ i = 1, 2, \dots, t-1$. The above edge labeling function g give rise graceful labeling for G. Thus G is graceful.

Illustration-2.4 **:** Semi smooth graceful labeling for St_5 and graceful labeling for path union of 5 copies of St_5 are shown in *figure*-3 and *figure*-4 respectively.



Figure−3 Smooth graceful labeling for St_5.



Figure−4 Graceful labeling for path union of 5 copies of St_5.

Theorem-2.5 **:** Star of a semi smooth graceful graph is graceful.

Proof : Let H be a semi smooth graceful graph and $G = H^{\star}$, where $p = |V(H)|$, $q = |E(H)|$. Let l be an arbitrary non-negative integer and $f : V(H) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$ be a semi smooth graceful labeling for for H, where $t \in \{0, 1, \ldots, q\}$. Let $V(H) = \{v_1, v_2, \ldots, v_p\}$.

Let $v_{0,1} = v_1, v_{0,2} = v_2, \ldots, v_{0,p} = v_p$ be vertices of the central copy H in G. Take $v_{i,j}$ $(1 \leq j \leq p)$ as vertices for i^{th} copy $H^{(i)}$ in G, $\forall\ i = 1, 2 \ldots, p$. Define the vertex labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$, where $Q = pq + p + q$ as follows

$$\begin{aligned} g(v_{0,j}) &= f(v_j), && \text{if } f(v_j) < t \\ &= f(v_j) + (Q - q) - l, && \text{if } f(v_j) \geq t, \qquad \forall\ j = 1, 2, \ldots, p; \\ g(v_{1,j}) &= g(v_{0,j}) + (Q - q), && \text{if } g(v_{0,j}) < \tfrac{Q}{2} \\ &= g(v_{0,j}) - (Q - q), && \text{if } g(v_{0,j}) > \tfrac{Q}{2}, \qquad \forall\ j = 1, 2, \ldots, p; \\ g(v_{i,j}) &= g(v_{i-2,j}) + (q + 1), && \text{if } g(v_{i-2,j}) < \tfrac{Q}{2} \\ &= g(v_{i-2,j}) - (q + 1), && \text{if } g(v_{i-2,j}) > \tfrac{Q}{2}, \end{aligned}$$

$$\forall\ i = 2, 3, \ldots, p,\ \forall\ j = 1, 2, \ldots, p.$$

Now join each vertex of central copy $H^{(0)}$ with its corresponding vertex of other copies $H^{(i)}$ by an edge, $\forall\ i = 1, 2, \ldots, p$. Above labeling pattern give rise graceful labeling to the graph G and so it is graceful.

Illustration-2.6 **:** Semi smooth graceful labeling for $H = C_{12}$ with twin chords and graceful labeling for the star of H are shown in *figure*-5 and *figure*-6 respectively.

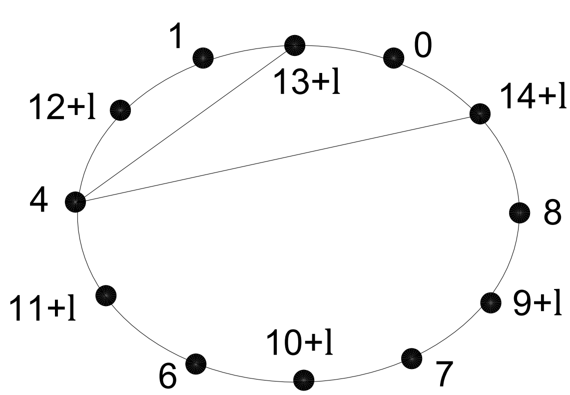
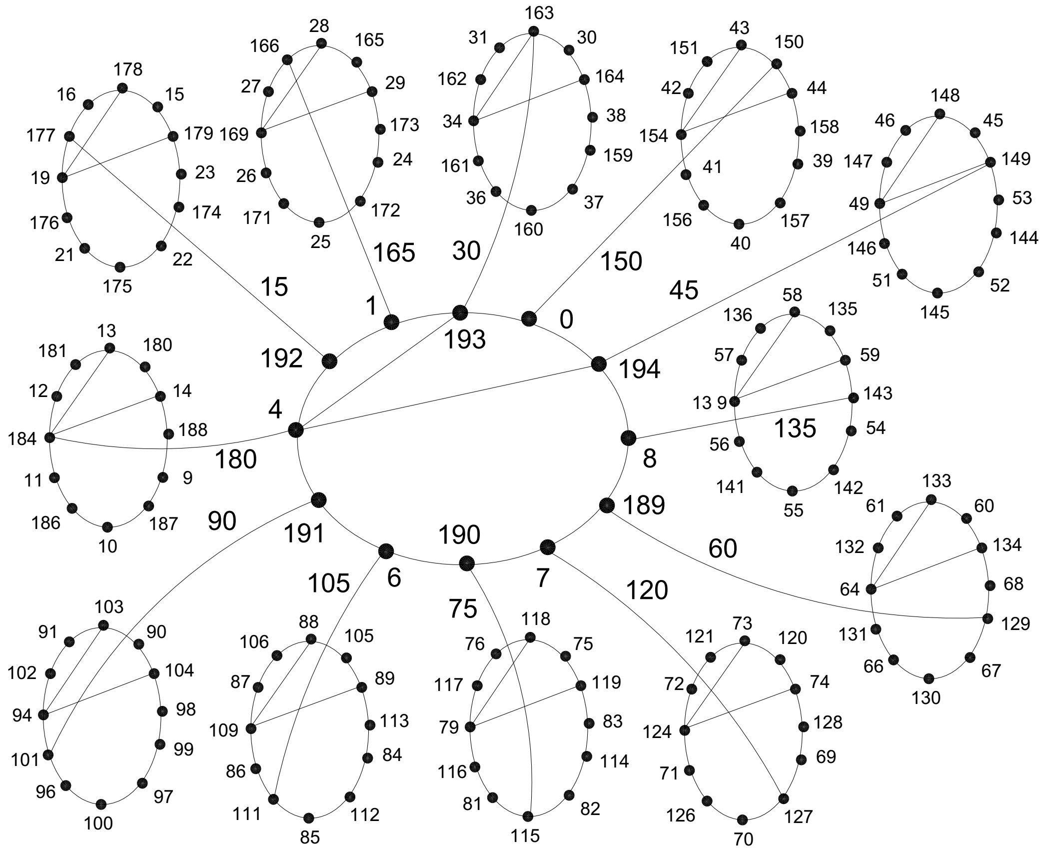


Figure-5 Semi smooth graceful labeling for $H = C_{12}$ with twin chords.



Figure−6 Graceful labeling for $H^{\star}$, where H is a cycle C_{12} with twin chords.

Theorem−2.7 **:** $C(t\cdot H)$ is graceful, where H is a semi smooth graceful graph and $t\equiv 0,3 \pmod 4$.

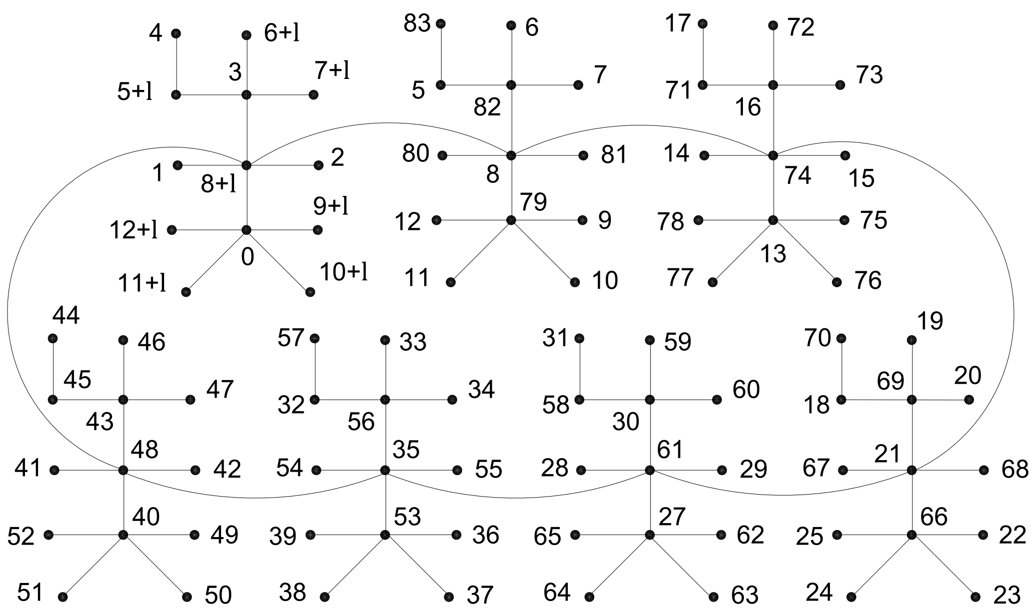
Proof : Let G be a cycle graph formed by t copies of a semi smooth graceful graph H, $t\equiv 0,3 \pmod 4$. Let $p=|V(H)|$, $q=|E(H)|$, $V(H)=\{v_1,v_2,\ldots,v_p\}$. For an arbitrary non-negative integer l, let $f:V(H)\longrightarrow\{0,1,\ldots,t-1,t+l,t+l+1,\ldots,q+l\}$ be a semi smooth graceful labeling for some $t\in\{0,1,\ldots,q\}$.

Obviously $P=|V(G)|=pt$ and $Q=|E(G)|=t(q+1)$. Let $u_{i,j}$ $(1\le j\le p)$ be vertices of i^{th} copy of $H^{(i)}$ in G, $\forall\ i=1,2,\ldots,t$ with $u_{1,j}=v_j$, $\forall\ j=1,2,\ldots,p$. Join $u_{i,k}$ with $u_{i+1,k}$ by an edge, $\forall\ i=1,2,\ldots,t-1$ and $u_{t,k}$ with $u_{1,k}$ to form cycle graph $C(t\cdot H)$, for some $k\in\{1,2,\ldots,p\}$. Define vertex labeling function $g:V(G)\longrightarrow\{0,1,\ldots,Q\}$ as follows

$$
\begin{aligned}
g(u_{1,j}) &= f(v_j), && \text{if } f(v_j) < t \\
&= f(v_j) + (Q-q) - l, && \text{if } f(v_j) \geq t, \qquad \forall\, j = 1,2,\ldots,p; \\
g(u_{2,j}) &= g(u_{1,j}) + (Q-q), && \text{if } g(u_{1,j}) < \tfrac{Q}{2} \\
&= g(u_{1,j}) - (Q-q), && \text{if } g(u_{1,j}) > \tfrac{Q}{2}, \qquad \forall\, j = 1,2,\ldots,p; \\
g(u_{i,j}) &= g(u_{i-2,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\
&= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2}, \\
& && \forall\, i = 3,4,\ldots,\lceil \tfrac{t}{2} \rceil, \ \forall\, j = 1,2,\ldots,p; \\
g(u_{\lceil \frac{t}{2} \rceil+1,j}) &= g(u_{\lceil \frac{t}{2} \rceil-1,j}) + (q+2), && \text{if } g(u_{\lceil \frac{t}{2} \rceil-1,j}) < \tfrac{Q}{2} \\
&= g(u_{\lceil \frac{t}{2} \rceil-1,j}) - (q+1), && \text{if } g(u_{\lceil \frac{t}{2} \rceil-1,j}) > \tfrac{Q}{2}, \qquad \forall\, j = 1,2,\ldots,p; \\
g(u_{\lceil \frac{t}{2} \rceil+2,j}) &= g(u_{\lceil \frac{t}{2} \rceil,j}) + (q+2), && \text{if } g(u_{\lceil \frac{t}{2} \rceil,j}) < \tfrac{Q}{2} \\
&= g(u_{\lceil \frac{t}{2} \rceil,j}) - (q+1), && \text{if } g(u_{\lceil \frac{t}{2} \rceil,j}) > \tfrac{Q}{2}, \qquad \forall\, j = 1,2,\ldots,p; \\
g(u_{i,j}) &= g(u_{i-2,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\
&= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2}, \\
& && \forall\, i = \lceil \tfrac{t}{2} \rceil + 3, \lceil \tfrac{t}{2} \rceil + 4, \ldots, t, \ \forall\, j = 1,2,\ldots,p.
\end{aligned}
$$

Above labeling pattern give rise a graceful labeling to the graph $C(t \cdot H)$ and so it is graceful.

Illustration−2.8 : Cycle of a tree on 13 vertices with 7 copies and its graceful labeling shown in *figure*−7.



Figure−7 Graceful labeling for $C(7 \cdot H)$, where H is a tree on 13 vertices and take $l = (Q - q) = 79$.

Theorem-2.9 **:** Every semi smooth graceful graph is odd graceful.

Proof : Let G be a semi smooth graceful graph with semi smooth vertex labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$, whose induced edge labeling function $g^{\star} : E(G) \longrightarrow \{1+l, 2+l, \ldots, q+l\}$ defined by $g^{\star}(e) = |g(u)-g(v)|$, $\forall\ e = (u, v) \in E(G)$, for some $t \in \{1, 2, \ldots, q\}$ and an arbitrary non-negative integer l.

Since G is a bipartite graph, we will take $V(G) = V_1 \cup V_2$ (where $V_1 \neq \phi$, $V_2 \neq \phi$ and $V_1 \cap V_2 = \phi$) and there is no edge $e \in E(G)$ whose both end vertices lies in V_1 or V_2. Moreover

$$\{g(u)\ /\ u \in V_1\} \subseteq \{1, 2, \ldots, t-1\}$$
$$\{g(u)\ /\ u \in V_2\} \subseteq \{t+l, t+l+1, \ldots, q+l\}.$$

Otherwise by taking l sufficiently large, the induced edge function $g^{\star}$ produce edge label which is less than l, gives a contradiction that G admits a semi smooth graceful labeling g.

Now define $h : V(G) \longrightarrow \{0, 1, 2, \ldots, 2q-1\}$ as follows

$$h(u) = 2 \cdot g(u), \qquad \forall\ u \in V_1 \text{ and}$$
$$h(v) = 2 \cdot g(v) - 1 - 2l, \qquad \forall\ v \in V_2.$$

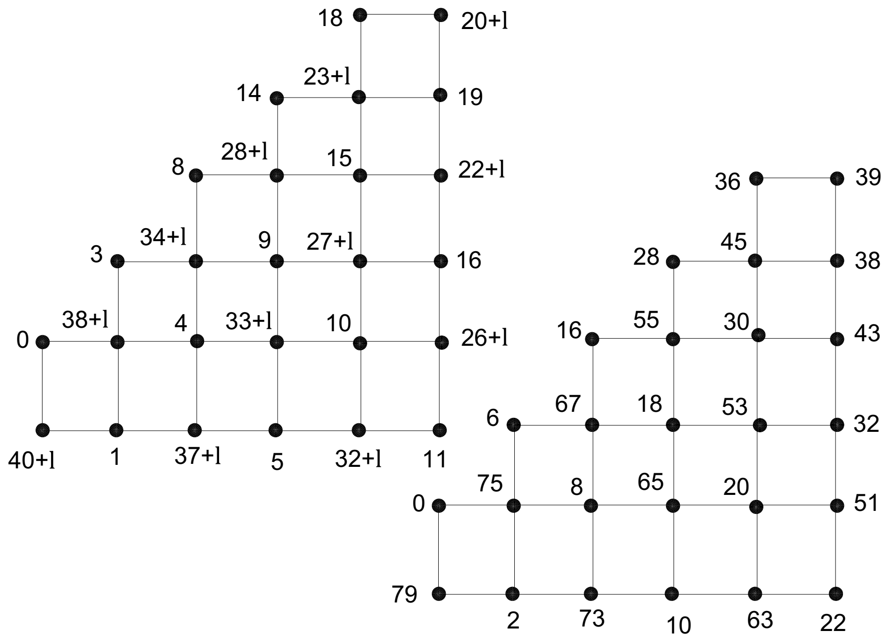
Above labeling function h give rise odd graceful labeling to the graph G. Because for any edge $e = (u, v) \in E(G)$ [where $u \in V_1$ and $v \in V_2$], $g^{\star}(e) = i + l$, for some $i \in \{1, 2, \ldots, q\}$.

Also for any $e = (u, v) \in E(G)$

$$\begin{aligned} h^{\star}(e) &= h(v) - h(u) \\ &= 2g(v) - (1+2l) - 2g(u) \\ &= 2(g(v) - g(u)) - (1+2l) \\ &= 2|g(v) - g(u)| - (1+2l) \\ &= 2g^{\star}(e) - (1+2l) \\ &= 2(i+l) - (1+2l) \\ &= 2i - 1. \end{aligned}$$

Thus G is an odd graceful graph.

Illustration-2.10 **:** Semi smooth graceful labeling and odd graceful labeling for St_6 are shown in *figure*-8.



Figure−8 Semi smooth graceful labeling and odd graceful labeling for St_6.

Theorem−2.11 **:** Cycle graph $C(t \cdot H)$ is smooth graceful, when $t \equiv 0 \pmod 4$ and H is a semi smooth graceful graph.

Proof : It is obvious that if we join two bipartite graphs by a path then the resultant graph is also bipartite graph. So $P(k \cdot H)$ are bipartite graphs, $\forall\ k = 2, 3, \ldots, t$, as H is a bipartite graph. To get $C(t \cdot H)$, we have to add one more edge in $P(t \cdot H)$ between first and last copy of H in $P(t \cdot H)$. Thus we have to add t edge in $\bigcup_{i=1}^{t} H^{(i)}$ for the construction of $C(t \cdot H)$. If these added edges does not form a cycle of odd length then $C(t \cdot H)$ is also a bipartite graph.

Here we will add t edges to $\bigcup_{i=1}^{t} H^{(i)}$ between corresponding vertices in each copy to a vertex v from H to form the cycle graph $C(t \cdot H)$, where $t \equiv 0 \pmod 4$. Thus $C(\cdot H)$ is a bipartite graph.

In *Theorem*−2.7 we proved that $C(t \cdot H)$ is a graceful graph with the vertex labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ defined as follows

$$\begin{aligned}
g(u_{1,j}) &= f(v_j), && \text{if } f(v_j) < t \\
&= f(v_j) + (Q - q) - l, && \text{if } f(v_j) \geq t, \qquad \forall\ j = 1, 2, \ldots, p; \\
g(u_{2,j}) &= g(u_{1,j}) + (Q - q), && \text{if } g(u_{1,j}) < \tfrac{Q}{2} \\
&= g(u_{1,j}) - (Q - q), && \text{if } g(u_{1,j}) \geq \tfrac{Q}{2}, \qquad \forall\ j = 1, 2, \ldots, p;
\end{aligned}$$

$$
\begin{aligned}
g(u_{i,j}) &= g(u_{i-2,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\
&= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2}, \\
& && \forall\ i = 3, 4, \ldots, \tfrac{t}{2}, \forall\ j = 1, 2, \ldots, p; \\
g(u_{\frac{t}{2}+1,j}) &= g(u_{\frac{t}{2}-1,j}) + (q+2), && \text{if } g(u_{\frac{t}{2}-1,j}) < \tfrac{Q}{2} \\
&= g(u_{\frac{t}{2}-1,j}) - (q+1), && \text{if } g(u_{\frac{t}{2}-1,j}) > \tfrac{Q}{2}, \quad \forall\ j = 1, 2, \ldots, p; \\
g(u_{\frac{t}{2}+2,j}) &= g(u_{\frac{t}{2},j}) + (q+2), && \text{if } g(u_{\frac{t}{2},j}) < \tfrac{Q}{2} \\
&= g(u_{\frac{t}{2},j}) - (q+1), && \text{if } g(u_{\frac{t}{2},j}) > \tfrac{Q}{2}, \quad \forall\ j = 1, 2, \ldots, p; \\
g(u_{i,j}) &= g(u_{i-2,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\
&= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2}, \\
& && \forall\ i = \tfrac{t}{2}+3, \tfrac{t}{2}+4, \ldots, t, \forall\ j = 1, 2, \ldots, p.
\end{aligned}
$$

Where $Q = t(q+1)$, $q = |E(H)|$, $p = |V(H)|$ and $f : V(H) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$ ($t \in \{1, 2, \ldots, q\}$ and l be an arbitrary non-negative integer) be a semi smooth graceful labeling for the semi smooth graceful graph H.

From above defined labeling patter g on $C(t \cdot H)$, we can see that for any $e = (u, v) \in E(C(t \cdot H))$, either $g(u) < \frac{Q}{2}$ and $g(v) \geq \frac{Q}{2}$ or $g(u) \geq \frac{Q}{2}$ and $g(v) < \frac{Q}{2}$. Thus if we define $h : V(C(t \cdot H)) \longrightarrow \{0, 1, \ldots, \frac{Q}{2}-1, \frac{Q}{2}+l, \frac{Q}{2}+l+1, \ldots, Q+l\}$ as follows.

$h(v) = g(v)$, when $g(v) < \frac{Q}{2}$

$= g(v) + l$, when $g(v) \geq \frac{Q}{2}$, for any arbitrary non-negative integer l, we can observe that the induced edge labeling function $h^\star : E(C(t \cdot H)) \longrightarrow \{1+l, 2+l, \ldots, Q+l\}$ defined by $h^\star(e = (u, v)) = |h(u) - h(v)| = |g(u) - g(v)| + l = g^\star(e) + l$, becomes a bijection, as $h^\star(E(C(t \cdot H))) = \{1+l, 2+l, \ldots, Q+l\}$ and so h is a smooth graceful labeling to the graph $C(t \cdot H)$ and hence $C(t \cdot H)$ is a smooth graceful graph.

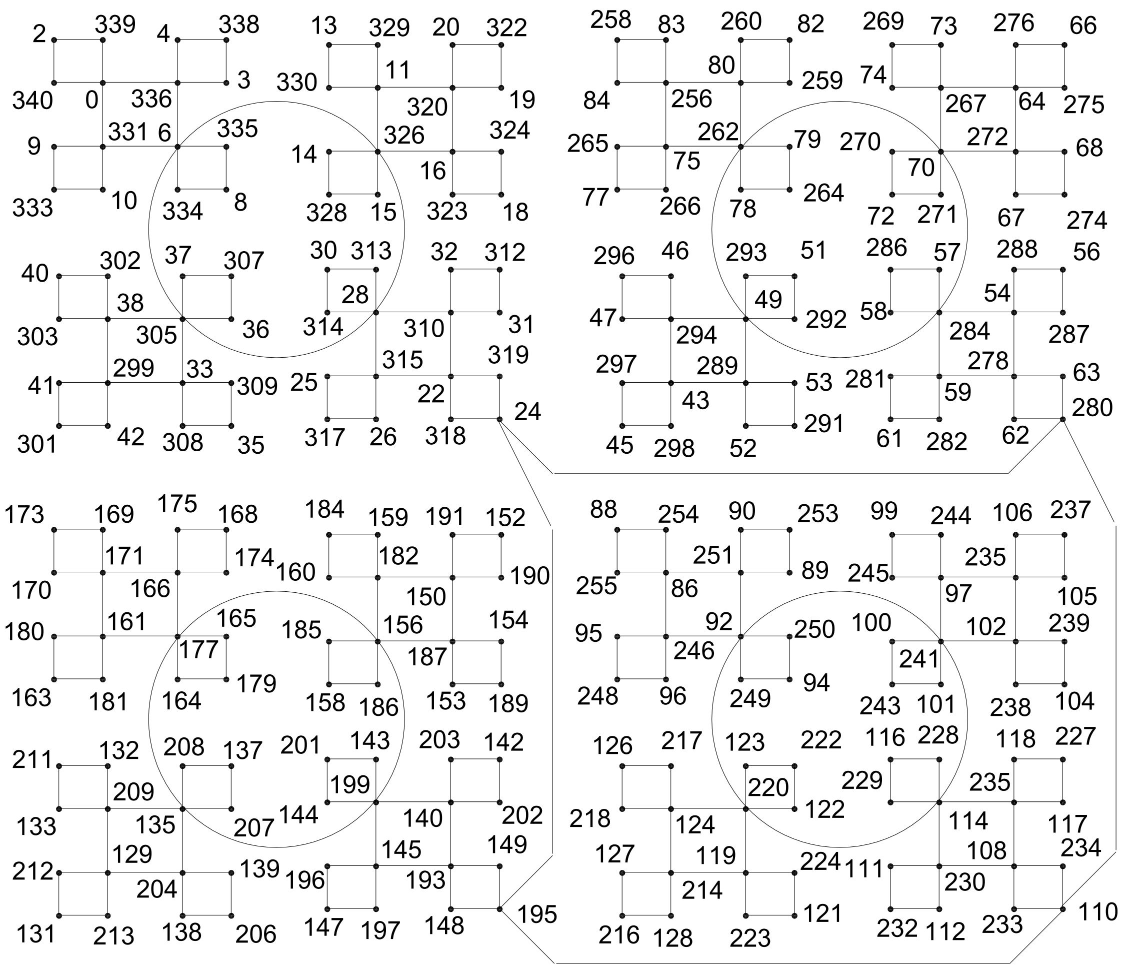
Theorem-2.12 **:** $C^t(m \cdot C_n)$ is graceful, when $m, n \equiv 0 \pmod 4$ and $t \in N$.

Proof : Since C_n ($n \equiv 0 \pmod 4$) is a smooth graceful graph, by *Theorem*-2.11 $C(m \cdot C_n)$ is also a smooth graceful graph, where $m, n \equiv 0 \pmod 4$. By applying same argument $C(m \cdot C(m \cdot C_n)) = C^2(m \cdot C_n)$ is a smooth graceful graph.

Similarly $C(m \cdot C^2(m \cdot C_n)) = C^3(m \cdot C_n)$, …, $C(m \cdot C^{t-1}(m \cdot C_n)) = C^t(m \cdot C_n)$ are smooth graceful graphs, where $m, n \equiv 0 \pmod 4$ and $t \in N$.

Particularly $C^t(m \cdot C_n)$ is graceful graph, where $m, n \equiv 0 \pmod 4$ and $t \in N$.

Illustration−2.13 : $C^3(4 \cdot C_4)$ and its graceful labeling shown in *figure*−9, where $Q = |E(C^3(4 \cdot C_4))| = 340$ and $P = |V(C^3(4 \cdot C_4))| = 256$.



Figure−9 $C^3(4 \cdot C_4)$ and its graceful labeling.

Theorem−2.14 : $P(t \cdot H)$ is a semi smooth graceful graph, where H is a semi smooth graceful graph and $t \in N$.

Proof : It is obvious that $P(t \cdot H)$ is a bipartite graph as H is bipartite. In *Theorem*−2.3 we proved that path union of t copies of semi smooth graceful graph is also graceful. Let $G = P(t \cdot H)$ and $V(H) = \{v_1, v_2, \ldots, v_p\}$ with $p = |V(H)|$, $q = |E(H)|$.

Since H is a semi smooth graceful graph, it admits a semi smooth graceful labeling say $f : V(H) \longrightarrow \{0, 1, \ldots, y-1, y+l, y+l+1, \ldots, q+l\}$ for some $y \in \{1, 2, \ldots, q\}$ and an arbitrary non-negative integer $l \in N$.

Let $Q = tq + t - 1$ and k be an non-negative integer. Take $w = \frac{tq}{2}$, when t is even or $\lfloor \frac{t}{2} \rfloor q + y$, when t is odd. Define the vertex labeling function $h : V(G) \longrightarrow \{0, 1, \ldots, w-1, w+k, w+k+1, \ldots, Q+k\}$ as follows

$$\begin{aligned}
h(v_{1,j}) &= f(v_j), && \text{if } f(v_j) < y \\
&= f(v_j) + (Q - q) + (k - y), && \text{if } f(v_j) \geq y, \forall\, j = 1, 2, \ldots, p; \\
h(v_{2,j}) &= h(v_{1,j}) + (Q - q) + k, && \text{if } h(v_{1,j}) < \tfrac{Q}{2} \\
&= h(v_{1,j}) - (Q - q + k), && \text{if } h(v_{1,j}) > \tfrac{Q}{2}, \forall\, j = 1, 2, \ldots, p; \\
h(v_{i,j}) &= h(v_{i-2,j}) + (q + 1 + k), && \text{if } h(v_{i-2,j}) < \tfrac{Q}{2} \\
&= h(v_{i-2,j}) - (q + 1 + k), && \text{if } h(v_{i-2,j}) > \tfrac{Q}{2}, \\
& && \forall\, i = 3, 4, \ldots, t, \forall\, j = 1, 2, \ldots, p.
\end{aligned}$$

Above labeling pattern give rise semi smooth graceful labeling to the graph $G = P(t \cdot H)$. Thus G is a semi smooth graceful graph.

Theorem-2.15 **:** $P^t(k \cdot T)$ is graceful tree, where T is a semi smooth graceful tree and $t, k \in N$.

Proof : We have T is a semi smooth graceful tree. By last *Theorem*-2.14 $P(k \cdot T)$ is also semi smooth graceful tree, as path union of k copies of a tree is also a tree. Applying similar theory $P^t(k \cdot T)$ is semi smooth graceful. Therefore it is a graceful tree.

Illustration-2.16 **:** $P^3(2 \cdot T)$ and its graceful labeling shown in *figure*-10, where $Q = |E(P^3(2 \cdot T))| = 103$ and $P^3(2 \cdot T)$ contains 8 copies of T inside it.

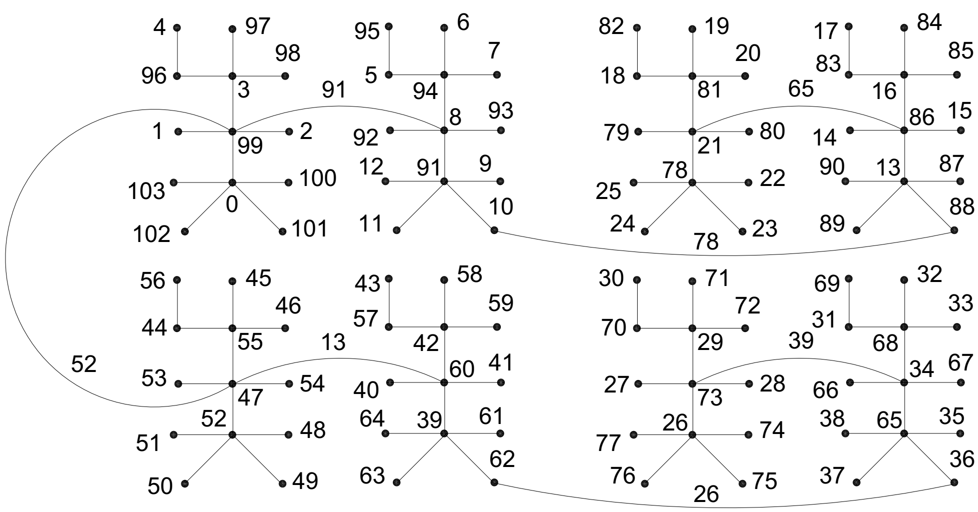
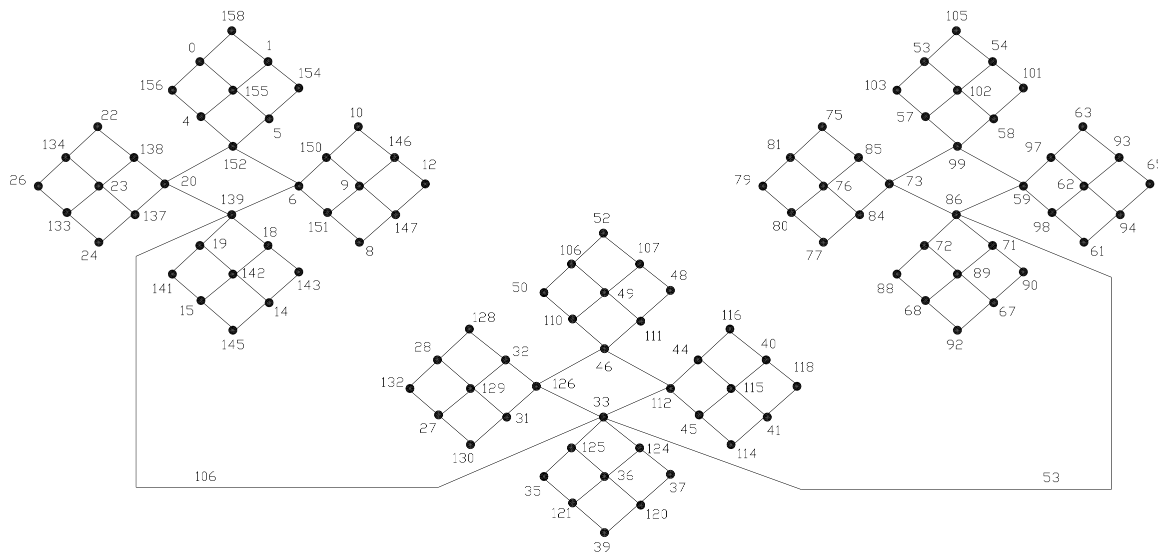


Figure-10 $P^3(2 \cdot T)$ and its graceful labeling, where T is a semi smooth graceful tree given in *figure*-2.

Corollary−2.17 : $P(k\cdot C(t\cdot P_n\times P_m))$ is graceful, where $t\equiv 0 \pmod 4$ and $m, n, k \in N$.

Proof : This follows from *Theorem*−1.11 and 1.14, as $P_n \times P_m$ is a smooth graceful graph proved by Kaneria and Jariya[4].

Illustration−2.18 : $P(3\cdot C(4\cdot P_3\times P_3))$ and its graceful labeling shown in *figure*−11, where $|E(P(3\cdot C(4\cdot P_3\times P_3)))| = 158$ and $|V(P(3\cdot C(4\cdot P_3\times P_3)))| = 108$.



Figure−11 $P(3\cdot C(4\cdot P_3\times P_3))$ and its graceful labeling.

Theorem−2.19 : $\bigcup_{i=1}^{t} P_{n_i}\times P_{m_i}$ is graceful, where $n_i (1\leq i\leq t)$, $m_i (1\leq i\leq t)$, $t\in N$.

Proof : Let $G=\bigcup_{i=1}^{t}(P_{n_i}\times P_{m_i})$, where $P_{n_i}\times P_{m_i}$ is the grid graph on $n_i\times m_i$ vertices and $q_i=|E(P_{n_i}\times P_{m_i})|=2m_in_i-(m_i+n_i)$, $\forall\ i=1,2,\dots,t$.

Let $u_{i,j,k}$ $(1\leq j\leq n_i, 1\leq k\leq m_i)$ be the vertices of $P_{n_i}\times P_{m_i}$ (assuming $m_i\geq n_i$), $\forall\ i=1,2,\dots,t$. Obviously $P=|V(G)|=\sum_{i=1}^{t}p_i$, where $p_i=|V(P_{n_i}\times P_{m_i})|=m_in_i$, $\forall\ i=1,2,\dots,t$ and $Q=|E(G)|=\sum_{i=1}^{t}q_i$. Kaneria and Jariya [4] proved that $P_{n_i}\times P_{m_i}$ $(i\leq i\leq t)$ are smooth graceful graphs.

We know that the vertex labeling functions $f_i: V(P_{n_i}\times P_{m_i})\longrightarrow\{0,1,\dots,q_i\}$ defined by

$$
\begin{aligned}
f(u_{i,j,1}) &= q_i-\tfrac{(j-1)^2}{2}, && \text{when } j \text{ is odd,}\\
&= \tfrac{j(j-2)}{2}, && \text{when } j \text{ is even, } \forall\ j=1,2,\dots,n_i;\\
f(u_{i,j,m_i}) &= \tfrac{q_i}{2}-\tfrac{1}{4}+(-1)^{m_i+j}[\tfrac{(n_i-j)^2}{2}+\tfrac{1}{4}], && \forall\ j=n_i,n_i-1,\dots,1;\\
f(u_{i,j,k}) &= f(u_{i,j-1,k+1})+(-1)^{j+k}, && \forall\ k=m_i-1,m_i-2,\dots,m_i+1-n_i,\\
& && \forall\ j=n_i,n_i-1,\dots,m_i+1-k;
\end{aligned}
$$

$f(u_{i,n_i,k}) = f(u_{i,n_i,1}) + (-1)^{n_i}[\frac{(2n_i-1)(k-1)}{2}]$, when k is odd,

$= f(u_{i,n_i-1,1}) - (-1)^{n_i}[\frac{(2n_i-1)k}{2}]$, when k is even,

$\forall\ k = 2, 3, \ldots, m_i - n_i$;

$f(u_{i,j,k}) = f(u_{i,j+1,k-1}) - (-1)^{j+k}$, $\forall\ k = 2, 3, \ldots, m_i - 1$,

$\forall\ j = 1, 2, \ldots, min\{n_i, m_i - k\}$; $\forall\ i = 1, 2, \ldots, t$ are graceful.

Using these we shall define $g_i : V(P_{n_i} \times P_{m_i}) \longrightarrow \{0, 1, \ldots, \lceil \frac{q_i}{2} \rceil - 1, \lceil \frac{q_i}{2} \rceil + l, \lceil \frac{q_i}{2} \rceil + 1 + l, \ldots, q_i + l\}$ by

$g_i(u) = f_i(u)$, when $f_i(u) < \frac{q_i}{2}$,

$= f_i(u) + l$, when $f_i(u) \geq \frac{q_i}{2}$, $\forall\ u \in V(P_{n_i} \times P_{m_i})$

and $\forall\ i = 1, 2, \ldots, t$,

where l is an arbitrary non negative integer. Which are smooth graceful labeling function $\forall\ i = 1, 2, \ldots, t$.

Define for each $k = 2, 3, \ldots, t$, $h_k : V(\bigcup_{i=1}^{k} P_{n_i} \times P_{m_i}) \longrightarrow \{0, 1, \ldots, \sum_{i=1}^{k} q_i\}$ as follows, assuming $h_1 = f_1$.

$h_k(w_k) = g_k(w_k)$, when $g_k(w_k) < \frac{q_k}{2}$,

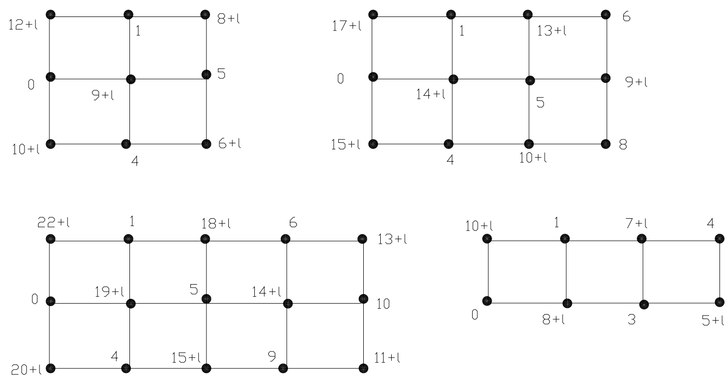
$= g_k(w_k) + \sum_{i=1}^{k-1} q_i - l$, when $g_k(w_k) \geq \frac{q_k}{2}$, $\forall\ w_k \in V(P_{n_i} \times P_{m_i})$;

$h_k(w) = h_{k-1}(w) + (\frac{q_k}{2} + 1)$, when q_k is even,

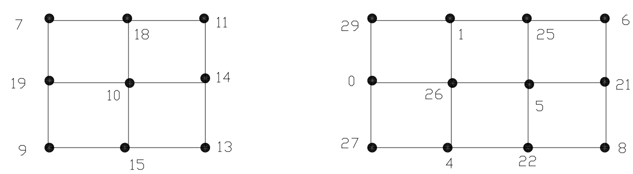
$= \sum_{i=1}^{k-1} q_i - h_{k-1}(w) + (\frac{q_k-3}{2})$, when q_k is odd, $\forall\ w \in V(\bigcup_{i=1}^{k-1} P_{n_i} \times P_{m_i})$.

Above defined labeling pattern give rise graceful labeling h_k to each disconnected graph $\bigcup_{i=1}^{k} P_{n_i} \times P_{m_i}$, $\forall\ k = 2, 3, \ldots, t$. Thus $\bigcup_{i=1}^{k} P_{n_i} \times P_{m_i}$ are disconnected graceful graphs, $\forall\ k = 2, 3, \ldots, t$. Particularly $\bigcup_{i=1}^{t} P_{n_i} \times P_{m_i}$ is graceful.

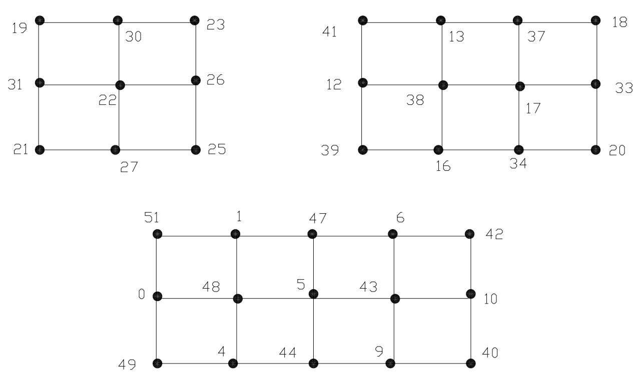
Illustration-2.20 **:** To get graceful labeling for $G = (P_3 \times P_3) \cup (P_3 \times P_4) \cup (P_3 \times P_5) \cup (P_2 \times P_4)$, we have $Q = 12 + 17 + 22 + 10 = 61, q_1 = 12, q_2 = 17, q_3 = 22, q_4 = 10$, we have computed smooth graceful labelings for $P_3 \times P_3$, $P_3 \times P_4$, $P_3 \times P_5$ and $P_2 \times P_4$ in *figure*-12 by smooth vertex labeling functions $g_i (1 \leq i \leq 4)$ respectively, graceful labeling for $(P_3 \times P_3) \cup (P_3 \times P_4)$ by h_2, in *figure*-13, graceful labeling for $(P_3 \times P_3) \cup (P_3 \times P_4) \cup (P_3 \times P_5)$ by h_3 in *figure*-14 and graceful labeling for G by h_4 in *figure*-15.



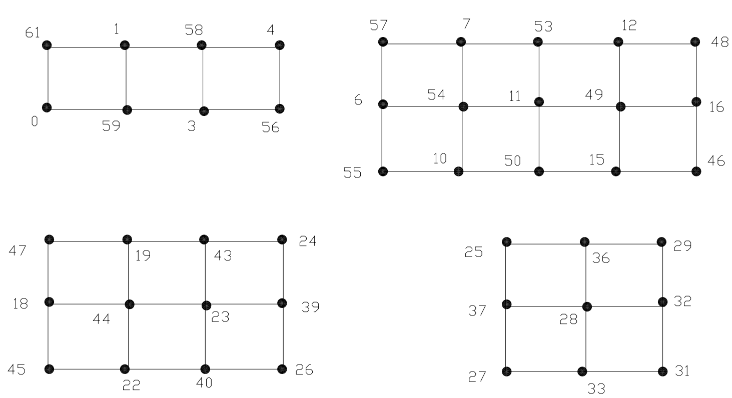
Figure−12 smooth graceful labeling for $P_3 \times P_3$, $P_3 \times P_4$, $P_3 \times P_5$, and $P_2 \times P_4$.



Figure−13 Graceful labeling h_2 for the graph $(P_3 \times P_3) \cup (P_3 \times P_4)$.



Figure−14 Graceful labeling h_3 for the graph $(P_3 \times P_3) \cup (P_3 \times P_4) \cup (P_3 \times P_5)$.



Figure−15 Graceful labeling h_4 for G.

Theorem−2.21 : $< C_{n_1}, P_{n_2}, C_{n_3}, \ldots, P_{n_{2t}}, C_{n_{2n+1}} >$ is graceful, when $n_i \equiv 0 \pmod 4$, $\forall\ i = 1, 3, \ldots, 2t+1$ and $n_i \in N$, $\forall\ i = 1, 2, \ldots, 2t+1$.

Proof : Obviously $p_i = q_i = |V(C_{n_i})| = |E(C_{n_i})| = n_i$, $\forall\ i = 1, 3, \ldots, 2t+1$ and $p_j = |V(P_{n_j})| = n_j$, $q_j = |E(P_{n_j})| = n_j - 1$, $\forall\ j = 2, 4, \ldots, 2t$. Let $G =< C_{n_1}, P_{n_2}, C_{n_3}, \ldots, P_{n_{2t}}, C_{n_{2n+1}} >$, $V(C_{n_i}) = \{u_{i,j}/1 \le j \le n_i\}$, $\forall\ i = 1, 3, \ldots, 2t+1$ and $V(P_{n_j}) = \{v_{j,k}/ 1 \le k \le n_j\}$, $\forall\ j = 2, 4, \ldots, 2t$ with $u_{i,1} = v_{i+1,n_{i+1}}$ and $v_{i+1,1} = u_{i+2,n_{i+2}}$ for every $i = 1, 3, \ldots, 2t-1$ to from the connected graph G. Here $P = |V(G)| = \sum_{i=1}^{2t+1} p_i - 2t$ and $Q = |E(G)| = \sum_{j=1}^{2t+1} q_j$.

Let $f_{2i} : V(P_{n_{2i}}) \longrightarrow \{0, 1, \ldots, q_{2i}\}$ be graceful labeling for $P_{n_{2i}}$ defined by

$$\begin{aligned} f_{2i}(v_{2i,k}) &= q_{2i} - (\tfrac{k-1}{2}), && \text{when } k \text{ is odd,} \\ &= (\tfrac{k-2}{2}), && \text{when } k \text{ is even,} \end{aligned}$$

$$\forall\ k = 1, 2, \ldots, p_{2i},\ \forall\ i = 1, 2, \ldots, t.$$

Let $f_{2i+1} : V(C_{n_{2i+1}}) \longrightarrow \{0, 1, \ldots, q_{2i+1}\}$ be graceful labeling for $C_{n_{2i+1}}$ defined by

$$\begin{aligned} f_{2i+1}(u_{2i+1,j}) &= q_{2i+1} - (\tfrac{j-1}{2}), && \text{when } j \text{ is odd,} \\ &= (\tfrac{j-2}{2}), && \text{when } j \text{ is even, and } j \le \tfrac{v_{2i+1}}{2}, \\ &= (\tfrac{j}{2}), && \text{when } j \text{ is even, and } j > \tfrac{v_{2i+1}}{2}, \end{aligned}$$

$$\forall\ j = 1, 2, \ldots, p_{2i+1},\ \forall\ i = 0, 1, \ldots, t.$$

Define for each $k = 2, 4, \ldots, 2t$ (assuming $g_1 = f_1$) $g_k : V(< C_{n_1}, P_{n_2}, \ldots, P_{n_k} >) \longrightarrow \{0, 1, \ldots, \sum_{i=1}^{k} q_i\}$ and $g_{k+1} : V(< C_{n_1}, P_{n_2}, \ldots, C_{n_{k+1}} >) \longrightarrow \{0, 1, \ldots, \sum_{j=1}^{k+1} q_j\}$ as follows.

$$\begin{aligned} g_k(u) &= f_k(u), && \text{when } f_k(u) < \tfrac{q_k}{2}, \\ &= f_k(u) + \textstyle\sum_{i=1}^{k-1} q_i, && \text{when } f_k(u) \ge \tfrac{q_k}{2}\ \forall\ u \in V(P_{n_k}); \\ g_k(w) &= g_{k-1}(w) + \tfrac{q_k}{2}, && \text{when } g_k \text{ is even,} \\ &= \textstyle\sum_{i=1}^{k-1} q_i + (\tfrac{g_k - 1}{2}) - g_{k-1}(w), && \text{when } g_k \text{ is odd;} \end{aligned}$$

$$\forall\ w \in V(< C_{n_1}, P_{n_2}, \ldots, C_{n_{k-1}} >);$$

$$\begin{aligned} g_{k+1}(v) &= f_{k+1}(v), && \text{when } f_{k+1}(v) \le \tfrac{q_{k+1}}{2}, \\ &= f_{k+1}(v) + \textstyle\sum_{j=1}^{k} q_j, && \text{when } f_{k+1}(v) > \tfrac{q_{k+1}}{2},\ \forall\ v \in V(C_{n_{k+1}}); \\ g_{k+1}(w) &= \textstyle\sum_{j=1}^{k} q_j + (\tfrac{g_{k+1}}{2}) - g_k(w), && \forall\ w \in V(< C_{n_1}, P_{n_2}, \ldots, C_{n_{k-1}}, P_{n_k} >). \end{aligned}$$

Above labeling pattern give rise graceful labelings g_k, g_{k+1} to the graphs $< C_{n_1}, P_{n_2}$, $\ldots$, $P_{n_k} >$ and $< C_{n_1}, P_{n_2}, \ldots, C_{n_{k+1}} >$ respectively and so they are graceful graphs, $\forall\ k = 2, 4, \ldots, t$. Particularly $< C_{n_1}, P_{n_2}, \ldots, P_{n_{2t}}, C_{n_{2t+1}} >$ is a graceful graph.

Illustration-2.22 **:** $< C_8, P_{10}, C_4, P_7, C_{12}, P_3, C_8 >$ and its graceful labeling g_7 shown in *figure*-20, for this we have computed graceful labelings $f_1, f_2, \ldots, f_7$ for C_8, P_{10}, C_4, P_7, C_{12}, P_3, C_8 respectively in *figure*-16, $< C_8, P_{10} >$, $< C_8, P_{10}, C_4 >$ and their graceful labelings g_2, g_3 are shown in *figure*-17, $< C_8, P_{10}, C_4, P_7 >$, $< C_8, P_{10}, C_4, P_7, C_{12} >$ and their graceful labeling g_4, g_5 are shown in *figure*-18 and $< C_8, P_{10}, C_4, P_7, C_{12}, P_3 >$ and its graceful labeling g_6 shown in *figure*-19.

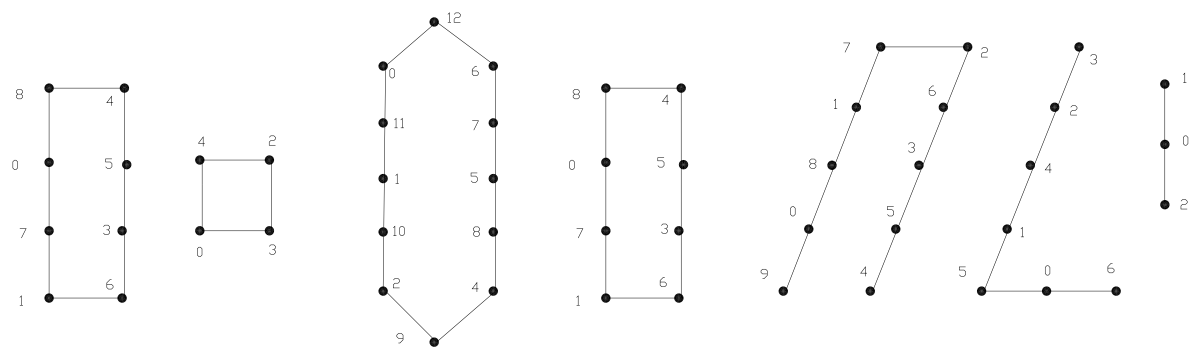


Figure-16 $C_8, P_{10}, C_4, P_7, C_{12}, P_3, C_8$ and its graceful labelings $f_1, f_2, \ldots, f_7$ respectively.

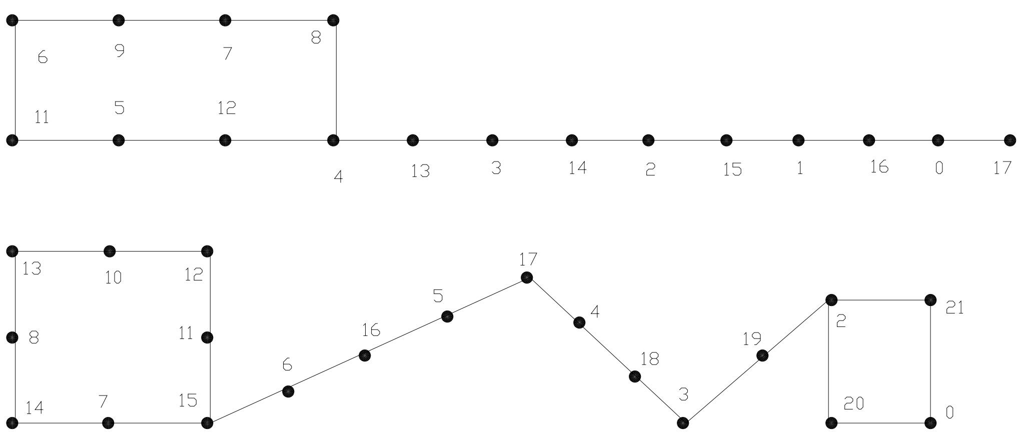
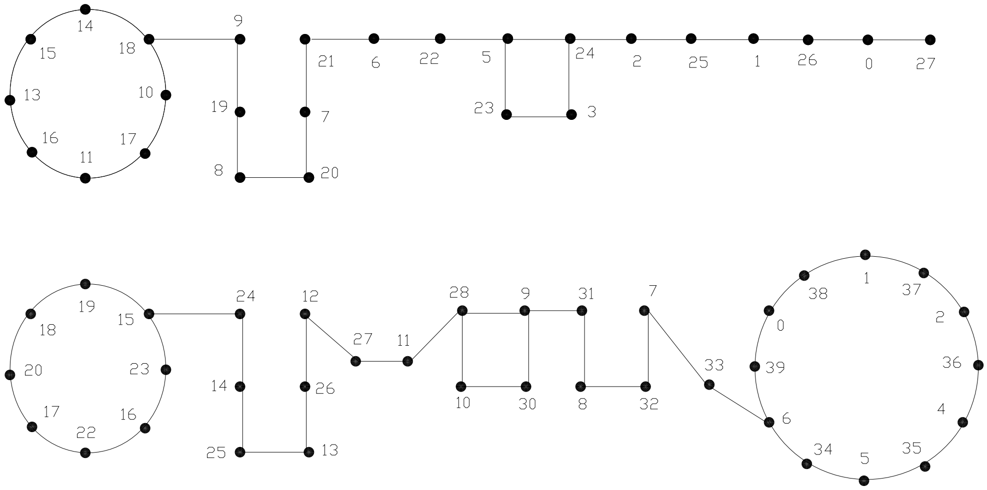
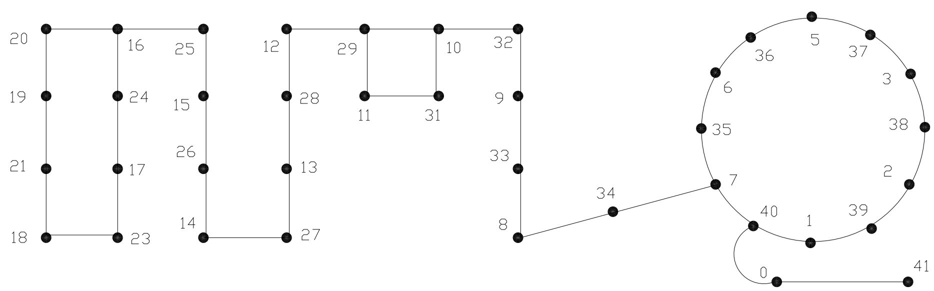


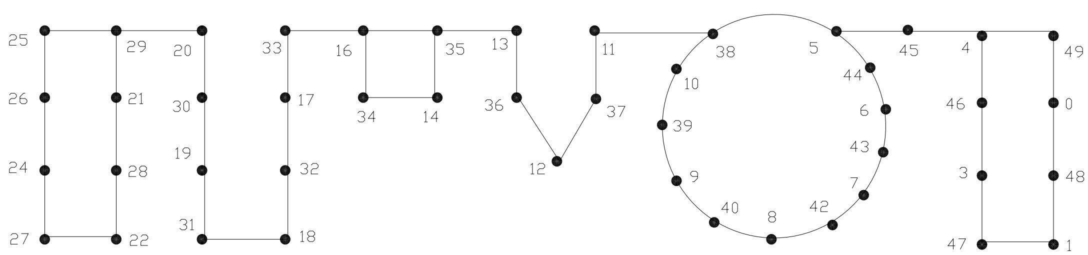
Figure-17 $< C_8, P_{10} >$, $< C_8, P_{10}, C_4 >$ and their graceful labelings g_2, g_3.



Figure−18 $< C_8, P_{10}, C_4, P_7 >$, $< C_8, P_{10}, C_4, P_7, C_{12} >$ and their graceful labelings g_4, g_5.



Figure−19 $< C_8, P_{10}, C_4, P_7, C_{12}, P_3 >$ and their graceful labeling g_6.



Figure−20 G and its graceful labeling g_7.

Theorem−2.23 : $< P_{n_1} \times P_{m_1}, P_{r_1}, P_{n_2} \times P_{m_2}, \ldots, P_{r_{t-1}}, P_{n_t} \times P_{m_t} >$ is graceful.

Proof : Let $G =< P_{n_1} \times P_{m_1}, P_{r_1}, P_{n_2} \times P_{m_2}, \ldots, P_{r_{t-1}}, P_{n_t} \times P_{m_t} >$. It is obvious that $p_i = |V(P_{n_i} \times P_{m_i})| = m_i n_i, \forall\ i = 1, 2, \ldots, t$ and $|V(P_{r_j})| = r_j, \forall\ j = 1, 2, \ldots, t-1$. Also $q_i = |E(P_{n_i} \times P_{m_i})| = 2p_j - (m_i + n_i), \forall\ i = 1, 2, \ldots, t$ and $|E(P_{r_j})| = q_j^{'} = r_j - 1$, $\forall\ j = 1, 2, \ldots, t-1$.

Let $V(P_{n_i} \times P_{m_i}) = \{u_{i,j,k}/1 \leq j \leq n_i, 1 \leq k \leq m_i\}$, $\forall\ i = 1, 2, \ldots, t$ and $V(P_{r_l}) = \{u_{l,k}/1 \leq k \leq r_l\}$, $\forall\, l = 1, 2, \ldots, t-1$ with $u_{i,1,1} = v_{i,r_i}$ and $v_{i,1} = u_{i+1,n_{i+1},m_{i+1}}$ for every $i = 1, 2, \ldots, t-1$ to from connected graph G. Here we see that $|V(G)| = \sum_{i=1}^{t} p_i + \sum_{i=1}^{t-1}(r_i - 2)$ and $|E(G)| = \sum_{i=1}^{t} q_i + \sum_{i=1}^{t-1}(q_i^{'})$. Let $f_l : V(P_{n_l} \times P_{m_l}) \longrightarrow \{0, 1, \ldots, q_l\}$ be vertex labeling function for$P_{n_l} \times P_{m_l}$ defined by

$$
\begin{aligned}
f(u_{l,j,1}) &= g_l - \tfrac{(j-1)^2}{2}, && \text{when } j \text{ is odd,}\\
&= \tfrac{j(j-2)}{2}, && \text{when } j \text{ is even, } \forall\ j = 1, 2, \ldots, n_l;\\
f(u_{l,j,m_l}) &= \tfrac{q_l}{2} - \tfrac{1}{4} + (-1)^{m_l+j}[\tfrac{(n_l-j)^2}{2} + \tfrac{1}{4}], && \forall\ j = n_l, n_{l-1}, \ldots, 1;\\
f(u_{l,j,k}) &= f(u_{l,j-1,k+1}) + (-1)^{j+k}, && \forall\ k = m_l - 1, m_l - 2, \ldots, m_l + 1 - n_l,\\
& && \forall\ j = n_l, n_l - 1, \ldots, m_l + 1 - k;\\
f(u_{l,n_l,k}) &= f(u_{l,n_l,1}) + (-1)^{n_l}[\tfrac{(2n_l-1)(k-1)}{2}], && \text{when } k \text{ is odd,}\\
&= f(u_{l,n_l,1}) - (-1)^{n_l}[\tfrac{(2n_l-1)(k)}{2}], && \text{when } k \text{ is even, } \forall\ k = 2, 3, \ldots, m_l - n_l;\\
f(u_{l,j,k}) &= f(u_{l,j+1,k-1}) + (-1)^{j+k}, \forall\ k = 2, 3, \ldots, m_l - 1, \forall j = 1, 2, \ldots, min\{n_l, m_l - k\},\\
& \qquad \forall\ l = 1, 2, \ldots, t.
\end{aligned}
$$

which is a graceful labeling function for $P_{n_l} \times P_{m_l}$, $\forall\ l = 1, 2, \ldots, t$.

Let $f_l^{'} : V(P_{r_l}) \longrightarrow \{0, 1, \ldots, q_l^{'}\}$ be vertex labeling function forP_{r_l} defined by

$$
\begin{aligned}
f_l^{'}(v_{l,k}) &= g_l - (\tfrac{k-1}{2}), && \text{when } k \text{ is odd,}\\
&= \tfrac{k-2}{2}, && \text{when } k \text{ is even;}\\
& && \forall\ k = 1, 2, \ldots, r_l, \forall\ l = 1, 2, \ldots, t-1.
\end{aligned}
$$

which is a graceful labeling function for P_{r_l}, $\forall\ l = 1, 2, \ldots, t-1$.

Define for each $l = 2, 3, \ldots, t$, $g_{l-1}^{'} : V(< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{n_{l-1}} \times P_{m_{l-1}}, P_{r_{l-1}} >) \longrightarrow \{0, 1, \ldots, \sum_{i=1}^{l-1}(q_i + q_i^{'})\}$ and $g_l : V(< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{r_{l-1}}, P_{n_l} \times P_{m_l} >) \longrightarrow \{0, 1, \ldots, q_l + \sum_{i=1}^{l-1}(q_i + q_i^{'})\}$ as follows (assuming $g_1 = f_1$).

$$
\begin{aligned}
g_{l-1}^{'}(u) &= f_{l-1}^{'}(u), && \text{when } f_{l-1}^{'}(u) < \tfrac{q_l^{'}}{2},\\
&= f_{l-1}^{'}(u) + g_{l-1} + \textstyle\sum_{i=1}^{l-2}(q_i + q_i^{'}), && \text{when } f_{l-1}^{'}(u) \geq \tfrac{q_l^{'}}{2},\\
& \qquad \forall\ u \in V(P_{l-1});\\
g_{l-1}^{'}(w) &= g_{l-1}(w) + \tfrac{q_{l-1}^{'}}{2}, && \text{when } g_{l-1}^{'} \text{ is even,}\\
&= q_{l-1} + \textstyle\sum_{i=1}^{l-2}(q_i + q_i^{'}) + (\tfrac{q_{l-1}^{'}-1}{2}) - g_{l-1}(w), && \text{when } g_{l-1}^{'} \text{ is odd,}\\
& \qquad \forall\ w \in V(< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{n_{l-1}} \times P_{m_{l-1}} >);
\end{aligned}
$$

$g_l(v) = f_l(v),$ when $f_l(v) < \frac{q_l}{2}$,

$= f_l(v) + \sum_{i=1}^{l-1}(q_i + q_i^{'}),$ when $f_l(v) \geq \frac{q_l}{2}$,

$\forall\ v \in V(P_{n_l} \times P_{m_l});$

$g_l(w) = g_{l-1}^{'}(w) + \frac{q_l}{2},$ when g_l is even,

$= \sum_{i=1}^{l-1}(q_i + q_i^{'}) + (\frac{q_l - 1}{2}) - g_{l-1}^{'}(w),$ when g_l is odd.

$\forall\ w \in V(< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{n_{l-1}} \times P_{m_{l-1}}, P_{r_{l-1}} >)$

Above labeling pattern give rise graceful labeling $g_{l-1}^{'}, g_l$ to the graphs $< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{n_{l-1}} \times P_{m_{l-1}}, P_{r_{l-1}} >$ and $< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{r_{l-1}}, P_{n_l} \times P_{m_l} >$ respectively, $\forall\ l = 2, 3, \ldots, t$. So they are graceful graphs. Particularly $G =< P_{n_1} \times P_{m_1}, P_{r_1}, \ldots, P_{r_{t-1}}, P_{n_t} \times P_{m_t} >$ is graceful.

Illustration-2.24 **:** $< P_3 \times P_3, P_5, P_3 \times P_4, P_8, P_4 \times P_4 >$ and its graceful labeling g_3 shown in *figure*-24, for this we have computed graceful labelings $f_1, f_2, f_3, f_1^{'}$ and $f_2^{'}$ for the graphs $P_3 \times P_3, P_3 \times P_4, P_4 \times P_4, P_5$ and P_8 respectively in *figure*-21, graceful labeling $g_1^{'}$ for $< P_3 \times P_3, P_5 >$, graceful labeling g_2 for $< P_3 \times P_3, P_5, P_3 \times P_4 >$ in *figure*-22, graceful labeling $g_2^{'}$ for $< P_3 \times P_3, P_5, P_3 \times P_4, P_8 >$ in *figure*-23.

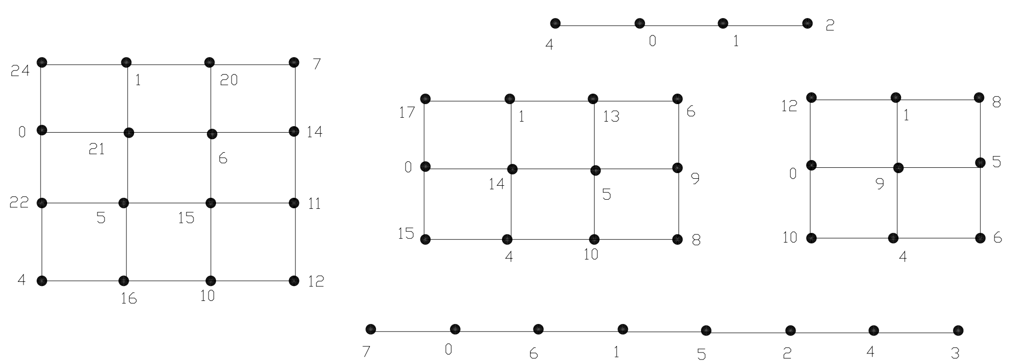


Figure-21 Graceful labeling f_1 for $P_3 \times P_3$, f_2 for $P_3 \times P_4$, f_3 for $P_4 \times P_4$, $f_1^{'}$ for P_5 and $f_2^{'}$ for P_8.

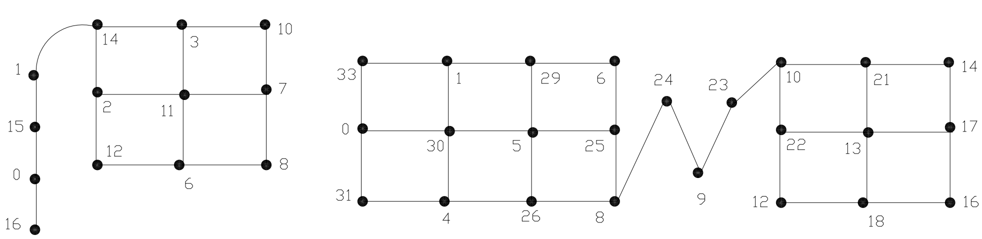
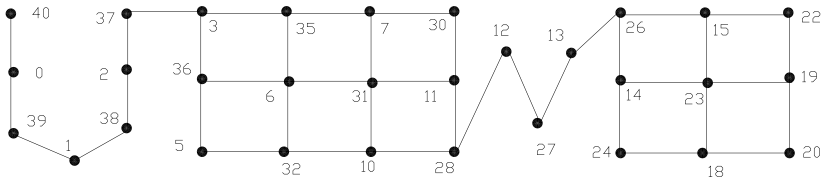
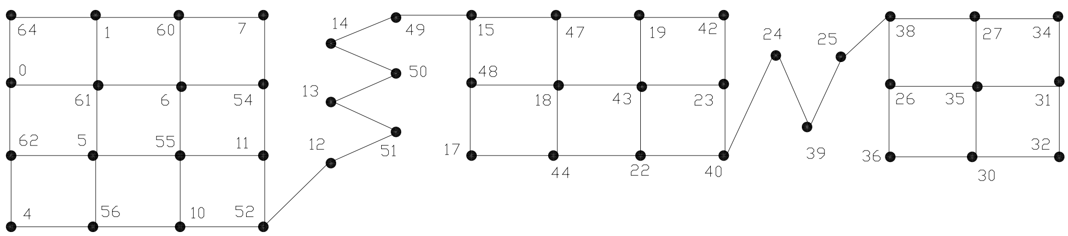


Figure-22 Graceful labeling $g_1^{'}$ for $< P_3 \times P_3, P_5 >$ and g_2 for $< P_3 \times P_3, P_5, P_3 \times P_4 >$.



Figure−23 Graceful labeling g_2' for the graph $< P_3 \times P_3, P_5, P_3 \times P_4, P_8 >$.



Figure−24 Graceful labeling for G.

Theorem−2.25 : Arbitrary path union of t complete bipartite graphs K_{m_i,n_i} $(1 \leq i \leq t)$ by $t-1$ paths of arbitrary length $r_j - 1$ $(1 \leq j \leq t-1)$ is graceful.

Proof : Let $G =< K_{m_1,n_1}, P_{r_1}, K_{m_2,n_2}, P_{r_2}, \ldots, K_{m_t,n_t} >$. Obviously $P = |V(G)| = \sum_{i=1}^{t}(m_i+n_i)+\sum_{i=1}^{t-1}(r_i-2)$ and $Q = |E(G)| = \sum_{i=1}^{t} q_i+\sum_{i=1}^{t-1} q_i'$, where $q_i = |E(K_{m_i,n_i}| = m_i n_i$ and $q_i' = |E(P_{r_j})| = r_j - 1$, $\forall\ i = 1, 2, \ldots, t$ and $\forall\ j = 1, 2, \ldots, t-1$.

Let $V(K_{m_i,n_i}) = \{u_{i,j}/1 \leq j \leq m_i\} \bigcup \{w_{i,k}/1 \leq k \leq n_i\}$, $\forall\ i = 1, 2, \ldots, t$ and $V(P_{r_l}) = \{v_{l,t}/1 \leq t \leq r_l\}$, with $u_{l,1} = v_{l,r_l}$, $u_{l+1,m_{l+1}} = v_{l,1}$ $\forall\ l = 1, 2, \ldots, t-1$.

Let $f_l : V(K_{m_l,n_l}) \longrightarrow \{0, 1, \ldots, q_l\}$ be graceful labeling function for K_{m_l,n_l} defined by

$f_l(u_{l,j}) = j - 1,$ $\forall\ j = 1, 2, \ldots, m_l;$

$f_l(w_{l,k}) = m_l \cdot k,$ $\forall\ k = 1, 2, \ldots, n_l$, $\forall\ l = 1, 2, \ldots, t.$

Let $f_l' : V(P_{r_l}) \longrightarrow \{0, 1, \ldots, q_l'\}$ be graceful labeling function for P_{r_l} defined by

$f_l'(v_{l,k}) = q_l' - (\frac{k-1}{2}),$ when k is odd,

$=(\frac{k-2}{2}),$ when k is even, $\forall\ k = 1, 2, \ldots, r_l;$

$\forall\ l = 1, 2, \ldots, t-1.$

Define for each $\forall\ l = 2, 3, \ldots, t$ (assuming $g_l = f_1$) $g_{l-1}' : V(< K_{m_1,n_1}, P_{r_1}, \ldots, K_{m_{l-1},n_{l-1}}, P_{r_{l-1}} >) \longrightarrow \{0, 1, \ldots, \sum_{i=1}^{l-1}(q_i + q_i')\}$ and $g_l : V(< K_{m_1,n_1}, P_{r_1}, \ldots, P_{r_{l-1}}, K_{m_l,n_l} >) \longrightarrow \{0, 1, \ldots, \sum_{i=1}^{l-1}(q_i + q_i') + q_l\}$ as follows

$$g'_{l-1}(u) = f'_{l-1}(u), \qquad \text{when } f'_{l-1}(u) < \tfrac{q'_l}{2},$$
$$= f'_{l-1}(u) + q_{l-1} + \textstyle\sum_{i=1}^{l-2}(q_i + q'_i), \qquad \text{when } f'_{l-1}(u) \geq \tfrac{q'_l}{2},$$
$$\forall\ u \in V(P_{l-1});$$
$$g'_{l-1}(w) = g_{l-1}(w) + (\tfrac{q'_{l-1}-1}{2}), \qquad \text{when } g'_{l-1} \text{ is odd},$$
$$= q_{l-1} + \textstyle\sum_{i=1}^{l-2}(q_i + q'_i) + (\tfrac{q'_{l-1}}{2}) - g_{l-1}(w), \qquad \text{when } q'_{l-1} \text{ is even},$$
$$\forall\ w \in V(< k_{m_1,n_1}, P_{r_1}, \ldots, K_{m_{l-1},n_{l-1}} >);$$
$$g_l(v) = f_l(v), \qquad \text{when } f_l(v) < m_l,$$
$$= f_l(v) + \textstyle\sum_{i=1}^{l-1}(q_i + q'_i), \qquad \text{when } f_l(v) \geq m_l,$$
$$\forall\ v \in V(K_{m_l,n_l});$$
$$g_l(w) = \textstyle\sum_{i=1}^{l-1}(g_i + g'_i) + m_l - g'_{l-1}(w) - 1,$$
$$\forall\ w \in V(< k_{m_1,n_1}, P_{r_1}, \ldots, K_{m_{l-1},n_{l-1}}, P_{r_l} >).$$

Above defined labeling pattern give rise graceful labelings g'_{l-1}, g_l to the graphs $< K_{m_1,n_1}, P_{r_1},\ldots, P_{r_{l-1}} >$ and $< K_{m_1,n_1}, P_{r_1}, \ldots, K_{m_l,n_l} >$ respectively, $\forall\ l = 2, 3, \ldots, t$. So these are graceful graphs, $\forall\ l = 2, 3, \ldots, t$. Particularly $G =< K_{m_1,n_1}, P_{r_1}, \ldots, K_{m_t,n_t} >$ is graceful.

Illustration-2.26 **:** $< K_{3,5}, P_6, K_{3,3}, P_5, K_{3,4} >$ and its graceful labeling g_3 shown in *figure*-28, for this we have computed graceful labelings $f_1, f_2, f_3, f'_1, f'_2$ for the graphs $K_{3,5}, K_{3,3}, K_{3,4}, P_6, P_5$ respectively in *figure*-25, graceful labeling g'_1 for $< K_{3,5}, P_6 >$, g_2 for $< K_{3,5}, P_6, K_{3,3} >$ in *figure*-26 and graceful labeling g'_2 for $< K_{3,5}, P_6, K_{3,3}, P_5 >$ in *figure*-27.

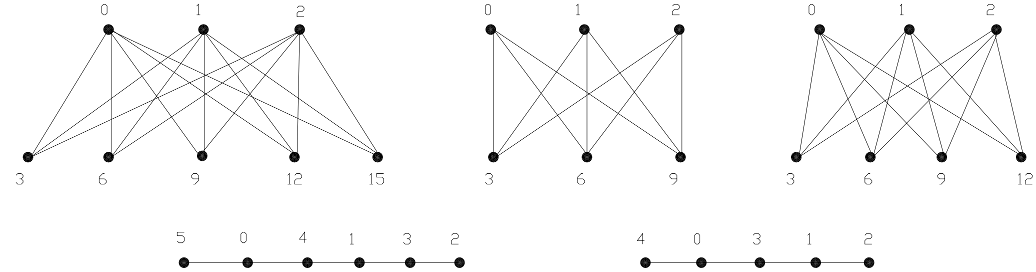
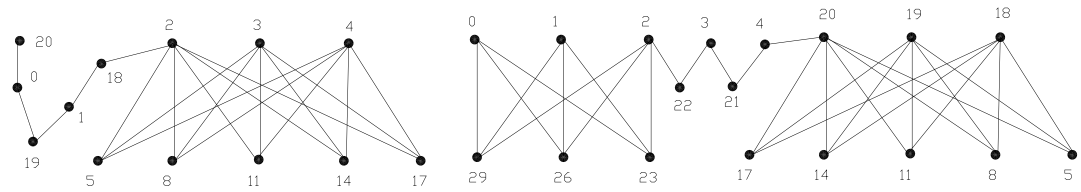
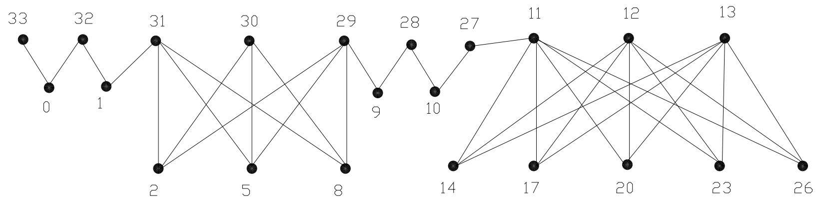


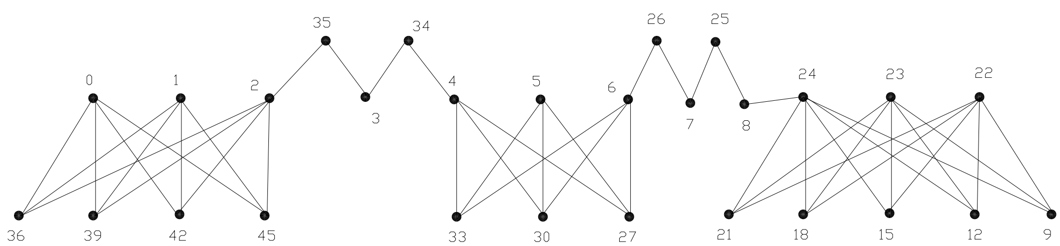
Figure-25 Graceful labeling for $k_{3,5}, K_{3,3}, K_{3,4}, P_6, P_5$.



Figure−26 Graceful labeling g_1' for $< K_{3,5}, P_6 >$ and g_2 for $< K_{3,5}, P_6, K_{3,3} >$.



Figure−27 Graceful labeling for $< K_{3,5}, P_6, K_{3,3}, P_5 >$.



Figure−28 Graceful labeling g_3 for G.

References

[1] A. Rosa, On certain valuation of graph, *Theory of Graphs (Rome, July* 1966*)*, Goden and Breach, N. Y. and Paris, (1967) pp. 349 − 355.

[2] H.K.Ng, $\alpha-$valuations and $k-$gracefulness, *Notices AMS*, 7, 1986, pp. 247.

[3] V.J. Kaneria and M.M. Jariya, Smooth Graceful Graphs And Its Application To Construct Graceful Graphs, *I.J. of Sci. and Research* Vol−3 (8), Aug. 2014, pp. 909 − 912. http://www.ijsr.net/archive/v3i8/MDIwMTU0ODg=.pdf

[4] V.J. Kaneria and M.M. Jariya, Semi smooth graceful graph and construction of new graceful trees, *Elixir Applied Mathematics* Vol−76, 2014, pp. 28536 − 28538.

[5] J. A. Gallian, *The Electronics Journal of Combinatorics*, 17, ♯DS6(2014). http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS6

[6] F. Harary, Graph theory *Addition Wesley, Massachusetts*, 1972.

[7] V. J. Kaneria and H. M. Makadia, Graceful Labeling for Step Grid Graph, *J. of advance in Maths* Vol−9 (5), Nov. 2014, pp. 2647 − 2654. http://cirworld.org/journals/index.php/jam/article/view/3199