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The upper forcing edge-to-vertex geodetic number of a graph

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Abstract

For a connected graph $G = (V, E)$, a set $S \subseteq E$ is called an *edge-to-vertex geodetic set* of G if every vertex of G is either incident with an edge of S or lies on a geodesic joining some pair of edges of S. The minimum cardinality of an edge-to-vertex geodetic set of G is $g_{ev}(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *edge-to-vertex geodetic basis* of G. A subset $T \subseteq S$ is called a *forcing* subset for S if S is the unique minimum edge-to-vertex geodetic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The *forcing edge-to-vertex geodetic number* of S, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of S. The *upper forcing edge-to-vertex geodetic number* of G, denoted by $f^{+}_{ev}(G)$, is $f^{+}_{ev}(G) = max\,\{f_{ev}(S)\}$, where the maximum is taken over all minimum edge-to-vertex geodetic sets S in G. It is shown that the upper forcing edge-to-vertex geodetic number lies between 0 and $g_{ev}(G)$. Also, the upper forcing edge-to-vertex geodetic number of certain classes of graphs such as cycle, tree, complete graph and complete bipartite graph are determined.

Keywords: edge-to-vertex geodetic number, forcing edge-to-vertex geodetic number, upper forcing edge-to-vertex geodetic number.
AMS Subject Classification(2010): 05C12.

1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic definitions and terminology we refer to [1]. For vertices u and v in a connected graph G, the *distance* $d\,(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d\,(u, v)$ is called a $u - v$ *geodesic*. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices of G. The *geodetic number* $g(G)$ of G is the minimum order of a geodetic set and any geodetic set of order $g(G)$ is called a *geodetic basis* of G. The geodetic number of a graph was introduced in [1] and further studied in [5]. A set $S \subseteq E(G)$ is called an *edge-to-vertex geodetic set* of G if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The minimum cardinality of an edge-to-vertex geodetic set of G is $g_{ev}\,(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}\,(G)$ is called an *edge-to-vertex geodetic basis* of G or a g_{ev}-set of G. The edge-to-vertex geodetic number of a graph was introduced in [12] and further studied in [7]. A vertex v is an *extreme vertex* of a graph

G if the subgraph induced by its neighbors is complete. An edge of a connected graph G is called an *extreme edge* of G if one of its ends is an extreme vertex of G. For any edge e in a connected graph G, the *edge-to-edge eccentricity* $e_3(e)$ of e is $e_3(e)$= *max* $\{d(e, f) : f \in E(G)\}$. Any edge e for which $e_3(e)$ is minimum is called an *edge-to-edge central edge* of G and the set of all edge-to-edge central edges of G is the *edge-to-edge center* of G. The minimum eccentricity among the edges of G is the *edge-to-edge radius*, *rad* G and the maximum eccentricity among the edges of G is the *edge-to-edge diameter*, *diam* G of G. Two edges e and f are *antipodal* if $d(e, f)$ = *diam* G or $d(G)$. This concept was studied in [10]. The forcing concept was first introduced and studied in minimum dominating sets in [2] and the same in geodetic number was introduced and studied by Chartrand and Zhang in[3]. Then the forcing concept is applied in various graph parameters viz. hull sets, matching's, edge coverings and Steiner sets in [4, 6, 9, 8, 11] by several authors. In this paper we study the upper forcing concept in minimum edge-to-vertex geodetic set of a connected graph.

Throughout the paper G denotes a connected graph with at least three vertices . The following theorems are used in the sequel.

Theorem 1.1 (12)**.** Let G be a connected graph with size q. Then every end-edge of G belongs to every edge-to-vertex geodetic set of G.

Theorem 1.2 (12)**.** For the complete bipartite graph $G = K_{n,n}(n \geq 2)$, a set S of edges of G is a minimum edge-to-vertex geodetic set if and only if S consists of n independent edges of G.

Theorem 1.3 (12)**.** For the complete bipartite graph $G = K_{m,n}(2 \leq m < n)$, a set S of edges of G is a minimum edge-to-vertex geodetic set if and only if S consists of $m - 1$ independent edges of G and $n - m + 1$ adjacent edges of G.

Theorem 1.4 (12)**.** For the complete graph $G = K_p(p \geq 4)$ with p even, a set S of edges of G is a minimum edge-to-vertex geodetic set of G if and only if S consists of $\frac{p}{2}$ independent edges.

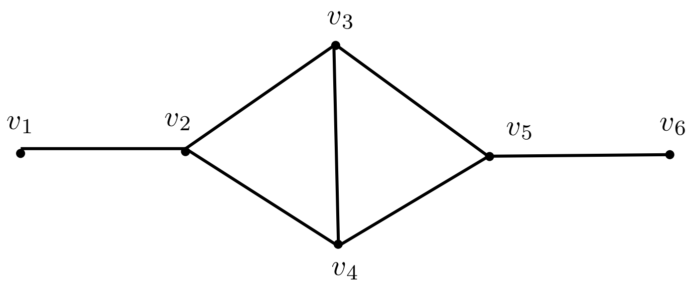
Theorem 1.5 (12)**.** For the complete graph $G = K_p(p \geq 5)$ with p odd, a set S of edges of G is a minimum edge-to-vertex geodetic set of G if and only if S consists of $\frac{p-3}{2}$ independent edges and two adjacent edges of G.

2 The Forcing Edge-to-vertex Geodetic Number of a Graph

For each minimum edge-to-vertex geodetic set S in a connected graph G, there is always some subset T of S such that S is the unique minimum edge-to-vertex geodetic set containing T. The maximum of such subsets T of S is considered in this section.

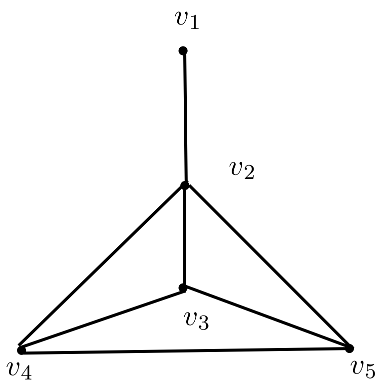
Definition 2.1. Let G be a connected graph and S an edge-to-vertex geodetic set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum edge-to-vertex geodetic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing edge-to-vertex geodetic number of S, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of S. The *upper forcing edge-to-vertex geodetic number* of G, denoted by $f_{ev}^{+}(G)$, is $f_{ev}^{+}(G) = max\,\{f_{ev}(S)\}$, where the maximum is taken over all minimum edge-to-vertex geodetic sets S in G.

Example 2.2. For the graph G given in Figure 1, $S = \{v_1v_2, v_5v_6\}$ is the unique minimum edge-to-vertex geodetic set of G so that $f_{ev}^{+}(G) = 0$. For the graph G given in Figure 2, $S_1 = \{v_1v_2, v_3v_4, v_3v_5\}$, $S_2 = \{v_1v_2, v_3v_4, v_4v_5\}$ and $S_3 = \{v_1v_2, v_3v_5, v_4v_5\}$, $S_4 = \{v_1v_2, v_3v_4, v_2v_5\}$, $S_5 = \{v_1v_2, v_2v_3, v_4v_5\}$ and $S_6 = \{v_1v_2, v_3v_5, v_2v_4\}$ are the only g_{ev}-sets of G, such that $f_{ev}(S_1) = f_{ev}(S_2) = f_{ev}(S_3) = 2$, and $f_{ev}(S_4) = f_{ev}(S_5) = f_{ev}(S_6) = 1$ so that $f_{ev}^{+}(G) = max\,\{f_{ev}(S)\} = max\,\{2, 2, 2, 1, 1, 1\} = 2$.



G

Figure 1



G

Figure 2

The next theorem follows immediately from the definition of the edge-to-vertex geodetic number and the upper forcing minimum edge-to-vertex geodetic number of a connected graph G.

Theorem 2.3. For every connected graph G, $0 \le f_{ev}^{+}(G) \le g_{ev}(G)$.

Proof: Since every connected graph G has one or more minimum edge-to-vertex geodetic sets and every minimum edge-to-vertex geodetic set contains at least two edges, it follows that $f^+_{ev}(G) \geq 0$. Let S be a minimum edge-to-vertex geodetic set of G and T a forcing subset of S. By definition, $T \subseteq S$. This implies that, the cardinality of T is less than or equal to the cardinality of S. That is $f^+_{ev}(G) \leq g_{ev}(G)$. ∎

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 1, $f^+_{ev}(G) = 0$ and for the graph $G = K_3$, $f^+_{ev}(G) = g_{ev}(G) = 2$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2, $f^+_{ev}(G) = 2$ and $g_{ev}(G) = 3$ so that $0 < f^+_{ev}(G) < g_{ev}(G)$.

In the following, we characterize graphs G for which bounds in Theorem 2.3 attained and also graph for which $f^+_{ev}(G) = 1$.

Theorem 2.5. Let G be a connected graph. Then
$(a) f^+_{ev}(G) = 0$ if and only if G has a unique minimum edge-to-vertex geodetic set.
$(b) f^+_{ev}(G) = 1$ if and only if G has at least two minimum edge-to-vertex geodetic sets, in which one element of each minimum edge-to-vertex geodetic set of G does not belong to any other minimum edge-to-vertex geodetic set of G. and
$(c) f^+_{ev}(G) = g_{ev}(G)$ if and only if there exists a minimum edge-to-vertex geodetic set of G which does not contain any proper forcing subsets.

Proof: (a) Let $f^+_{ev}(G) = 0$ Then, by definition, $f_{ev}(S) = 0$ for some minimum edge-to-vertex geodetic set S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique minimum edge-to-vertex geodetic set of G. Conversely, Let S be the unique minimum edge-to-vertex geodetic set of G. It is clear that $f_{ev}(S) = 0$ and hence $f^+_{ev}(G) = 0$.
(b) Let $f^+_{ev}(G) = 1$. Then by Theorem 2.5(a), G has at least two minimum edge-to-vertex geodetic sets. Also, since $f^+_{ev}(G) = 1$,then by definition $f_{ev}(S) = 1$ for all S.Therefore there is a singleton subset T of a minimum edge-to-vertex geodetic set S of G such that T is not a subset of any other minimum edge-to-vertex geodetic sets of G. Thus one element of each S does not belong to any other minimum edge-to-vertex geodetic set of G. Conversely, suppose that G has at least two minimum edge-to-vertex geodetic sets, in which one element of each minimum edge-to-vertex geodetic set not containing any other minimum edge-to-vertex geodetic sets. It is clear that $f_{ev}(S) = 1$ for all minimum edge-to-vertex geodetic set S in G. Hence $f^+_{ev}(G)$ = max$\{f_{ev}(S)\} = 1$.
(c) Let $f^+_{ev}(G) = g_{ev}(G)$. Then $f_{ev}(S) = g_{ev}(G)$ for some minimum edge-to-vertex geodetic set S in G. Since, $q \geq 2$, $g_{ev}(G) \geq 2$ and hence $f_{ev}(S) \geq 2$. Then by Theorem 2.5(a), G has at least two minimum edge-to-vertex geodetic sets and so the empty set ϕ is not a forcing subset for any minimum edge-to-vertex geodetic set of G. Since $f_{ev}(S) = g_{ev}(G)$ for some S, there exists some minimum edge-to-vertex geodetic sets S such that no proper subset of S is a forcing subset of S. Thus there

exists at least one minimum edge-to-vertex geodetic set of G which does not contain any proper forcing subsets.Conversely, the data implies that G contains more than one minimum edge-to-vertex geodetic sets such that at least one minimum edge-to-vertex geodetic set S other than S is a forcing subset for S. Hence it follows that $f^+_{ev}(G) = g_{ev}(G)$. ∎

Definition 2.6. An edge e of a connected graph G is an edge-to-vertex geodetic edge of G if e belongs to every edge-to-vertex geodetic basis of G. If G has a unique edge-to-vertex geodetic basis S, then every edge of S is an edge-to-vertex geodetic edge of G.

Example 2.7. For the graph G given in Figure 1, $S = \{v_1v_2, v_5v_6\}$ is the unique minimum edge-to-vertex geodetic set of G so that both the edges in S are edge-to-vertex geodetic edges of G.

Remark 2.8. By Theorem 1.1, each end edge of G is an edge-to-vertex geodetic edge of G. In fact there are certain edge-to-vertex geodetic edges, which are not end edges as shown in the following example.

Example 2.9. For the graph G given in Figure 3, $S_1 = \{v_1v_2, v_6v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only g_{ev}-sets of G so that every g_{ev}-set contains the edge v_1v_2. Hence the edge v_1v_2 is the unique edge-to-vertex geodetic edge of G, which is not an end edge of G.

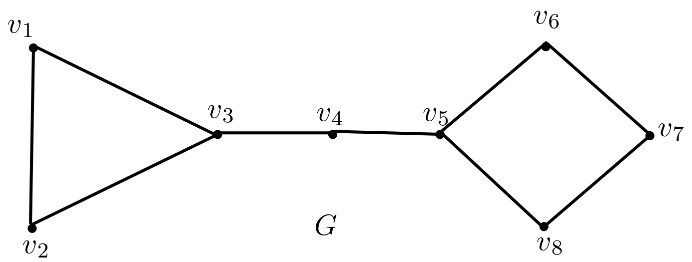


Figure 3

Theorem 2.10. Let G be a connected graph and S a minimum edge-to-vertex geodetic set of G. Then no edge-to-vertex geodetic edge of G belongs to any minimum forcing set of S.

Proof: Let S be a minimum edge-to-vertex geodetic set of G. Let T be a unique minimum forcing subset of S. Let e be an edge-to-vertex geodetic edge of G. By the definition $e \in S$ for all S. We show that $e \notin T$ for all T contained in S. Suppose e is in any forcing subset T of S, then e does not belong to any other minimum edge-to-vertex geodetic set of G. This implies that e is not an edge-to-vertex geodetic edge of G. Thus $e \notin T$ for all $T \subset S$. ∎

Theorem 2.11. Let G be a connected graph and W be the set of all edge-to-vertex geodetic edges of G. Then $f^+_{ev}(G) \leq g_{ev}(G) - |W|$.

Proof: Let S be a minimum edge-to-vertex geodetic set of G. Then $g_{ev}(G) = |S|$, $W \subseteq S$ and S is the unique minimum edge-to-vertex geodetic set containing $S - W$. Thus $f^+_{ev}(G) \leq |S - W| \leq |S| - |W| = g_{ev}(G) - |W|$. ∎

Corrolary 2.12. If G is a connected graph with k end edges, then $f_{ev}^{+}(G) \leq g_{ev}(G) - k$.

Proof: This follows from Theorems 1.1 and 2.11. ∎

Remark 2.13. The bound in Theorem 2.11 is sharp. For the graph G given in Figure 3, $S_1 = \{v_1v_2, v_6v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only g_{ev}-sets of G such that $f_{ev}(S_1) = 2$ and $f_{ev}(S_2) = f_{ev}(S_3) = 1$ so that $f_{ev}^{+}(G) = \max\{f_{ev}(S)\} = 2$ and $g_{ev}(G) = 3$. Also, every g_{ev}-set contains the edge v_1v_2 so that $|W| = 1$ hence $f_{ev}^{+}(G) = g_{ev}(G) - |W|$. Also, the inequality in Theorem 2.11 can be strict. For the graph G given in Figure 4, $S_1 = \{v_1v_2, v_3v_4, v_5v_6\}$, $S_2 = \{v_1v_4, v_2v_3, v_5v_6\}$ are the only two g_{ev}-sets of G such that $f_{ev}(S_1) = f_{ev}(S_2) = 1$ so that $f_{ev}^{+}(G) = 1$. Also $g_{ev}(G) = 3$. Here, v_5v_6 is the only edge-to-vertex geodetic edge of G and so $f_{ev}^{+}(G) < g_{ev}(G) - |W|$.

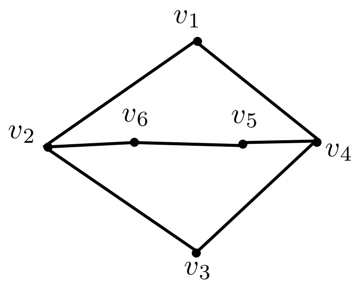


Figure 4

In the following we determine the upper forcing edge-to-vertex geodetic number of some standard graphs.

Theorem 2.14. For an even cycle $C_p (p \geq 4)$, a set $S \subseteq E(G)$ is a minimum edge-to-vertex geodetic set if and only if S consists of antipodal edges.

Proof: Let $p = 2k$ and let $C_p : v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_1$ be the cycle. Then the edges v_1v_2 and $v_{k+1}v_{k+2}$ are antipodal edges. Let $S = \{v_1v_2, v_{k+1}v_{k+2}\}$. Clearly, S is a minimum edge-to-vertex geodetic set of C_p. Conversely, let S be a minimum edge-to-vertex geodetic set of C_p. Then $g_{ev}(C_p) = |S|$. Let S' be any set of pair of antipodal edges of C_p. Then as in the first part of this theorem, S' is a minimum edge-to-vertex geodetic set of C_p. Hence $|S'| = |S|$. Thus $S = \{uv, xy\}$. If uv and xy are not antipodal, then any vertex that is not on the $uv - xy$ geodesic does not lie on the $uv - xy$ geodesic. Thus S is not a minimum edge-to-vertex geodetic set, which is a contradiction. ∎

Theorem 2.15. For an even cycle $C_p (p \geq 4)$, $f_{ev}^{+}(C_p) = 1$.

Proof: If p is even, then by Theorem 2.14, every minimum edge-to-vertex geodetic set of C_p consists of pair of antipodal edges. Hence C_p has $p/2$ independent minimum edge-to-vertex geodetic sets and it is clear that each singleton set is the minimum forcing set for exactly one minimum edge-to-vertex geodetic set of C_p. Hence it follows from Theorem 2.5 (a) and (b) that $f_{ev}^{+}(C_p) = 1$. ■

Theorem 2.16. For an odd cycle $C_p(p > 5)$, $f_{ev}^{+}(C_p) = 3$.

Proof: Let p be odd. Let $p = 2n + 1$, $n = 2, 3, ...$ Let the cycle be $C_p : v_1, v_2, v_3, ..., v_{2n+1}, v_1$. If $S = \{uv, xy\}$ is any set of two edges of C_p, then no edge of the $uv - xy$ longest path lies on the $uv - xy$ geodesic in C_p and so no two element subset of C_p is an edge-to-vertex geodetic set of C_p. Now, it is clear that the sets $S_1 = \{v_1v_2, v_{n+1}v_{n+2}, v_{2n}v_{2n+1}\}$, $S_2 = \{v_1v_2, v_{n+1}v_{n+2}, v_{2n+1}v_1\}$, $S_3 = \{v_2v_3, v_{n+2}v_{n+3}, v_{2n+1}v_1\}$,..., $S_{2n} = \{v_nv_{n+1}, v_{2n}v_{2n+1}, v_{n-1}v_n\}$, $S_{2n+1} = \{v_{n+1}v_{n+2}, v_{2n+1}v_1, v_{n-1}v_n\}$ are the minimum edge-to-vertex geodetic sets of C_p. (Note that there are more minimum edge-to-vertex geodetic sets of C_p, for example $S = \{v_{n+2}v_{n+3}, v_1v_2, v_nv_{n+1}\}$ is a minimum edge-to-vertex geodetic set different from these). It is clear from the minimum edge-to-vertex geodetic sets S_i $(1 \leq i \leq 2n + 1)$ that each $\{v_iv_{i+1}\}$ $(1 \leq i \leq 2n)$ and $\{v_{2n+1}v_1\}$ is a subset of more than one minimum edge-to-vertex geodetic set $S_i(1 \leq i \leq 2n+1)$. Hence it follows from Theorem 2.5 (b) and (c) that $f_{ev}^{+}(C_p) \leq 3$. Since S_2 is the unique minimum edge-to-vertex geodetic set containing $T = \{v_1v_2, v_{2n+1}v_1\}$, it follows that $f_{ev}(S_2) = 2$. But it is easily verified that the two element subsets of S_1 are contained in more than one minimum edge-to-vertex geodetic set $S_i(1 \leq i \leq 2n + 1)$ so that $f_{ev}(S_1) \neq 2$ and hence $f_{ev}(S_1) = 3$. Thus $f_{ev}^{+}(C_p) = 3$. ■

Theorem 2.17. For the complete bipartite graph $G = K_{n,n}(n \geq 2)$, $f_{ev}^{+}(G) = n - 1$.

Proof: Let $X = \{u_1, u_2, ...u_n\}$ and $Y = \{v_1, v_2, ...v_m\}$ be a partition of G. Let S be a minimum edge-to-vertex geodetic set of G. Then by Theorem 1.2, every element of S are independent and $|S| = n$. We show that $f_{ev}^{+}(G) = n - 1$.

Case(i): Suppose that $f_{ev}^{+}(G) \leq n - 2$. Then there exists a forcing subset T of S such that S is the unique minimum edge-to-vertex geodetic set of G containing T and $|T| \leq n - 2$. Hence there exists at least two edges u_iv_j, $u_lv_m \in S$ such that $u_iv_j, u_lv_m \notin T$ and $i \neq l, j \neq m$. Then $S_1 = S - \{u_iv_j, u_lv_m\} \cup \{u_iv_m, u_lv_j\}$ is a set of n independent edges of G. By Theorem 1.2, S_1 is a minimum edge-to-vertex geodetic set of G which is a contradiction to T is a forcing subset of S. Hence $f_{ev}^{+}(G) \leq n - 2$ is not possible.

Case(ii): Suppose that $f_{ev}^{+}(G) > n - 1$. By Theorem 2.5(c), $f_{ev}^{+}(G) = n$. Then there exists a forcing subset T of S such that S is the unique minimum edge-to-vertex geodetic set of G containing T and $|T| = n$. Hence all the proper subsets of S having a single element, two elements, three elements,..., $n - 1$ elements are contained in more than one minimum edge-to-vertex geodetic sets of G. Let F be a proper subset of S with cardinality $n - 1$. Let S_1 and S_2 be the two minimum edge-to-vertex geodetic sets of G containing F. Since S_1 and S_2 have $n - 1$ elements as common, the other n^{th} element of S_1

and S_2 is also same. Thus we get more than one minimum edge-to-vertex geodetic set with the same n independent edges, which is a contradiction to T is a forcing subset of S. Hence $f^+_{ev}(G) = n$ is not possible. Thus $f^+_{ev}(G) = n - 1$. ∎

Theorem 2.18. For the complete bipartite graph $G = K_{m,n}(2 \leq m < n)$, $f^+_{ev}(G) = n - 1$.

Proof: Let $X = \{u_1, u_2, ...u_n\}$ and $Y = \{v_1, v_2, ...v_m\}$ be a partition of G. Let S be a minimum edge-to-vertex geodetic set of G. Then by Theorem 1.3, $S = S_1 \cup S_2$, where S_1 consists of $m-1$ independent edges and S_2 consists of $n - m + 1$ adjacent edges and $|S| = n$. We show that $f^+_{ev}(G) = n - 1$.
Case(i): Suppose that $f^+_{ev}(G) \leq n - 2$. Then there exists a forcing subset T of S such that S is the unique minimum edge-to-vertex geodetic set of G containing T and $|T| \leq n - 2$. Hence there exists at least two edges $x, y \in S$ such that $x, y \notin T$. Let us assume that $S_2 = \{u_k v_{l1}, u_k v_{l2}, ..., u_k v_{ln-m+1}\}$. Suppose that $x, y \in S_1$. Then $x = u_i v_j$ and $y = u_l v_m$ such that $i \neq l$ and $j \neq m$. Now, $S_3 = S - \{x, y\} \cup \{u_i v_m, u_l v_j\}$ consists of $m - 1$ independent edges and $n - m + 1$ adjacent edges of G and also containing T. By Theorem 1.3, S_3 is a minimum edge-to-vertex geodetic set of G, which is a contradiction to T is a forcing subset of G. Suppose that $x, y \in S_2$. Let $x = u_k v_{l1}$ and $y = u_k v_{l2}$. Let $u_i v_j$ be an edge of S_1. Now, join the vertices $v_{l2}, v_{l3}, ..., v_{ln-m+1}$ to u_i . Now $S_4 = S_1 - \{u_i v_j\} \cup \{u_k v_{l1}\} \cup \{u_i v_j, u_i v_{l2}, u_i v_{l3}, ..., u_i v_{ln-m+1}\}$ consists of $m - 1$ independent edges and $n - m + 1$ adjacent edges of G. By Theorem 1.3, S_4 is a minimum edge-to-vertex geodetic set of G containing T, which is a contradiction. Suppose that $x \in S_1$ and $y \in S_2$. Let $x = u_i v_j$ and $y = u_k v_{l1}$. $S_5 = S_1 - \{u_i v_j\} \cup \{u_i v_{l1}\} \cup \{u_k v_j, u_k v_{l2}, u_k v_{l3}, ..., u_i v_{ln-m+1}\}$ consists of $m - 1$ independent edges and $n - m + 1$ adjacent edges of G and also containing T. By Theorem 1.3, S_5 is a minimum edge-to-vertex geodetic set of G, which is a contradiction to that T is a forcing subset of G. Hence $f^+_{ev}(G) \leq n - 2$ is not possible.
Case(ii): Suppose that $f^+_{ev}(G) > n - 1$. This implies that, by Theorem 2.5(c), $f^+_{ev}(G) = n$. Then there exists a forcing subset T of S such that S is the unique minimum edge-to-vertex geodetic set of G containing T and $|T| = n$. Hence all the proper subsets of S containing a single element, two elements, three elements,..., $n - 1$ elements are contained in more than one minimum edge-to-vertex geodetic sets of G. Consider a proper subset F of cardinality $n - 1$($m - 2$ independent edges and $n - m + 1$ adjacent edges). Since $f^+_{ev}(G) = n$, it is clear that the proper subset F lies more than one minimum edge-to-vertex geodetic sets of G, say S_1 and S_2. Now S_1 and S_2 have $n - 1$ elements in common. This implies that the other n^{th} independent edge of S_1 and S_2 is also same. Thus we get more than one minimum edge-to-vertex geodetic set of G with the same n independent edges which is a contradiction to that T is a forcing subset of S. Hence $f^+_{ev}(G) = n - 1$. ∎

Theorem 2.19. For the complete graph $G = K_p(p \geq 4)$ with p even, $f^+_{ev}(G) = \frac{P-2}{2}$.

Proof: The proof is similar to the proof of Theorem 2.17. ∎

Theorem 2.20. For the complete graph $G = K_p(p \geq 5)$ with p odd, $f^+_{ev}(G) = \frac{P-1}{2}$.

Proof: The proof is similar to the proof of Theorem 2.18. ∎

Theorem 2.21. For a non trivial tree of size $q \geq 2$, $f^{+}_{ev}(G) = 0$.

Proof: Let G be a tree of size q. Then by Theorem 1.1, every pendent edge of G belongs to every edge-to-vertex geodetic set of G. But it is clear that, in a tree, the set of all pendent edges of G is the unique minimum edge-to-vertex geodetic set of G. Now, it follows from Theorem 2.5(a) that $f^{+}_{ev}(G) = 0$. ∎

Theorem 2.22. For a star $G = K_{1,q}$, $f^{+}_{ev}(G) = 0$.

Proof: This follows from Theorem 2.21. ∎

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